On Monotonic Patterns in Periodic Boundary Solutions of Cylindrical and Spherical Kortweg–De Vries–Burgers Equations

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Equations

The well known KdV-Burgers equation for flat waves is of the form

$$u_t = -2uu_x + \varepsilon^2 u_{xx} + \delta u_{xxx}. \tag{1}$$

Its cylindrical and spherical analogues are

$$u_t + \frac{1}{2t} u = -2uu_x + \varepsilon^2 u_{xx} + \delta u_{xxx}. \tag{2}$$

and

$$u_t + \frac{1}{t} u = -2uu_x + \varepsilon^2 u_{xx} + \delta u_{xxx}. \tag{3}$$

respectfully, [1] – [2]:

Blacktock D.T., *On plane, cylindrical and spherical sound waves of finite amplitude in lossless fluids*, Techn. Rep. AF, 49 (638), **1965**, General Dynamics, Rochester, N.Y.

Sachdev P.L., Seebas R., *Propagation of spherical and cylindrical N-waves*. Journ. of Fluid . Mech., 58, 197 (**1973**).

Alternative presentations

The equations (1)–(3) may be put in the form

$$w_t + \frac{n}{2t} w = \gamma w_{xx} - 2ww_x + w_{xxx}$$

by the change of variables $t \to t\sqrt{\delta}$, $x \to x\sqrt{\delta}$, $u \to -\frac{u}{2}$. Here $\gamma = \frac{\varepsilon^2}{\sqrt{\delta}}$ and $n = 0, 1/2, 1$ for flat, cylindrical and spherical waves correspondingly.
Still another form is obtained by the change
$t = (0.5z + 1)^2$, $u = \frac{v}{\sqrt{t}}$ for (2);
and by $t = e^y$, $u = \frac{v}{t}$ for (3). The transformed equations are

$$v_y = -2vv_x + (1 + 0.5y)(\varepsilon^2 v_{xx} + \delta v_{xxx}) \text{ and}$$

$$v_y = -2vv_x + e^y(\varepsilon^2 v_{xx} + \delta v_{xxx}) \text{ respectfully.}$$

Initial value - boundary conditions

We consider the initial value - boundary problem: 1

$$u(x,0) = f(x),\ u(a,t) = l(t),\ u(b,t) = 0,\ u_x(b,t) = 0,\ x \in [a,b]. \tag{4}$$

In the case $\delta = 0$ (that is, for Burgers equation), it comes to

$$u(x,0) = f(x),\ u(a,t) = l(t),\ u(b,t) = 0,\ x \in [a,b]. \tag{5}$$

Below, $l(t) = A\sin(\omega t)$ and $b \gg 1$.

Traveling waves for flat KdVB

For $t \gg 1$ equations (2) and (3) tend to (1) as well as their solutions.
Recall that the explicit form of traveling wave solutions for the flat KdV-Burgers (1) is as follows

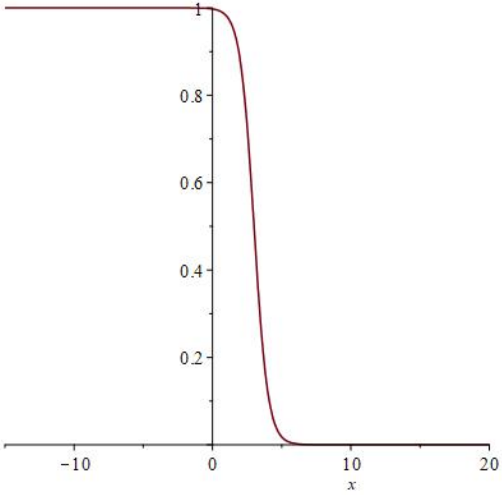
$$u_{\mathrm{tws}}(x,t) = \frac{3\varepsilon^4 \tanh^2(\frac{\varepsilon^2(x-Vt-s)}{10\delta})}{50\delta} - \frac{3\varepsilon^4 \tanh(\frac{\varepsilon^2(x-Vt-s)}{10\delta})}{25\delta} + \frac{V}{2} - \frac{3\varepsilon^4}{50\delta} \tag{6}$$

Since $u|_{x=+\infty} = 0$ the travelling wave has a velocity $V = \frac{6\varepsilon^4}{25\delta}$.
The Burgers equation ($\delta = 0$) has travelling wave solutions, vanishing at $x \to +\infty$. They are given by the formula

$$u_{\mathrm{Btws}}(x,t) = \frac{V}{2}\left[1 - \tanh\left(\frac{V}{2\varepsilon^2}(x - Vt + s)\right)\right] \tag{7}$$

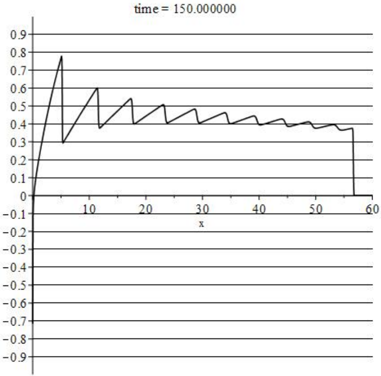
The head part of solutions to (2), (3) ultimately become similar to the latter shock, shown below.

Traveling wave for flat Burgers



Typical examples, Cylindrical Burgers

Here we demonstrate typical graphs for cylindrical and spherical Burgers waves, figure 1, 2.



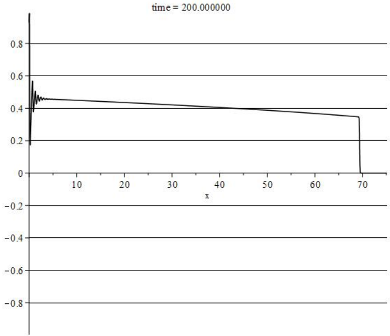


Figure: 1. *Cylindrical Burgers*, $\varepsilon = 0.1$,
Left: $u_0 = \sin t$, $t = 150$. **Right**: $u_0 = \sin 10t$, $t = 200$.

Typical examples, Spherical Burgers

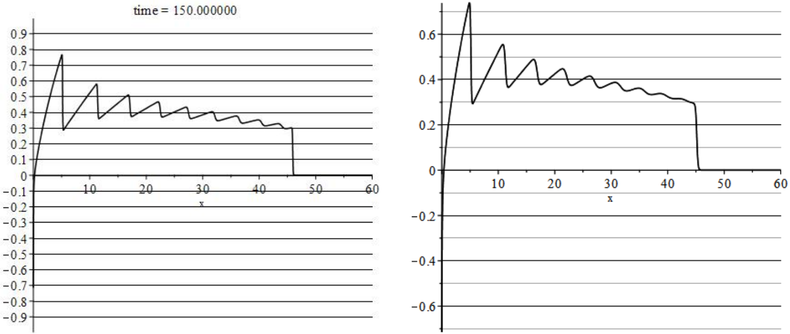


Figure: 2. *Spherical Burgers*, $u_0 = \sin t$,
Left: $\varepsilon = 0.1, t = 150$ **Right**: $\varepsilon^2 = 0.3, t = 150$

Typical examples, Cylindrical KdV-Burgers

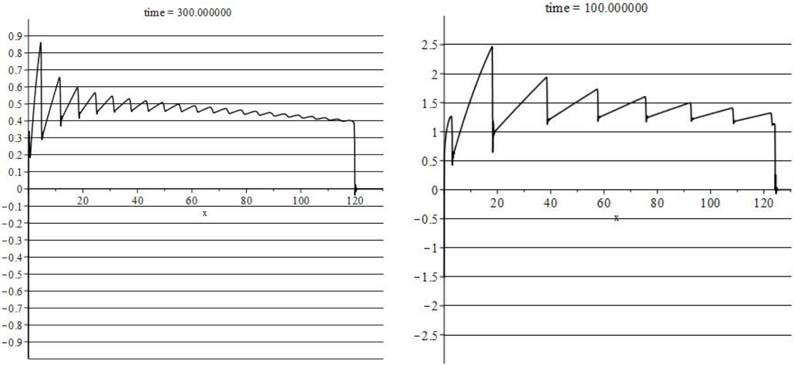


Figure: 3. *Cylindrical KdV-Burgars,*
Left: $u_0 = \sin t, t = 300, \varepsilon = 0.1, \delta = 0.001.$
Right: $u_0 = 3\sin t, t = 100, \varepsilon = 0.1, \delta = 0.001.$

Typical examples, Spherical KdV-Burgers

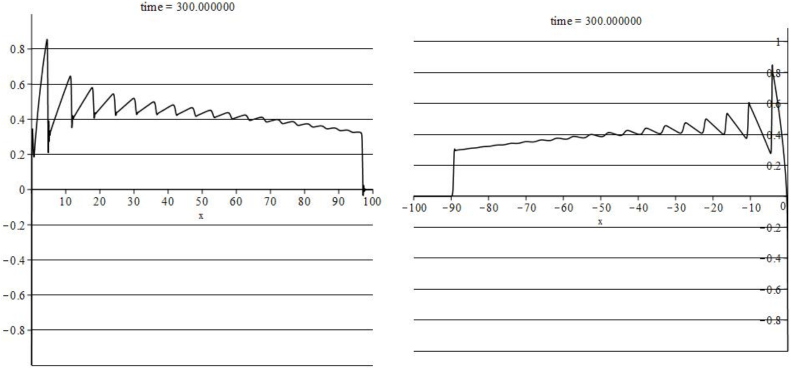


Figure: 4. *Spherical KdV-Burgers,* $u_0 = \sin t$,
Left: $t = 300, \varepsilon = 0.1, \delta = 0.001$.
Right: $x \leftrightarrow -x, t = 300, \varepsilon^2 = 0.02, \delta = 0.001\ \varepsilon^2 = 0.2$

ANIMATION, cylindrical Burgers

ANIMATION, cylindrical KdV-Burgers

Overview of examples

- For a wide variety of parameters of periodic border conditions, the perturbation starts as a sawtooth profile.
- Stronger viscosity effectively damps oscillation and may result in absence of sawtooth effects.
- Cylindrical wave moves faster and decay slower than the spherical wave with the same periodical border condition.
- Greater dispersion coefficient δ leads to a more prominent oscillations at the bottom of each tooth (at the place of the derivative breaks).
- After the decay of initial oscillations, graphs become monotonic declining convex lines, terminating by a constant amplitude shock, that travels with constant velocity. This velocity is numerically equal to the amplitude

Symmetries

Since cylindrical and spherical equations explicitly depend on time, their stock of symmetries is scarce.

The algebras of classical symmetries are generated by vector fields:

$$
\begin{aligned}
X &= \frac{\partial}{\partial x}, \\
Y &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \\
Z &= \sqrt{t}\frac{\partial}{\partial x} + \frac{1}{4\sqrt{t}}\frac{\partial}{\partial u}, \\
W &= \ln(t)\frac{\partial}{\partial x} + \frac{1}{2t}\frac{\partial}{\partial u}.
\end{aligned}
$$

It is not hard to find invariant solutions for X, Z and W symmetries. Results on invariant solutions are collected in the table.

Symmetries and invariant solutions

For the Y symmetry an invariant solution must have a form $x^{-1}f(\frac{t}{x^2})$ where $f(\xi)$ is a solution of the nonlinear differential equation

$$f'' + \frac{1}{\varepsilon^2\xi} ff' + \left(\frac{2.5}{\xi} - \frac{1}{4\xi^2\varepsilon^2}\right) f' + \frac{1}{2\xi^2\varepsilon^2} f^2 + \left(\frac{1}{2\xi^2} - \frac{1}{4\xi^3\varepsilon^2}\right) f = 0.$$

Equation	Symmetries	Invariant solutions
Cylindrical Burgers	X, Y, Z	$\frac{C}{\sqrt{(t)}}, \frac{(x+4C)}{4t}, x^{-1}f(\frac{t}{x^2})$ for some f
Cylindrical KdV-Burgers	X, Z	$\frac{C}{\sqrt{(t)}}, \frac{(x+4C)}{4t}$
Spherical Burgers	X, Y, W	$\frac{C}{t}, \frac{x+2C}{2t\ln(t)}, x^{-1}f(\frac{t}{x^2})$ for some f
Spherical KdV-Burgers	X, W	$\frac{C}{t}, \frac{x+2C}{2t\ln(t)}$

Conservation laws

First rewrite equations (1) – (3) into an appropriate, conservation law form

$$[t^n \cdot u]_t = [t^n \cdot (-u^2 + \varepsilon^2 u_x + \delta u_{xx})]_x, \tag{8}$$

$n = 0,\ 1/2,\ 1$ for flat, cylindrical and spherical cases correspondingly.
Hence for solutions of the above equations we have

$$\oint\limits_{\partial\mathcal{D}} t^n \cdot [u\, dx + (\varepsilon^2 u_x - u^2 + \delta u_{xx})\, dt] = 0, \tag{9}$$

where $\mathcal{D}$ is a rectangle

$$\{0 \le x \le L,\ 0 \le t \le T\}.$$

Conservation laws, continued

Bearing in mind the initial value/boundary conditions $u(x,0) = u(+\infty, t) = 0$, for $L = +\infty$ the integrals read

$$\int_{+\infty}^{0} T^n u(x, T)\, dx + \int_{T}^{0} t^n(\varepsilon^2 u_x(0, t) - u^2(0, t) + \delta u_{xx}(0, t))\, dt = 0.$$

Thus

$$\int_{0}^{+\infty} u(x, T)\, dx = \frac{1}{T^n} \int_{0}^{T} t^n(-\varepsilon^2 u_x(0, t) + u^2(0, t) - \delta u_{xx}(0, t))\, dt. \tag{10}$$

Subsequently

$$\frac{1}{T}\int_0^{+\infty} u(x,T)\,dx = \frac{1}{T}\int_0^T \frac{1}{T^n} t^n(-\varepsilon^2 u_x(0,t)+u^2(0,t)-\delta u_{xx}(0,t))\,dt. \tag{11}$$

The right-hand side of (11) can be computed in some simple cases or estimated. Assume that $\varepsilon^2 u_x(0,t)+\delta u_{xx}(0,t)$ is negligible compared to $u^2(0,t)$. Then

$$\frac{1}{T}\int_0^{+\infty} u(x,T)\,dx \approx \frac{1}{T}\int_0^T \frac{1}{T^n} t^n(u^2(0,t))\,dt = \frac{1}{T^{n+1}}\int_0^T t^n(A\sin^2(\omega t))\,dt. \tag{12}$$

Simple examples

It follows that

- $\lim\limits_{T\to\infty} \frac{1}{T}\int\limits_0^T A^2 \sin^2(\omega t)\, dt = \lim\limits_{T\to\infty} \frac{A^2}{2\omega T}(\omega T - 0.5\sin(2\omega T)) =$
  $\frac{A^2}{2}, \quad n = 0;$
- $\lim\limits_{T\to\infty} \frac{1}{T}\int\limits_0^T \frac{1}{T^{\frac{1}{2}}} t^{\frac{1}{2}}(A\sin^2(\omega t)\, dt =$
  $\lim\limits_{T\to\infty} \frac{A^2}{3\sqrt{\omega^3 T^3}}\left((\omega T)^{\frac{3}{2}} - \frac{3}{4}\sqrt{\omega T}\sin(2\omega T) + \frac{3}{8}\sqrt{\pi}\,\mathrm{FresnelS}(\frac{2\sqrt{\omega T}}{\sqrt{\pi}})\right) =$
  $\frac{A^2}{3}, \quad n = \frac{1}{2};$
- $\lim\limits_{T\to\infty} \frac{1}{T}\int\limits_0^T \frac{1}{T} t(A^2\sin^2(\omega t)\, dt =$
  $\lim\limits_{T\to\infty} \frac{A^2}{4\omega^2 T^2}\left(-\omega T\sin(2\omega T) + \omega^2 T^2 + \sin^2(\omega T)\right) =$
  $\frac{A^2}{4}, \quad n = 1.$

Solutions with constant boundary conditions

Take $u(0,t) = M$. The graphs of solutions for $M = 1$ are shown below.
For the resulting compression wave, $u_x(0,t) = 0$ and the right-hand side of (11) equals

$$\frac{1}{T}\int_0^T \frac{M^2}{T^n} t^n \, dt = \frac{M^2}{n+1} \tag{13}$$

As the figures 1 — 4 show, for periodic boundary condition, after the decay of initial oscillations, graphs become monotonic convex lines that begin approximately at the hight $A/2$ and breaking at $x = V \cdot T$ and at the height V.

These monotonic lines are similar to the graphs or constant-boundary solutions.

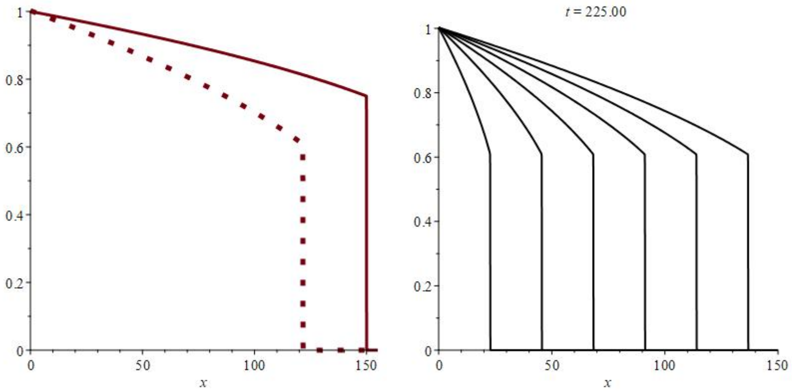


Figure: 5. *Constant boundary solutions to Burgers equation,* $\varepsilon = 0.1, t = 200$. **Left**: *Solid line — cylindrical, dots line — spherical.* **Right**: *A trace of movement to the right of the spherical solution at moments* $t = 37.5 \cdot k, k = 1 \ldots 6$

Self-similar or "homothetic" solutions

Looking at the solution's graph animation one can clearly see (eg, on figure 5, right) that the monotonic part and its head shock develops as a homothetic transformation of the initial configuration. So we seek solutions of the form $u(x,t) = y(\frac{x}{t})$. Substituting it into equations (1) – (3) we get the equation

$$-y'\frac{x}{t^2} + \frac{ny}{t} = \frac{2yy'}{t} + \frac{\varepsilon^2 y''}{t^2} + \frac{\delta y'''}{t^3}, \qquad (14)$$

or

$$-\xi y' + ny = 2yy' + \frac{\varepsilon^2 y''}{t} + \frac{\delta y'''}{t^2}, \qquad (15)$$

for $y = y(\xi)$ and $n = 0,\ 1/2,\ 1.$

For t large enough we may omit last two summands. It follows that appropriate solutions of the above ordinary differential equations are

$$u_1(x,t) = C_1, \; C_1 \in \mathbb{R}, \; n = 0, \text{ for flat waves equation;}$$

$$u_2(x,t) = -\frac{2+\sqrt{C_2\xi+4}}{C_2}, \; C_2 \in \mathbb{R}, \; n = \frac{1}{2}, \; \text{ for cylindrical and}$$

$$u_3(x,t) = \exp\left(\mathrm{LambertW}\left(-\frac{\xi}{2}e^{-\frac{C_3}{2}}\right) + \frac{C_3}{2}\right), \; C_3 \in \mathbb{R}, \; n = 1$$

for spherical equation.

Remark: Lambert function

The *Lambert W* function, also called the *omega function* or *product logarithm,* is a multivalued function, namely, the branches of the inverse relation of the function $f(w) = we^w$, where w is any complex number.
For each integer k there is one branch, denoted by $W_k(z)$, which is a complex-valued function of one complex argument.
W_0 is known as the principal branch. When dealing with real numbers the $W_0 = \mathrm{LambertW}$ function satisfies

$$\mathrm{LambertW}(x) \cdot e^{\mathrm{LambertW}(x)} = x.$$

The *Lambert W* function, introduced in 1758, has numerous applications in solving equations, mathematical physics, statistics, etc.

Finding constants

Since, at the head shock, $x = Vt$ and $u = V$ we have the condition $y(V) = V$. It follows then that
$C_1 = V,\ C_2 = -\frac{3}{V},\ C_3 = \ln(V) + \frac{1}{2}$.
For flat waves it corresponds to a travelling wave solution of the classical Burgers equation.
For the cylindrical waves the monotonic part is given by

$$u_2 = \frac{1}{3}\left(2V + V\sqrt{4 - \frac{3x}{Vt}}\right);$$

for spherical waves

$$u_3 = V\sqrt{e}\exp\left(\mathrm{LambertW}\left(-\frac{x}{2Vt\sqrt{e}}\right)\right).$$

Note that

$$u_2|_{x=0} = \frac{4V}{3} \text{ and } u_3|_{x=0} = V\sqrt{e} \approx 1.65V. \qquad (16)$$

These formulas show that the velocity is proportional to the amplitude at the start of oscillation. And it does not depend on frequency that together with amplitude define the oscillating part of solutions; more on that below.

The corresponding graphs visually coincide with the graphs obtained by numerical modelling; for instance see comparison to the solution at $(t = 100)$ for the problem

$$u_t = 0.01u_{xx} - 2uu_x - u/t, \;\; u(0,t) = 1, u(75,t) = 0, u(x,0) = 0 \tag{17}$$

on the next figure, left.

Partial approximation

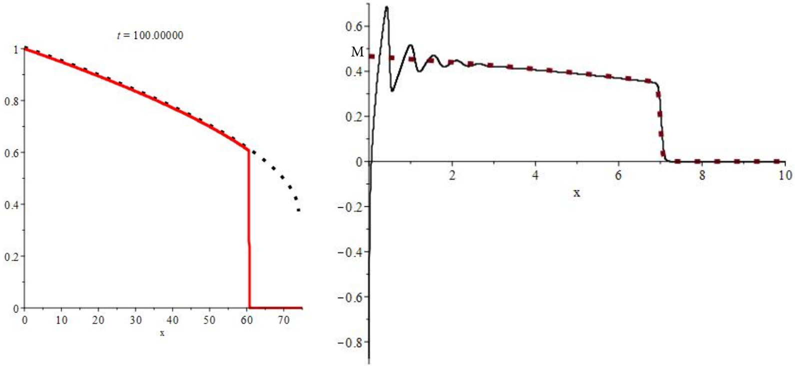


Figure: 6. **Left**: *Solid line — solution to* (17), *dots line — its* u_2 *approximation.*
Right: *Solid line— solution to* (20), *dots line — its* $\tilde{u}_2$ *approximation; both at* $t = 20$.

Approximation

Yet the smooth part of the periodic boundary solution ends with a break, which travels with a constant velocity and amplitude, very much like a head of the Burgers' travelling wave solution (7).

A rather natural idea is to truncate a homothetic solution, multiplying it by a (normalized) Burgers TWS. Namely, put or the cylindrical waves

$$\tilde{u}_2 = \frac{1}{2}[1 - \tanh(\frac{V}{\varepsilon^2}(x - Vt))] \cdot \frac{1}{3}\left(2V + V\sqrt{4 - \frac{3x}{Vt}}\right); \qquad (18)$$

for spherical waves

$$\tilde{u}_3 = \frac{1}{2}[1 - \tanh(\frac{V}{\varepsilon^2}(x - Vt))] \cdot V\sqrt{e}\exp\left(\mathrm{LambertW}\left(-\frac{x}{2Vt\sqrt{e}}\right)\right). \qquad (19)$$

Head shock approximation

This construction produces an approximation of an astonishing accuracy, see the previous figure, right; this figure corresponds to the cylindrical Burgers problem

$$\varepsilon = 0.1, u(0,t) = \sin 10t, u(10,t) = 0, u(x,0) = 0. \tag{20}$$

Moreover, it is evident that the graphs of $\tilde{u}_2$, $\tilde{u}_3$ neatly represent the median lines of the approximated solutions on their whole range.

By median we mean, for $u(0,t) = \sin \omega t$,

$$M(x) = (2\pi n/\omega)^{-1} \int_0^{2\pi n/\omega} u(x,t)\,dt, \quad n \in \mathbb{N},\ n \gg 1.$$

VIDEO, Central approximation I

Spherical KdV-Burgers, $\varepsilon^2 = 0.025, \delta = 0.002$. The change $x \to -x$ results in perturbation moving to the left. Color lines correspond to approximations at the moments $t = 100,\ 200,\ 400$. See as the propagating perturbation coincides consequently with these central approximation lines in the head part.

VIDEO, Central approximation II

Spherical KdV-Burgers, $\varepsilon^2 = 0.025, \delta = 0.002$. The change $x \to -x$ results in motion to the left.

See as the propagating perturbation coincides with the central approximation dots line in the head part.

Now evaluate the trapezoid area under $\tilde{u}_2$, $\tilde{u}_3$ graphs:

For cylindrical equation

$$\int_0^{Vt} \left[\frac{[1-\tanh(\frac{V}{\varepsilon^2}(x-Vt))]}{2} \frac{1}{3} \left(2V + V\sqrt{4-\frac{3x}{Vt}} \right) \right] dx = \frac{32}{27} V^2 t;$$

for spherical equation

$$\int_0^{Vt} \left[\frac{[1-\tanh(\frac{V}{\varepsilon^2}(x-Vt))]}{2} V\sqrt{e} \exp\left(\mathrm{LambertW}\left(\frac{-x}{2Vt\sqrt{e}} \right) \right) \right] dx = \frac{V^2 t \cdot e}{2}. \quad (21)$$

Hence the mean value of the left-hand side of (11) can be estimated as follows. Since the signal from $x = 0$ spreads, after decay of oscillations, to the right with a constant speed V and the same constant amplitude V at the shock, and it is very well approximated by an appropriate homothetic solution, we get

$$\frac{1}{T}\int\limits_0^{+\infty} u(x,T)\,dx = \frac{1}{T}\int\limits_0^{VT} u(x,T)\,dx \approx \begin{cases} \frac{32}{27}V^2 \text{ in cylindrical case;} \\ \frac{V^2\cdot e}{2} \text{ in spherical case,} \end{cases} \tag{22}$$

This mean value can be also evaluated numerically. In the case illustrated by figure 1 the direct numerical evaluation differs from the estimation (22) by 1%.

Conservation law revisited

For constant-boundary waves, it follows from (13) that

$$\frac{M^2}{n+1} = \begin{cases} \frac{32}{27} V^2 \text{ in cylindrical case;} \\ \frac{V^2 \cdot e}{2} \text{ in spherical case,} \end{cases} \tag{23}$$

So the mean value $M = M(0)$ at the start of oscillations (or in a vicinity of the oscillator) is linearly linked to the velocity of the head shock, which, in its turn, can be measured with great accuracy by the distance passed by the head shock after a sufficiently long time.

To find value M directly for an arbitrary border condition is a tricky task, because the integrands u_x and u_{xx} of the right-hand side of (11) have numerous breaks.
Numerical experiments show (eg, see figure 3), that for the $u|_{x=0} = A\sin(t)$ boundary condition such a value is $M \approx A \cdot a$, where $a \approx 0.467$ is the mean value for $1 \cdot \sin(t)$ condition.

Conclusion

- We obtained simple explicit formulas describing the monotonic part of the solution and its head break. These approximate formulas have great accuracy. Moreover, their graphs neatly represent the median lines of the approximated solutions on their entire ranges. (By median line we mean the level around which the periodical oscillations occur).
- To obtain these approximations we used self-similar solutions of the dissipationless and dispersionless KdV–Burgers equation and a traveling wave solution of the flat Burgers equation. Formulas depend on only one parameter: either on t he velocity of the signal propagation or on the median value of the solution in the vicinity of the periodic boundary.

Conclusion, continued & Acknowledgements

- Some open questions remain. Our approximations are very good for the one-parameter class of constant boundary solutions. The existence of a one-parameter family of solutions points to the existence of a suitable symmetry, but the classical symmetry analysis was, so far, unhelpful.
- Conservation laws allows us to assess the value of the approximation's parameter using the boundary condition, but the resulting estimation is too rough.

The results of the talk are available at https://www.preprints.org/manuscript/202012.0579/v1

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The graphs in this paper were obtained by numerical methods using the Maple PDETools package.

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THANK YOU FOR YOUR ATTENTION