On connection between the splitting parameters of KdV initial datum and its conserved quantities

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Equations

Korteweg-de Vries equation:

$$u_t = 2uu_x + u_{xxx}.$$

Korteweg-de Vries- Burgers equation:

$$u_t = u_{xx} + 2uu_x + u_{xxx}.$$

Generalised Korteweg-de Vries- Burgers equation (nonhomogeneous media):

$$u_t = g(x)u_{xx} + 2uu_x + f(x)u_{xxx}.$$

Here $g(x)$ and $f(x)$ are viscosity/dissipation and dispersion coefficients respectively.

KdV, $u_t = u_{xx} + 2uu_x + u_{xxx}$.

KdV possess

- Compact travelling wave solutions (solitons) $u(x,t) = 6a^2 \operatorname{sech}^2(4a^3 t + a(x+s))$;
- infinitely many conservation laws.
  The first four conserved quantities for KdV are

$$
\begin{aligned}
I_1(u)) &= \int_{-\infty}^{+\infty} u(x,t)\,dx \text{ — mass},\\
I_2(u) &= \int_{-\infty}^{+\infty} u^2(x,t)\,dx \text{ — momentum},\\
I_3(u) &= \int_{-\infty}^{+\infty} \left(2u^3(x,t) - 3(u_x(x,t))^2\right)\,dx \text{ — energy},\\
I_4(u) &= \int_{-\infty}^{+\infty} \left(5u^4 - 30u(u_x)^2 + 9(u_{xx})^2\right)\,dx,
\end{aligned}
$$

If $u(x,t)$ is a KdV solution then $\frac{\partial}{\partial t} I_k(u) = 0$, that is $u(x,0) = f(x) \Rightarrow I_k(u) = I_k(f)$, and $I_k(f)$ is constant in time.
In particular, for solitons
$u(x,t) = Sol_{a,s}(x,t) = 6a^2 \operatorname{sech}^2(a(x+s) + 4a^3 t)$,

$$
\begin{aligned}
I_1(Sol_{a,s}) &= \int_{-\infty}^{+\infty} 6a^2 \operatorname{sech}^2(ax)\,dx = 12a, \qquad (1)\\
I_2(Sol_{a,s}) &= \int_{-\infty}^{+\infty} (6a^2 \operatorname{sech}^2(ax))^2\,dx = 48a^3,\\
I_3(Sol_{a,s}) &= \frac{1728}{5} a^5,\\
I_4(Sol_{a,s}) &= \frac{20736}{7} a^7,\\
\dots \quad & \dots \quad \dots\\
I_l(Sol_{a,s}) &= K_l a^{2l-1}.
\end{aligned}
$$

There is a simple recurrent procedure to generate $I_k(u) \to I_{k+1}(u)$ using the bi-hamiltonian structure of KdV (see [?]).

Note that for the KdV of the form $u_t = u_{xxx} + 2uu_x$ the hamiltonian operators are D and $(D^3 + uD + u_x)$, where D is a total derivative with respect to x.

We assume that the initial data $u(x,0)$ is bounded and has a compact support.
Then the asymptotic form (at $t \to \infty$) of the N-soliton solution is

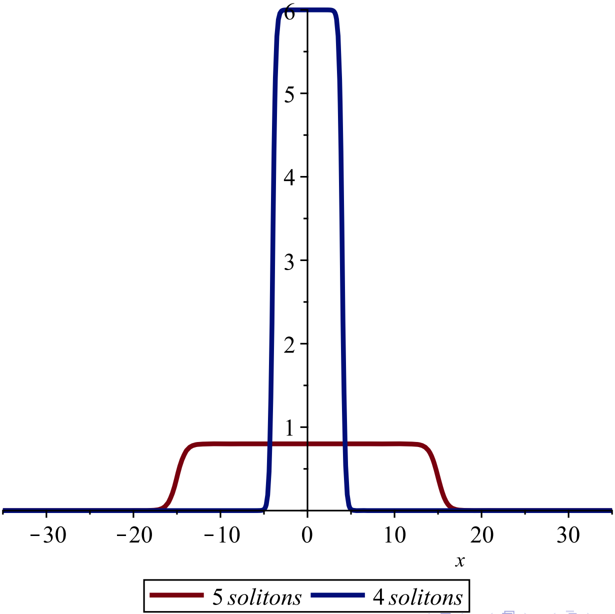
$$\sum_{i=1}^{N} 6a_i^2 \operatorname{sech}^2(a_i x + p_i + 4a_i^3 t) + R(x,t),$$

where $R(x,t)$ is a tail and phase shifts are given by the formula

$$p_i = \frac{1}{2} \log\left(\frac{\gamma_i}{2a_i} \cdot \prod_{j=i+1}^{N} \left(\frac{a_j - a_i}{a_j + a_i}\right)^2\right).$$

Here $\{-a_i^2\}$ is the the discrete specter of the differential operator $-\frac{d^2}{dx^2} + f(x)$ and γ_i are the norming constants from the inverse scattering procedure. For an arbitrary $f(x)$ this data is hard to obtain, so estimations based solely on conserved quantities may be useful.

Initial data ending in 4 and 5-soliton splitting



Splitting of $g(x) = 3(\tanh(3(x+4)) - \tanh(3(x-4)))$

Splitting of $f(x) = 0.4(\tanh(x+15) - \tanh(x-15))$, movie

Reflectionless splitting

Since after some deliberation $q(x)$ splits (at least numerically) into a disconnected sum of N different-speed solitons, we get

$$
\begin{aligned}
I_1(q)) &= \int_{-\infty}^{+\infty} q(x)\,dx = \sum_{i=1}^{N} \int_{-\infty}^{+\infty} Sol_{a_i,s}\,dx = 12\sum_{i=1}^{N} a_i \\
I_2(q) &= \int_{-\infty}^{+\infty} q^2(x)\,dx = 48\sum_{i=1}^{N} a_i^3 \\
I_3(q) &= \int_{-\infty}^{+\infty} \left(2q^3(x) - 3(q_x(x))^2\right)\,dx = \frac{1728}{5}\sum_{i=1}^{N} a_i^5 \\
I_4(q) &= \int_{-\infty}^{+\infty} \left(5q^4 - 30q(q_x)^2 + 9q_{xx}^2\right)\,dx \\
&= \frac{20736}{7}\sum_{i=1}^{N} a_i^7
\end{aligned} \tag{2}
$$

Thus we obtain the system on $a_i,\ i = 1 \dots N$:

$$K_j \sum_{i=1}^{N} a_i^{2j+1} = I_j(q),\ j = 1 \dots N,$$

where K_j is the constant specific to the j-th conserved quantity and $a_1 > a_2 > \dots a_N > 0$ is assumed.

The above equations hold for all $j = 1 \dots \infty$, but to find N solitons it suffice to consider only first N equations.

General case

$$
\begin{aligned}
I_1(q)) &= 12\sum_{i=1}^{N} a_i + \int_{-\infty}^{+\infty} R(x,t)\,dx \\
I_2(q) &= 48\sum_{i=1}^{N} a_i^3 + \int_{-\infty}^{+\infty} R^2(x,t)\,dx \\
I_3(q) &= \frac{1728}{5}\sum_{i=1}^{N} a_i^5 + \int_{-\infty}^{+\infty} \left(2R^3(x,t) - 3(R_x(x,t))^2\right)\,dx \\
\dots & \quad \dots
\end{aligned}
$$

It follows that the discrepancies $I_j(R(x,t)) = I_j(q) - K_j\sum_{i=1}^{N} a_i^{2j+1}$ are also constant.

Signs of disrepancies

The first four of $I_j(R)$ are alternating in sign.

Indeed, at least the initial perturbation mass is carried away by solitons, so $I_1(R) \leqslant 0$.

Since momentum of any part of solution is non-negative, it follows that $I_2(R) \geqslant 0$. The reflected tail is oscillating around zero value, therefore

$$\int_{-\infty}^{+\infty} \left(2R^3(x,t)\right)\, dx \text{ is small while } \int_{-\infty}^{+\infty} \left(-3(R_x(x,t))^2\right)\, dx$$

is negative and comparatively large; so $I_3(R) \leqslant 0$.

Signs of disrepancies

Similarly plausible argument can be applied to $I_4(R)$ if the conservation law is rewritten to equivalent quadratic form $5u^4 - 30uu_x^2 + 9u_{xx}^2 \sim$

$$5u^4 + 15u^2 u_{xx} + 9u_{xx}^2 = 9(u^2 + \frac{5+\sqrt{5}}{6}u_{xx})(u^2 + \frac{5-\sqrt{5}}{6}u_{xx});$$

Hypothesis The whole series of conserved quantities for a radiation tail is alternating in signs.

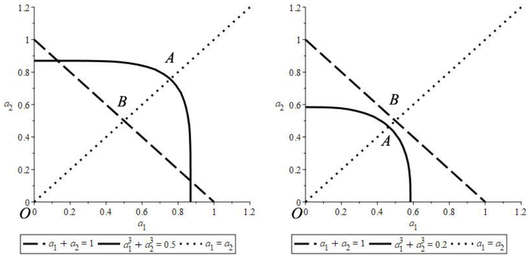
Admissible domain

Necessary conditions

$$
\begin{aligned}
I_1(q) &\leqslant 12\sum_{i=1}^{N} a_i\\
I_2(q) &\geqslant 48\sum_{i=1}^{N} a_i^3\\
I_3(q) &\leqslant \frac{1728}{5}\sum_{i=1}^{N} a_i^5\\
I_4(q) &\geqslant \frac{20736}{7}\sum_{i=1}^{N} a_i^7\\
\dots &\quad \dots;\\
a_i &\geqslant 0.
\end{aligned}
$$

Number of solitons after splitting

The system (3) defines the admissible domain in $\{x_1, x_2, \ldots, x_n\}$, the solitons'parameters space. In order for this domain not to be empty a number of inequalities must hold, for instance $I_1(q) \geqslant 12 \sum_{i=1}^{N} a_i, \; I_2(q) \leqslant 48 \sum_{i=1}^{N} a_i^3$. For $n = 2$ we have $a_1 + a_2 \geqslant p_1, \; a_1^3 + a_2^3 \leqslant p_3$ (here $p_k = K_j^{-1} I_j(q), j = 2k - 1$).



The admissible domain would be nonempty if $OA > OB$ as on the graph (above left), where $p_1 = 1, p_2 = 0.5$. For both points A and B $a_1 = a_2$, so $OA = \sqrt{2\left(\frac{p_1}{2}\right)^2} = \sqrt{\frac{1}{2}}$ and $OB = \sqrt{2\left(\frac{p_2}{2}\right)^2} = \sqrt{\frac{1}{8}}$

On the right graph $p_1 = 1, p_2 = 0.2$ and admissible domain is empty.Increase the number of solitons to n. Then $a_1 = a_2 = \dots a_n = \frac{1}{n}$ for A and $a_1^3 = a_2^3 = \cdots = a_n^3 = \frac{0.2}{n}$. If require $OA^2 = n\left(\frac{1}{n^2}\right) \geqslant OB^2 = n\left(\sqrt[3]{\frac{0.2}{n}}\right)^2 \Rightarrow n^2 > 5 \Rightarrow n = 3.$

For arbitrary p_1, p_2 the smallest number of solitons is the integer n such that

$$n \geqslant \sqrt{\frac{p_1^3}{p_2}}. \tag{3}$$

For other conserved quantities similar conditions of non-emptiness of the admissible domain lead to compare $n^{k-1} \vee \frac{p_1^k}{p_2}$. However usually (eg, for all examples below) it suffice to use (3) to predict the right number of resulting solitons.

Remark: The intersection points correspond to reflectionless splitting; their coordinates may be used as a estimations of the solitons' train parameters.

Back to videos

In the case of the second video
$I_1(q) = 24$ $I_2(q) = 18.56,$ $I_3(q) = 27.904,$ $I_4(q) = 55.637.$.
The number of solitons $n > \sqrt{\frac{p_1^3}{p_2}} = \sqrt{\frac{2^3}{0.3867}} \approx 4.5 \Rightarrow n = 5$ Note that the system for 4 solitons $\sum_{i=1}^4 a_i^{2j-1}$, $j = 1, \ldots, 4$ has no solutions.
Amplitudes $1.543 = 6a_1^2$, $1.385 = 6a_2^2$, $1.125 = 6a_3^2$, $0.775 = 6a_4^2$, $0.36 = 6a_5^2$ (it is easily measurable on display), and
$a_1 \approx 0.507$, $a_2 \approx 0.480$, $a_3 \approx 0.433$, $a_4 \approx 0.359$, $a_2 \approx 0.245$.
The inequalities are satisfied:

$$K_j \sum_{i=1}^{5} a_i^{2j-1} \geqslant I_j(q) \text{ for } j = 1, 3 \text{ и}$$

$$K_j \sum_{i=1}^{5} a_i^{2j-1} \leqslant I_j(q) \text{ for } j = 2, 4$$

Non-homogeneous media. Model

Consider equation

$$u_t(x,t) = 2uu_x + f'(x)u_{xx} + f(x)u_{xxx} = \left(u_x^2 + f(x)u_{xx}\right)_x. \quad (4)$$

Since the equation has a form $u_t = F_x(u)$, the mass $\int_{-\infty}^{+\infty} u\,dx$ is conserved.

But the momentum $\langle u^2 \rangle = \displaystyle\int_{-\infty}^{+\infty} u^2\,dx$ does not:

$$\tfrac{1}{2}\langle u^2 \rangle_t = \langle uu_t \rangle = \langle u(u^2 + f(x)u_{xx})_x \rangle = \langle 2u^2u_x \rangle + \langle u(f(x)u_{xx})_x \rangle$$

$$\tfrac{2}{3}u^3\Big|_{-\infty}^{+\infty} + f(x)uu_{xx}\Big|_{-\infty}^{+\infty} - \langle u_x f(x)u_{xx} \rangle = -\tfrac{1}{2}\, f(x)u_x^2\Big|_{-\infty}^{+\infty} + \langle f'(x)u_x^2 \rangle,$$

$$\langle u^2 \rangle_t = 2\langle f'(x)u_x^2 \rangle. \quad (5)$$

The choice of $f(x)$

We consider examples where $f(x) > 0$ is numerically constant outside a finite neighborhood of origin.

If $f(\pm\infty) = \gamma_\pm$, the equation at $x \to \pm\infty$ comes to

$$u_t = 2uu_x + \gamma_\pm u_{xxx}$$

These are KdVs, whose solitons are given by

$$6\gamma_\pm a^2 \operatorname{sech}^2(a(x+s) + 4\gamma_\pm a^3 t);$$

they move to the left.

If start with a KdV$_+$ soliton to the right of the above neighborhood, it crosses the transient region and becomes KdV$_-$ soliton or splits into a number of them.

The number of solitons and their parameters may be evaluated using the monotonicity of the momentum evolution (5).

Conservation laws restrictions, single soliton

Suppose the problem

$$\{u_t = (u^2 + f(x)u_{xx})_x, \quad f(+\infty) = \gamma_0, f(-\infty) = \gamma_1\}$$

has a solution $u(x,t)$, such that
$u(x,0) = 6a_0^2\gamma_0 \operatorname{sech}^2(a_0((x+s)+4a_0^2 t))|_{t=0}$
and at $t \gg 0$, $u(x,t)$ it coincides with a soliton, possibly plus reflected wave.
Let this soliton has the amplitude $6a_1^2\gamma_1$.
If it is plausible to ignore a reflected wave then

$$\begin{aligned}
12a_1\gamma_1 &= 12a_0\gamma_0 && -\text{mass conservation;} \\
48a_1^3\gamma_1^2 &> 48a_0^3\gamma_0^2 && -\text{impulse evolution if } f' \geqslant 0; \\
48a_1^3\gamma_1^2 &< 48a_0^3\gamma_0^2 && -\text{impulse evolution if } f' \leqslant 0.
\end{aligned} \tag{6}$$

Then $\frac{a_1}{a_0} = \frac{\gamma_0}{\gamma_1}$ and reflection coefficient $R = \frac{V_1}{V_0} = \frac{a_1^2\gamma_1}{a_0^2\gamma_0} = \frac{\gamma_0}{\gamma_1}$

Suppose the problem

$$\{u_t = (u^2 + f(x)u_{xx})_x, \quad f(+\infty) = \gamma_0, f(-\infty) = \gamma_1\}$$

has a solution $u(x,t)$, such that
$u(x,0) = 6a_0^2\gamma_0 \operatorname{sech}^2(a_0((x+s)+4a_0^2 t))|_{t=0}$
and at $t \gg 0$, $u(x,t)$ it coincides with a bi-soliton, possibly plus reflected wave.
Let bi-soliton consist of peaks with amplitude $6a_1^2\gamma_1$ and $6a_2^2\gamma_1$.
If it is plausible to ignore a reflected wave then

$$\begin{aligned} &12a_1\gamma_1 + 12a_2\gamma_1 = 12a_0\gamma_0 && -\text{mass conservation;} \\ &48a_1^3\gamma_1^2 + 48a_2^3\gamma_1^2 > 48a_0^3\gamma_0^2 && -\text{impulse evolution if } f' \geqslant 0; \\ &48a_1^3\gamma_1^2 + 48a_2^3\gamma_1^2 < 48a_0^3\gamma_0^2 && -\text{impulse evolution if } f' \leqslant 0. \end{aligned} \tag{7}$$

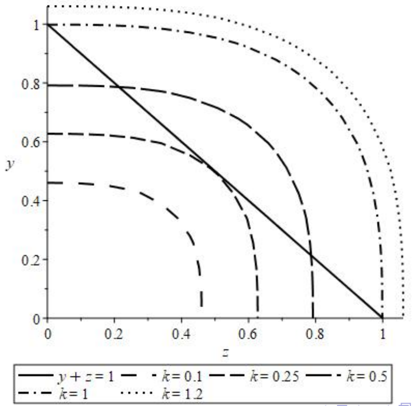
Conservation laws restrictions, 4

Denote
$$y=\frac{a_1\gamma_1}{a_0\gamma_0},\quad z=\frac{a_2\gamma_1}{a_0\gamma_0},\quad k=\frac{\gamma_1}{\gamma_0};$$
the (7) may be rewritten to the form
$$\begin{aligned} y+z&=1 && -\text{mass conservation;}\\ y^3+z^3&>k && -\text{impulse evolution if } k<1;\\ y^3+z^3&<k && -\text{impulse evolution if } k>1.\end{aligned}\tag{8}$$
The solution of the system $\{y+z=1,\ y^3+z^3=k\}$ is $\{\frac{1}{2}\pm\frac{1}{6}\sqrt{12k-3},\frac{1}{2}\mp\frac{1}{6}\sqrt{12k-3}\}$. Since obviously $0\leqslant y,\ z\leqslant 1$, it make sense only for $\frac{1}{4}\leqslant k\leqslant 1$, see the next figure.
In this case for the first (greater) peak it follows that
$$1>y=\frac{a_1\gamma_1}{a_0\gamma_0}>y_+=\frac{1}{2}+\frac{1}{6}\sqrt{12k-3}.$$

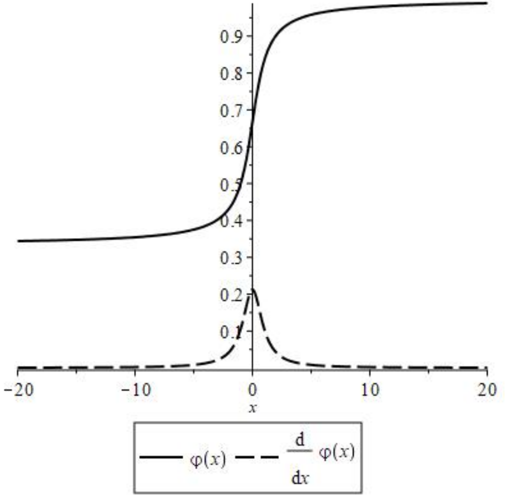
Conservation laws restrictions, 5

Since the refraction coefficient $R = \frac{V_1}{V_0} = \frac{4a_1^2\gamma_1}{4a_2^0\gamma_0}$ equals $\frac{y^2}{k}$ we obtain the restriction on the first peak refraction coefficient (it also coincides with the amplitudes ratio)

$$R > \frac{2k + 1 + \sqrt{12k - 3}}{6k}.$$



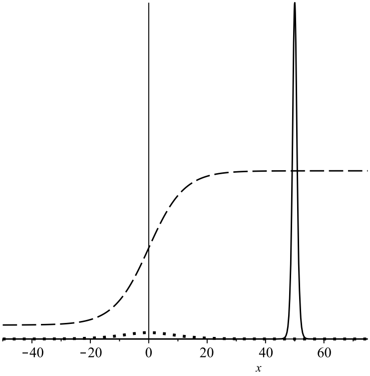
Example 1. $\varphi(x) = \frac{2}{3}\left(1 + \frac{1}{\pi}\arctan(x)\right)$.



Movie for $\varphi(x) = \frac{2}{3}\left(1 + \frac{1}{\pi}\arctan(x)\right)$.

Example 2

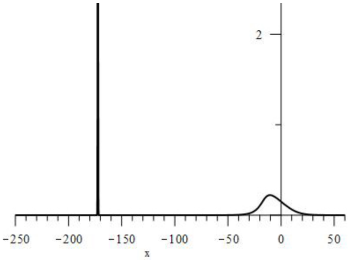
The decreasing (with respect to the direction of the soliton motion) dispersion coefficient $f(x) = \frac{1}{24}\left(13 + 11\tanh(\frac{x}{12})\right)$ in $u_t = \left(u_x^2 + f(x)u_{xx}\right)_x$. Thus $u_t = 2uu_x + u_{xxx}$ at $x = +\infty$ and $u_t = 2uu_x + \frac{1}{12}u_{xxx}$ at $x = -\infty$.



Nonhomogeneous distribution $f(x) = \frac{1}{24}\left(13 + 11\tanh(\frac{x}{12})\right)$ and the initial soliton $6\,\mathrm{sech}^2(4t + x - 50)$.

Soliton in the case of $f(x) = \frac{1}{24}\left(13 + 11\tanh(\frac{x}{12})\right)$

No reflection! Example 2, continued



Soliton $6\,\mathrm{sech}^2(4t + x - 50)$ transformed by the nonhomogeneous layer $\{f(x) = \frac{1}{24}\left(13 + 11\tanh(\frac{x}{12})\right), g(x) = f'(x)\}$, (enlarged part of the previous movie final frame).

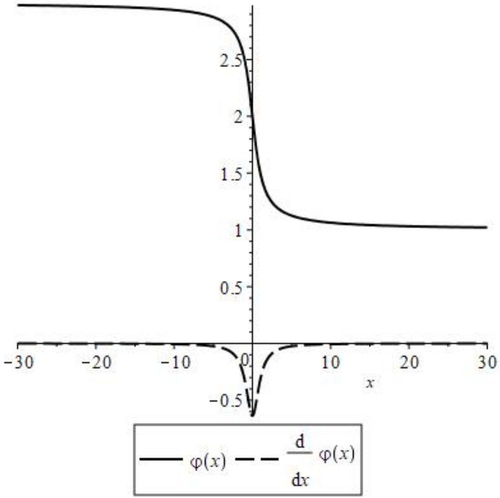
No reflected wave can be seen.

The stable height of the first peak is 18.5 high. The height of the second one is about 0.37 (one must be cautious since the peak is under formation: it have not wholly left the transition region).

Recall that the amplitude of the initial soliton is 6.

Example 3. Transmitted solitary wave

Increasing (with respect to the soliton motion) dispersion coefficient $\varphi(x) = \frac{2}{3}\left(1 + \frac{1}{\pi}\arctan(x)\right)$ in $u_t = \left(u_x^2 + \varphi(x)u_{xx}\right)_x$. Thus $u_t = 2uu_x + u_{xxx}$ at $x = +\infty$ and $u_t = 2uu_x + \frac{1}{3}u_{xxx}$ at $x = -\infty$.



$\varphi(x) = \frac{2}{3}\left(1 + \frac{1}{\pi}\arctan(x)\right)$

Thank you
for your attention