Continuum mechanics of media with innerstructures

A. Duyunova, V. Lychagin, S. Tychkov

May 25, 2020

Motivation

1. Diatomic gases. Configuration space of particle is $M = \mathbb{R}^3 \times \mathbb{RP}^2$ or $M = \mathbb{R}^3 \times \mathbb{S}^2$.
2. Atmosphere consisting of such gases. Configuration manifold is the total space of the bundle $\pi \colon M \to \mathbb{R}^+ \times \mathbb{S}^2$, fibers is diffeomorhic to $\mathbb{RP}^2$ or $\mathbb{S}^2$.
3. Water. The configuration space of particle is diffeomorphic to $\mathbb{RP}^3$.
4. Media composed with solids (Cosserat theory). Configuration space of such media is $\mathbb{R}^3 \times SO(3) \to \mathbb{R}^3$.

Configuration space

By a configuration space of such medium, we mean:

1. Smooth bundle $\pi\colon M \to B$, where M and B are Riemannian manifolds equipped with metrics g_M and g_B correspondingly.
2. To compare inner structures (i.e. fibers of π) at different points of B, we assume that π is equipped with a connection ∇_π, which splits tangent spaces T_mM into the vertical T_m^vM and horizontal $H_m \overset{\pi_*}{\simeq} T_{\pi(m)}B$ parts, i.e. $T_mM = T_m^vM \oplus H_m$, where spaces T_m and H_m are orthogonal. Also, the restriction of the metric g_M to H_m coincides with g_B.
3. Flow in the medium is given by a π-projectable vector field X on M. This field can be split due to the connection ∇_π into the sum $X = X_H + X_V$, where X_V is a π-vertical field and X_H is a horizontal lift of the vector field $\pi_*(X)$ on the base manifold B.

Thermodynamics of media with inner structure

The *extensive* quantities: the internal energy density ε and *the rate of deformation* Δ, where $\Delta = d_{\nabla} X \in \mathrm{End}\, T$, $X \in T$ is the flow velocity and ∇ is the Levi-Civita connection associated with the metric g_M.

The *intensive,* quantities are the temperature θ and the stress tensor $\sigma \in \mathrm{End}\, T^*$. We will use the duality of End T and End T^* given by the pairing $\langle \sigma, \Delta \rangle = \mathrm{Tr}\ \sigma \Delta^*$.

The thermodynamic phase space of the medium is

$$\Phi = \mathbb{R}^3 \times \mathrm{End}\, T^* \times \mathrm{End}\, T,$$

with coordinates

$$(s, \theta, \varepsilon, \sigma, \Delta),$$

where s is the entropy density.

Thermodynamics

Φ is a contact manifold equipped with the structure form [1]

$$\alpha = ds - \theta^{-1}\, d\varepsilon + \theta^{-1} \sum_{i,j} \sigma_{ij} d\Delta_{ij}.$$

The first law of thermodynamics: the *thermodynamic state* is a maximal integral manifold of α, i.e. a Legendrian manifold $L \subset (\Phi, \alpha)$ of dimension $\dim(\mathrm{End}\, T) + 1$.

The projection ϕ of the contact manifold Φ to the symplectic manifold $(\widetilde{\Phi}, d\alpha)$, where $\widetilde{\Phi} = \mathbb{R}^2 \times \mathrm{End}\, T^* \times \mathrm{End}\, T$ and $\phi(s, \theta, \varepsilon, \sigma, \Delta) = (\theta, \varepsilon, \sigma, \Delta)$.

The restriction of the mapping ϕ to the Legendrian manifold L is a local diffeomorphism on the image $\widetilde{L} = \phi(L)$, and, therefore, $\widetilde{L} \subset \widetilde{\Phi}$ is an immersed Lagrangian manifold in a $2(\dim(\mathrm{End}\, T) + 1)$-dimensional symplectic manifold equipped with the structure form

$$d\alpha = \theta^{-2}(d\theta \wedge d\varepsilon + \sum_{i,j} (\theta\, d\sigma_{ij} \wedge d\Delta_{ij} + \sigma_{ij}\, d\Delta_{ij} \wedge d\theta)).$$

Thermodynamics

The first law of thermodynamics: the thermodynamic state is a Lagrangian submanifold of the symplectic manifold $(\widetilde{\Phi}, d\alpha)$.
Also we require (see [1] for details) the quadratic differential form

$$\kappa = \theta^{-2}(d\theta \cdot d\varepsilon + \sum_{i,j}(\theta\, d\sigma_{ij} \cdot d\Delta_{ij} + \sigma_{ij} d\Delta_{ij} d\theta)).$$

Thermodynamics

The symplectic structure defines the Poisson bracket on functions on $\widetilde{\Phi}$ of the form

$$[F, G] = \frac{\theta}{2}\left(\frac{\partial G}{\partial \Delta}\cdot\frac{\partial G}{\partial \sigma} - \frac{\partial F}{\partial \Delta}\cdot\frac{\partial G}{\partial \sigma} + \theta\left(\frac{\partial G}{\partial \varepsilon}\frac{\partial F}{\partial \theta} - \frac{\partial G}{\partial \varepsilon}\frac{\partial G}{\partial \theta}\right) - \sigma\cdot\left(\frac{\partial F}{\partial \varepsilon}\frac{\partial G}{\partial \sigma} - \frac{\partial G}{\partial \varepsilon}\frac{\partial G}{\partial \sigma}\right)\right).$$

The thermodynamic state of the medium can be also defined by equations

$$F_k(\theta, \varepsilon, \sigma, \Delta) = 0, \quad k = 1, \ldots, \dim(\mathrm{End}\, T) + 1,$$

where all pair-wise brackets $[F_k, F_l]$ vanish.

Thermodynamics

We introduce the density of Helmholtz free energy $h = h(\theta, \Delta)$ on L, $h = \varepsilon - \theta s$, then we get the following equations for the Lagrangian manifold $\widetilde{L}$

$$\sigma = h_{\Delta}, \quad \varepsilon = (\theta h)_{\theta}\,. \qquad (1)$$

In these coordinates the quadratic form κ is

$$\kappa = \theta^{-2}\left((2h_{\theta} + \theta h_{\theta\theta})d\theta^2 + 2(h_{\Delta} + \theta h_{\Delta\theta})d\theta d\Delta + \theta \sum_{i,j,k,l} h_{\Delta_{ij}\Delta_{kl}} d\Delta_{ij} \otimes d\Delta_{kl}\right).$$

Thermodynamic invariants

Consider a medium that possesses a symmetry given by an algebraic group $G \subset GL(T)$.
G-action on the tangent space T can be prolonged to the contact G-action on thermodynamic phase space Φ, if we assume that this action is trivial on $\mathbb{R}^3 = (s, \theta, \varepsilon)$ and natural on $\mathrm{End}\, T^* \times \mathrm{End}\, T$.
Let $J_1, \ldots, J_N$ be a set of algebraically independent rational G-invariants on Φ, which generate the field of rational G-invariants and, therefore, separate regular G-orbits (Rosenlicht theorem [2]).
Then a regular G-invariant thermodynamic state, i.e. a G-invariant algebraic Legendrian manifold $L \subset \Phi$ such that almost all G-orbits in L are regular, can be written in the form of $h = h(J_1, \ldots, J_N)$, where h is a rational function.
We consider only 'Newtonian media', i.e. media with a symmetry group $G = O(g) \subset GL(T)$, where T is a Euclidean vector space with a metric g.

Thermodynamic invariants

Theorem (Procesi [3])

Algebra of polynomial $O(g)$-invariants on $A \in \operatorname{End} T$ is generated by the Artin-Procesi invariants

$$\mathcal{P}_{\alpha,\beta}(A) = \operatorname{Tr}\left(A^{\alpha_1} A'^{\beta_1} \cdots A^{\alpha_m} A'^{\beta_m}\right), \quad \sum_i (\alpha_i + \beta_i) \leq 2^n - 1,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$ are multi-indices.

The next theorem follows from the Procesi theorem, the Rosenlicht theorem [2] and the observation that codimension of regular orbits equals

$$\nu = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

Theorem

Field of rational invariants of the $O(g)$-action on $\operatorname{End} T$ is generated by any ν algebraically independent Artin-Procesi invariants. This field separates regular orbits.

Thermodynamic invariants

In this case, the following state equation:

$$\sigma = \frac{\partial h}{\partial \Delta} = \sum_{\alpha,\beta} \frac{\partial h}{\partial \mathcal{P}_{\alpha,\beta}} \frac{\partial \mathcal{P}_{\alpha,\beta}}{\partial \Delta}.$$

If we consider media, which satisfy 'Hooke's law', the Helmholtz free energy is a quadratic function of Δ and, therefore, has the form:

$$h = \frac{1}{2}\left(a(\theta)\mathcal{P}_2(\Delta) + b(\theta)\mathcal{P}_{11}(\Delta) + c(\theta)\mathcal{P}_1^2(\Delta)\right) + d(\theta)\mathcal{P}_1(\Delta),$$

where a, b, c, d are some functions.
In this case the state equations take the form:

$$\sigma = a(\theta)\Delta' + b(\theta)\Delta + (c(\theta)\operatorname{Tr}\Delta + d(\theta))1.$$

Thermodynamic invariants of media with inner structure

Let a Euclidean vector space (T, g) be the orthogonal direct sum of a vertical (V, g_F) and a horizontal (H, g_B) Euclidean spaces, that is

$$(T, g) = (V, g_F) \oplus (H, g_B), \qquad (2)$$

where $\dim V = m$, $\dim H = n$.

Now we study invariants of the natural $O(g_F) \times O(g_B)$-action on End T.

Let Π_V be the orthogonal projector to V.

Theorem

Algebra of polynomial $O(g_F) \times O(g_B)$-invariants on $A \in \mathrm{End}\, T$ is generated by Artin-Procesi invariants

$$\mathcal{P}_{\alpha,\epsilon,\beta}(A) = \mathrm{Tr}\,(A^{\alpha_1}\Pi_V^{\epsilon_1}A'^{\beta_1}\cdots A^{\alpha_k}\Pi_V^{\epsilon_k}A'^{\beta_k}), \quad \sum_i(\alpha_i+\epsilon_i+\beta_i) \leq 2^{n+m}-1$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$, $\beta = (\beta_1, \ldots, \beta_m)$ are multi-indices.

Thermodynamic invariants of media with inner structure

Similar to the ordinary Newtonian media, for the Newtonian media with inner structure, we have the following state equations:

$$\sigma = \frac{\partial h}{\partial \Delta} = \sum_{\alpha,\epsilon,\beta} \frac{\partial h}{\partial \mathcal{P}_{\alpha,\epsilon,\beta}} \frac{\partial \mathcal{P}_{\alpha,\epsilon,\beta}}{\partial \Delta}.$$

In the case when the media satisfy 'Hooke's law', the Helmholtz free energy is a quadratic function of Δ and therefore has the form:

$$h = \frac{1}{2}\left(a_1(\theta)\,\mathrm{Tr}\ \Delta^2 + a_2(\theta)\,\mathrm{Tr}\,(\Delta\Delta') + a_3(\theta)\,\mathrm{Tr}^2\ \Delta + a_4(\theta)\,\mathrm{Tr}^2\,(\Delta\Pi_V) + a_5(\theta)\,\mathrm{Tr}\,(\Delta'\Delta\Pi_V) + a_6(\theta)\,\mathrm{Tr}\,(\Delta\Delta'\Pi_V)\right) + b_1(\theta)\,\mathrm{Tr}\ \Delta + b_2(\theta)\,\mathrm{Tr}\ \Delta\Pi_V,$$

where $a_1, \ldots, a_6, b_1, b_2$ are some functions.
In this case the state equations take the form:

$$\sigma = a_1(\theta)\Delta' + a_2(\theta)\Delta + (a_3(\theta)(\mathrm{Tr}\ \Delta) + b_1(\theta))1 + (a_4(\theta)\,\mathrm{Tr}\,(\Delta\Pi_V) + b_2(\theta))\Pi_V + a_5(\theta)\Delta\Pi_V + a_6(\theta)\Pi_V\Delta.$$

We consider a Riemannian manifold (M, g) and write down the conservation laws for an arbitrary vector field X on M. We will assume that M is an oriented manifold and $\Omega = \Omega_g$ is the volume form associated with the metric g. We denote by ∇ and d_∇ the Levi-Civita connection and the covariant differential also associated with metric g.
By

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \nabla_X,$$

we denote the material derivative.

Divergence

The divergence of a vector field, $\operatorname{div} X$, is defined in the standard way in terms of Lie derivative:

$$\mathcal{L}_X \Omega = (\operatorname{div} X)\, \Omega.$$

On the other hand, the covariant differential $d_\nabla X \in T \otimes T^*$ is a field of linear operators acting in the tangent spaces and we get

$$\operatorname{div} X = \operatorname{Tr}(d_\nabla X).$$

To see that we get an equivalent construction, let us write down the latter in local coordinates $x_1, \ldots, x_n$.
We get

$$d_\nabla X = \sum_{i,j} \left(\frac{\partial X_i}{\partial x_j} + \sum_k \Gamma^i_{kj} X_k \right) \frac{\partial}{\partial x_i} \otimes dx_j,$$

where $X = \sum X_i \frac{\partial}{\partial x_i}$, and Γ^i_{kj} are the Christoffel symbols.

Divergence

The important case for us is the case of linear operators $A \in \mathrm{End}\, T = T \otimes T^*$.
In this case, $d_\nabla A \in \mathrm{End}\, T \otimes T^* = T \otimes T^* \otimes T^*$ and, by taking $(1,3)$-contraction $c_{1,3}$, we get a differential 1-form that we call the divergence of the operator A:

$$\mathrm{div}\, A = c_{1,3}(d_\nabla A) \in T^*.$$

In local coordinates, we have

$$A = \sum a_i^k \frac{\partial}{\partial x_i} \otimes dx_k, d_\nabla \left(\frac{\partial}{\partial x_i} \right) = \sum \Gamma_{ij}^k \frac{\partial}{\partial x_k} \otimes dx_j,$$
$$d_\nabla (dx_i) = -\sum \Gamma_{jk}^i dx_k \otimes dx_j$$

and, therefore,

$$\mathrm{div}\, A = \sum_{i,k} \left(\frac{\partial a_i^k}{\partial x_i} + \sum_j (a_j^k \Gamma_{ij}^i - a_i^j \Gamma_{ik}^j) \right) dx_k.$$

Density if internal force

We consider the stress tensor $\sigma \in \mathrm{End}\, T$ as the surface force $\widehat{\sigma} = g(\sigma(\nu), \cdot) \in T^*$ applied to an imaginary surface orthogonal to a normal vector ν. In our case we cannot directly find the 'integral sum' of all forces applied to a volume, since each of the 'applied forces' belongs to different spaces.
It can be shown that the density of internal force is $\mathrm{div}\, \sigma$ (see [4]).

Conservation laws

We have the following system of differential equations describing media with inner structures:

$$\begin{cases} \dfrac{d\rho}{dt} + \rho \operatorname{div} X = 0, \\ \rho \dfrac{dX}{dt} = \operatorname{div}^\flat \sigma, \\ \dfrac{d\varepsilon}{dt} + \varepsilon \operatorname{div} X + \operatorname{div}(\mathcal{J}_q) + \langle \sigma, \Delta \rangle = 0, \end{cases}$$

where

$$\sigma = \frac{\partial h}{\partial \Delta}, \quad \varepsilon = h + \theta \frac{\partial h}{\partial \theta},$$

and X is a π-projectable vector field.

References

- Lychagin, Valentin, Contact geometry, measurement, and thermodynamics. Nonlinear PDEs, their geometry, and applications, 3–52, Tutor. Sch. Workshops Math. Sci., Springer, 2019.
- Rosenlicht M., A remark on quotient spaces, An. Acad. Brasil. Cienc. 35 (1963) 487–489.
- Procesi C., Lie groups: an approach through invariants and representations, Springer, (2005).
- arXiv:2005.05840 [math-ph]