On the second Painlevé equation and its higher analogues

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Contents

The Painlevé property

Example 1.

No singularity points	Simple pole	Branch singularity point
$\dfrac{dw}{dz} = w$	$\dfrac{dw}{dz} = w^2$	$\dfrac{dw}{dz} = \dfrac{1}{2} w^3$
$w(z) = z_0 e^z$	$w(z) = \dfrac{1}{z_0 - z}$	$w(z) = \pm \dfrac{1}{\sqrt{z_0 - z}}$

Definition 1. If singular points of the solution of the differential equation depend on the initial data, then such points are called *movable*.

The Painlevé property

Definition 2. A differential equation satisfies *the Painlevé property*, if its explicit solution has movable simple poles only.

Problem 1. Classify all equations satisfied the Painlevé property in the given form

$$\frac{d^2 w}{dz^2} = P\left(z, w, \frac{dw}{dz}\right), \qquad (1)$$

where $P(z, w, w')$ is a meromorphic function in z and a rational function in w and w'.

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The Painlevé property and isomonodromic problem

the linear problem

$$\begin{cases} \left(\dfrac{d}{d\lambda} + A(\lambda, z)\right) Y(\lambda, z) = 0, \quad \det Y(\lambda, z) = 1; \\ \left(\dfrac{d}{dz} + B(\lambda, z)\right) Y(\lambda, z) = 0, \quad \operatorname{tr} B(\lambda, z) = 0,\ B(\lambda, z) = -\dfrac{d}{dt} Y \cdot Y^{-1}. \end{cases}$$

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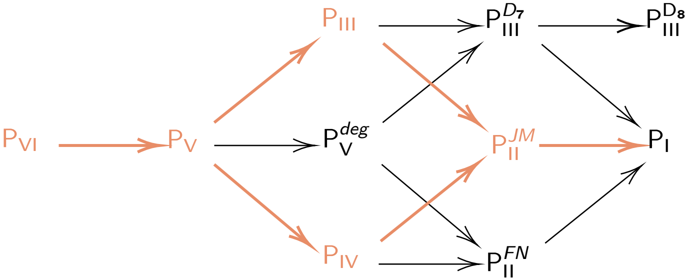

The confluence scheme for the Painlevé equations

Six Painlevé equations

PVI $\left[w(z);\alpha,\beta,\gamma,\delta\right]$; $(0,0,0,0)_{(0,1,\infty,z)}$

$$w'' = \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{w-z}\right)w' + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha+\beta\frac{z}{w^2}+\gamma\frac{z-1}{(w-1)^2}+\delta\frac{z(z-1)}{(w-z)^2}\right).$$

$\vdots$

PII $\left[w(z);\alpha\right]$; $(3)_{(\infty)}$

$$w'' = 2w^3 + wz + \alpha.$$

PI $\left[w(z)\right]$; $(5/2)_{(\infty)}$

$$w'' = 6w^2 + z.$$

Example 2. (from P_{II} to P_I)

$$w'' = 2w^3 + wz + \alpha \quad \left\{w \mapsto \varepsilon w + \varepsilon^{-5},\ z \mapsto \varepsilon^2 z - 6\varepsilon^{-10},\ \alpha \mapsto 4\varepsilon^{-15}\right\}$$

$$w'' = 2\varepsilon^6 w^3 + 6w^2 + \varepsilon^6 zw + z \quad \{\varepsilon \to 0\} \quad w'' = 6w^2 + z$$

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The Hamiltonian structure

$$\mathrm{P_{II}}\left[w\left(z\right);b\right]: \quad w'' = 2w^3 + zw + \left(b - \frac{1}{2}\right). \tag{3}$$

Okamoto variables

$$q = w, \quad p = w' + w^2 + \frac{z}{2}; \tag{4}$$

$$\{q, p\} = 1, \quad \{q, q\} = \{p, p\} = 0.$$

$$\mathrm{H_{II}}\left[b\right] = \frac{1}{2}p^2 - \left(q^2 + \frac{z}{2}\right)p - bq \quad \Leftrightarrow \quad \begin{cases} q' = \dfrac{\partial H}{\partial p} = p - q^2 - \dfrac{z}{2}, \\ p' = -\dfrac{\partial H}{\partial q} = 2pq + b. \end{cases} \tag{5}$$

The Lax pair and the isomonodromic problem

The Jimbo-Miwa pair

$$A(\lambda, z) = \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & u \\ -2u^{-1}v & 0 \end{pmatrix} + \begin{pmatrix} v + z/2 & -uw \\ -2u^{-1}(wv - b) & -v - z/2 \end{pmatrix},$$

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Remark. $\mathrm{P}_{34}[v(z); b] : \dfrac{d^2 v}{dz^2} = \dfrac{1}{2z} \left(\dfrac{dv}{dz} \right)^2 - 2v^2 - zv - \dfrac{b^2}{2v}.$

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Bäcklund transformations

Definition 3.

- *A Bäcklund transformation* transforms one PDE solution to another its solution.
- *An auto-Bäcklund transformation* leaves a PDE invariant.

Example 3. $w \mapsto -w$: $-w'' = -2w^3 - zw + b \Rightarrow \mathrm{P_{II}}[w; b] \mapsto \mathrm{P_{II}}[w; -b]$.

Claim 1.

1. *The transformation s in the given form preserves the solution of the* $\mathrm{P_{II}}$ *equation*

$$s\left(\tilde{q}, \tilde{p}, \tilde{z}; \tilde{b}\right) = s\left(q + \frac{b}{p}, p, z; -b\right), \quad (6)$$

2. *The coordinates* $\tilde{q}$ *and* $\tilde{p}$ *are canonical.*

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1. *The transformation r in the given form preserves the solution of the* $\mathrm{P_{II}}$ *equation*

$$r\left(\hat{q},\hat{p},\hat{z};\hat{b}\right)=r\left(-q,-p+2q^2+z,z;1-b\right). \tag{7}$$

2. *The coordinates $\hat{q}$ and $\hat{p}$ are canonical.*

Theorem 1. *The Bäcklund transformation for the second Painlevé equation is the following one*

$$\tilde{w}=w-\frac{1}{2}\frac{2\alpha-\varepsilon}{\varepsilon w'-w^2-\frac{z}{2}},\quad \alpha=1-\tilde{\alpha},\quad \varepsilon=\pm 1. \tag{8}$$

Remark.

$$\varepsilon=1,\quad \text{then (8) coincides with the transformation } s;$$
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$$T_k = (rs)^k, \quad S_k = (rs)^k s, \quad k \in \mathbb{Z}. \tag{9}$$

k	...	-2	-1	0	1	2	...
T_k	...	$srsr$	sr	1	rs	$rsrs$	...
S_k	...	$srsrs$	srs	s	r	rsr	...

Compositions of Bäcklund transformations

$$s : b \mapsto -b, \quad r : b \mapsto 1 - b$$

Claim 3.

1. $T_k(b) = b - k$, $S_k(b) = k - b$, $k \in \mathbb{Z}$.
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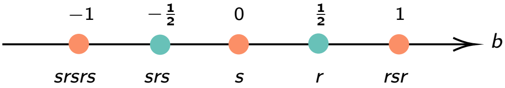

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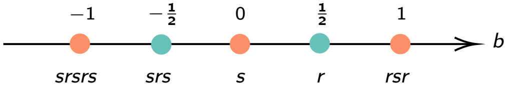

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The affine Weyl group and its extension

Theorem 2. *Bäcklund transformations r and s are generators of the affine Weyl group of type $A_1^{(1)}$,*

$$W = \{ T_k, S_k : k \in \mathbb{Z} \} = \langle r, s \rangle, \tag{10}$$

with fundamental relations $r^2 = 1$, $s^2 = 1$.

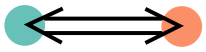
Remark. If we set $s_0 = s$, $s_1 = rsr$, and $\pi = r$, we obtain the extension of W:

$$\tilde{W} = \langle s_0, s_1; \pi \rangle, \quad s_0^2 = 1, \quad s_1^2 = 1, \quad \pi s_0 = s_1 \pi. \tag{11}$$

The Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

The Dynkin diagram



The discrete dynamics d-P$\left(A_1^{(1)}/E_7^{(1)}\right)$ [KNY17]:

$$\begin{cases} \overline{p} &= -p + 2q^2 + z, \\ \overline{q} &= -q + \dfrac{1-b}{\overline{p}}. \end{cases} \tag{12}$$

$E_7^{(1)}$ root system

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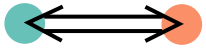

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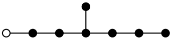

$E_7^{(1)}$ root system

Generating rational solutions

The particular rational solution

$$(q, p; b) = \left(0, \frac{z}{2}; \frac{1}{2}\right) \tag{13}$$

	…	T_{-1}	T_0	T_1	…	…	S_{-1}	S_0	S_1	…
q	…	$-\frac{1}{z}$	0	$\frac{1}{z}$	…	…	$\frac{2z^3-4}{z\left(z^3+4\right)}$	$\frac{1}{z}$	0	…
p	…	$\frac{z^3+4}{2z^2}$	$\frac{z}{2}$	$\frac{z}{2}$	…	…	$\frac{z^3+4}{2z^2}$	$\frac{z}{2}$	$\frac{z}{2}$	…
b	…	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	…	…	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	…

Claim 4. *If $m = 2b - k$ holds, rational solutions obtained by actions of T_k and S_m on $(q, p; b) = \left(0, \frac{z}{2}; \frac{1}{2}\right)$ are the same.*

Generating rational solutions

The particular rational solution

$$(q, p; b) = \left(0, \frac{z}{2}; \frac{1}{2}\right) \tag{13}$$

	...	T_{-1}	T_0	T_1	...	...	S_{-1}	S_0	S_1	...
q	...	$-\frac{1}{z}$	0	$\frac{1}{z}$	...	...	$\frac{2z^3-4}{z(z^3+4)}$	$\frac{1}{z}$	0	...
p	...	$\frac{z^3+4}{2z^2}$	$\frac{z}{2}$	$\frac{z}{2}$	...	...	$\frac{z^3+4}{2z^2}$	$\frac{z}{2}$	$\frac{z}{2}$	...
b	...	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	...	...	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	...

Claim 4. *If $m = 2b - k$ holds, rational solutions obtained by actions of T_k and S_m on $(q, p; b) = \left(0, \frac{z}{2}; \frac{1}{2}\right)$ are the same.*

Sigma-coordinates

$$\sigma(z) = \mathrm{H}_{\mathrm{II}}(b), \tag{14}$$

$$\sigma' = -\frac{1}{2}p, \quad \sigma'' = -pq - \frac{1}{2}b. \tag{15}$$

$$(\sigma'')^2 + 4(\sigma')^3 + 2\sigma'(z\sigma' - \sigma) - \frac{1}{4}b^2 = 0. \tag{16}$$

Claim 5. *Sigma – coordinates are* log *– **symplectic***

$$\Omega = d(\ln\sigma') \wedge d\sigma''.$$

	s	r
$\tilde{\sigma}$	σ	$\sigma + \mathcal{A}$
$\tilde{\sigma}'$	σ'	$-\sigma' + \mathcal{A}^2 + \frac{z}{2}$
$\tilde{\sigma}''$	σ''	$\sigma'' + 2\mathcal{A}^3 + z\mathcal{A} + \left(b - \frac{1}{2}\right)$

Table: The action of W on sigma – coordinates; $\mathcal{A} = \dfrac{\sigma'' + b/2}{2\sigma'}$

The mKdV equation: some symmetries

$$v_t - 6v^2 v_x + v_{xxx} = 0. \tag{17}$$

Remark. The mKdV equation is the result of the Bäcklund transformation of the KdV equation by the Miura transformation.

the KdV equation	the Miura transformation	the mKdV equation
$u_t + 6uu_x + u_{xxx} = 0 \Rightarrow$	$u = v_x - v^2$	$\Rightarrow v_t - 6v^2 v_x + v_{xxx} = 0.$

The symmetry	Independent coordinates
$X = x\partial_x + 3t\partial_t - v\partial_v$	$v(x,t) = \dfrac{w(z(x,t))}{(3t)^{1/3}}, \quad z(x,t) = \dfrac{x}{(3t)^{1/3}}$

$$\mathrm{P_{II}}[w(z);\alpha_1] : w'' = 2w^3 + zw + \alpha_1, \quad \alpha_1 = \text{const}.$$

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The stationary mKdV hierarchy

The stationary KdV hierarchy

$$\partial_{t_{2n+1}} u + \partial_x \ell_{n+1}[u] = 0, \quad n \geq 0, \tag{18}$$

$$\partial_x \ell_{n+1} = \left(\partial_x^3 + 4u\partial_x + 2u_x\right)\ell_n, \quad \ell_0 = \frac{1}{2}, \tag{19}$$

where ℓ_n is the Lenard operator.

Example 4. $(n = 2)$ $\qquad u_{t_5} + u_x^{(5)} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x = 0$

$\Downarrow$ the Miura transformation

The stationary mKdV hierarchy [Jos04]

$$\partial_{t_{2n+1}} v + (\partial_{xx} + 2\partial_x v)\,\ell_n\left[v_x - v^2\right] = 0, \quad n \geq 0, \tag{20}$$

$$\partial_x \ell_{n+1} = \left(\partial_x^3 + 4\left(v_x - v^2\right)\partial_x + 2\left(v_x - v^2\right)_x\right)\ell_n, \quad \ell_0 = \frac{1}{2}. \tag{21}$$

Example 5. $(n = 2)$ $\qquad v_{t_5} + v_x^{(5)} - 10v^2 v_{xxx} - 40v_x v_{xx} - 10v_x^3 + 30v^4 x_x = 0$

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The stationary PII hierarchy

$$\partial_{t_{2n+1}} v + (\partial_{xx} + 2\partial_x v)\, \ell_n \left[v_x - v^2\right] = 0, \quad n \geq 0. \tag{22}$$

The symmetry

$$X_n = x\partial_x + (2n+1)\,\partial_{t_{2n+1}} - v\partial_v$$

Independent coordinates

$$v\,(x, t_{2n+1}) = \frac{w\,(z\,(x,t))}{((2n+1)\,t_{2n+1})^{\frac{1}{2n+1}}},$$

$$z\,(x,t) = \frac{x}{((2n+1)\,t_{2n+1})^{\frac{1}{2n+1}}}$$

The stationary PII hierarchy [Jos04]

$$\left(\frac{d}{dz} + 2w\right) \mathcal{L}_n \left[w' - w^2\right] = zw + \alpha_n, \quad n \geq 1, \tag{23}$$

$$\partial_z \mathcal{L}_{n+1} = \left(\partial_z^3 + 4\left(w' - w^2\right)\partial_z + 2\left(w' - w^2\right)'\right)\mathcal{L}_n, \quad \mathcal{L}_1 = w' - w^2. \tag{24}$$

Example 6. $(n = 0)$ $\quad (z-1)\,w = -\alpha_0$

Example 7. $(n = 2)$ $\quad w'''' - 10w^2w'' - 10ww'^2 + 6w^5 = zw + \alpha_2$

The stationary PII hierarchy

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$X_n = x\partial_x + (2n+1)\,\partial_{t_{2n+1}} - v\partial_v$	$v\left(x, t_{2n+1}\right) = \dfrac{w\left(z\left(x,t\right)\right)}{\left(\left(2n+1\right) t_{2n+1}\right)^{\frac{1}{2n+1}}}$, $z\left(x,t\right) = \dfrac{x}{\left(\left(2n+1\right) t_{2n+1}\right)^{\frac{1}{2n+1}}}$

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$X_n = x\partial_x + (2n+1)\,\partial_{t_{2n+1}} - v\partial_v$	$v\,(x, t_{2n+1}) = \dfrac{w\,(z\,(x,t))}{((2n+1)\,t_{2n+1})^{\frac{1}{2n+1}}}, \quad z\,(x,t) = \dfrac{x}{((2n+1)\,t_{2n+1})^{\frac{1}{2n+1}}}$

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The non-stationary PII hierarchy

The Virasoro symmetry	Independent coordinates
$$T_n = \sum_{k=0}^{n} (2k+1)\, T_{k+1} \partial_{T_{k+1}}$$	$$t_0 = -z,$$ $$t_k = \frac{(2k+1)\, T_{k+1}}{((2n+1)\, T_{n+1})^{\frac{2k+1}{2n+1}}}$$

The non-stationary PII hierarchy [MM07]

$$\left(\frac{d}{dz} + 2w\right)\left(\mathcal{L}_n\left[w' - w^2\right] + \sum_{k=1}^{n-1} t_k \mathcal{L}_k\left[w' - w^2\right]\right) = zw + \alpha_n, \quad n \geq 1, \quad (25)$$

$$\partial_z \mathcal{L}_{n+1} = \left(\partial_z^3 + 4\left(w' - w^2\right)\partial_z + 2\left(w' - w^2\right)'\right)\mathcal{L}_n, \quad \mathcal{L}_1 = w' - w^2. \quad (26)$$

Example 8.

$(n = 2)$ $$\left(w'''' - 10w^2 w'' - 10 w w'^2 + 6w^5\right) + t_1\left(w'' - 2w^3\right) = zw + \alpha_2$$

The non-stationary PII hierarchy

The Virasoro symmetry

$$T_n = \sum_{k=0}^{n} (2k+1)\, T_{k+1} \partial_{T_{k+1}}$$

Independent coordinates

$$t_0 = -z,$$

$$t_k = \frac{(2k+1)\, T_{k+1}}{\left((2n+1)\, T_{n+1}\right)^{\frac{2k+1}{2n+1}}}$$

The non-stationary PII hierarchy [MM07]

$$\left(\frac{d}{dz} + 2w\right)\left(\mathcal{L}_n\left[w' - w^2\right] + \sum_{k=1}^{n-1} t_k \mathcal{L}_k\left[w' - w^2\right]\right) = zw + \alpha_n, \quad n \geq 1, \quad (25)$$

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$(n = 2)$ $$\left(w'''' - 10w^2 w'' - 10w w'^2 + 6w^5\right) + t_1\left(w'' - 2w^3\right) = zw + \alpha_2$$

References

[CMR17] Leonid O Chekhov, Marta Mazzocco, and Vladimir N Rubtsov.
Painlevé monodromy manifolds, decorated character varieties, and cluster algebras.
International Mathematics Research Notices, 2017(24):7639–7691, 2017.

[JM81] Michio Jimbo and Tetsuji Miwa.
Monodromy perserving deformation of linear ordinary differential equations with rational coefficients. II.
Physica D: Nonlinear Phenomena, 2(3):407–448, 1981.

[Jos04] Nalini Joshi.
The second Painlevé hierarchy and the stationary KdV hierarchy.
Publications of the Research Institute for Mathematical Sciences, 40(3):1039–1061, 2004.

[KNY17] Kenji Kajiwara, Masatoshi Noumi, and Yasuhiko Yamada.
Geometric aspects of Painlevé equations.
Journal of Physics A: Mathematical and Theoretical, 50(7):073001, 2017.

[MM07] Marta Mazzocco and Man Yue Mo.
The Hamiltonian structure of the second Painlevé hierarchy.
Nonlinearity, 20(12):28–45, 2007.

[Nou04] Masatoshi Noumi.
Painlevé equations through symmetry, volume 223.
Springer Science & Business, 2004.

[Oka86] Kazuo Okamoto.
Studies on the Painlevé equations. III: second and fourth Painlevé equations, PII and PIV.
Mathematische Annalen, 275(2):221–255, 1986.

[VDPS09] Marius Van Der Put and Masa-Hiko Saito.
Moduli spaces for linear differential equations and the Painlevé equations.
Annales de l'Institut Fourier, 59(7):2611–2667, 2009.