Presymplectic structures and intrinsic Lagrangians

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Based on:

M.G. 2016

M.G. A. Kotov, to appear

M.G. V. Gritsaenko, in progress

Also:

K. Alkalaev, M.G. 2013

M.G. 2012

July 6, 2020,
Krasil'shchik's seminar

Motivations

- Lagrangians (or their substitutes) are inevitable for quantization and very useful in constructing new models and interactions.

- Lagrangian formulation depends on embedding into one or another jet-bundle. Lack of an invariant understanding of the structures underlying Lagrangian formulation. Invariant approach in terms of equation manifold?

- In $n = 1$ (mechanics) the answer is given by (constrained) Hamiltonian mechanics or its presymplectic version. *Henneaux*

- Intrinsic Lagrangian – originates from gauge theories and AKSZ construction and extend the analysis in $n = 1$ to generic n.

Jet-bundles

Fiber bundle: $\mathcal{F}_X \to X$. Locally: fiber F, local coordinates (fields) ϕ^i, base X (space-time), local coordinates (independent variables): x^a, $a = 1, \ldots, n$.

Jet-bundle $J^k(\mathcal{F}_X) \to X$. Local coordinates

$J^0 : \quad x^a, \phi^i, \qquad J^1 : \quad x^a, \phi^i, \phi^i_a, \qquad J^2 : \quad x^a, \phi^i, \phi^i_a, \phi^i_{ab}, \qquad \ldots$

Projections:

$$\ldots \to J^N \to J^{N-1} \to J^{N-2} \to \ldots \to J^1 \to J^0$$

Useful to work with J^∞. A local diff. form on $J^\infty(\mathcal{F}_X)$ – a form on J^N for some N pulled back to $J^\infty(\mathcal{F}_X)$.

$J^\infty(\mathcal{F}_X)$ is equipped with the Cartan distribution. Seen as a flat connection it determines covariant derivative, total derivative D. In coordinates:

$$D_a = \frac{\partial}{\partial x^a} + \phi^i_a \frac{\partial}{\partial \phi^i} + \phi^i_{ab} \frac{\partial}{\partial \phi^i_b} + \dots$$

Defining property: if σ is a section $X \to \mathcal{F}_X$ and σ_{pr} its prolongation $\sigma_{pr} : X \to J^\infty(\mathcal{F}_X)$:

$$\sigma^*_{pr}(D_a f) = \frac{\partial}{\partial x^a}(\sigma^*_{pr} f)$$

for any local function f.

Space time differentials dx^a. Horizontal differential

$$d_{\mathsf{h}} = dx^a \partial_a^T \,, \qquad d_{\mathsf{h}}^2 = 0 \,.$$

Differential forms $\Omega^{p,r}(\mathcal{F}_X)$:

$$\alpha = \alpha_{a_1 \dots a_p | I_1 \dots I_r} dx^{a_1} \dots dx^{a_p} d_{\mathsf{v}} \phi^{I_1} \dots d_{\mathsf{v}} \phi^{I_r} \,, \qquad \phi^I = \phi^i_{a\dots}$$

Vertical differential:

$$d_{\mathsf{v}} = d - d_{\mathsf{h}} = d_{\mathsf{v}} \phi^I \frac{\partial}{\partial \phi^I}$$

Variational bicomplex:

$$d_{\mathsf{v}}^2 = 0, \qquad d_{\mathsf{v}} d_{\mathsf{h}} + d_{\mathsf{h}} d_{\mathsf{v}} = 0 \,, \qquad d_{\mathsf{h}}^2 = 0$$

Bidegree (l, p).

Def: PDE is a collection of local functions

$$E_\alpha[\phi, x]\,.$$

The equation manifold (stationary surface) is $\mathcal{E}_X \subset J^\infty(\mathcal{F}_X)$ singled out by:

$$D_{a_1} \dots D_{a_l} E_\alpha = 0\,, \qquad l = 0, 1, 2, \dots$$

It is usually assumed that x^a are not constrained so that $\mathcal{E}_X$ is a bundle over X.

D_a are tangent to $\mathcal{E}_X$ and hence restricts to $\mathcal{E}_X$. So do the differentials d_h and d_v, defining the variational bicomplex on the equation manifold.

Moreover, d_{h} defines involutive distribution on $\mathcal{E}_X$ that projects to TX, called Cartan distribution.

Definition: *[simpified version of the Vinogradov's one]* *A PDE is a bundle $\mathcal{E}_X \to X$ equipped with a Cartan distribution.*

In addition one typically assumes regularity, constant rank. Use notation $(\mathcal{E}_X, d_{\mathsf{h}})$.

In this form it is clear which PDEs are to be considered isomorphic.

Scalar field Example: Start with:

$$L = \frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi)$$

$\mathcal{E}$ is coordinatized by $x^a, \phi, \phi_a, \phi_{ab}, \ldots$. Already ϕ_{ab} are not independent. One can e.g. take $\phi_{abc\ldots}$ traceless. The d_{h} differential on $\mathcal{E}$ reads as

$$d_{\mathsf{h}} x^a = dx^a\,, \qquad d_{\mathsf{h}}\phi = dx^a\phi_a,, \qquad d_{\mathsf{h}}\phi_a = dx^b(\phi_{ab} - \frac{1}{n}\eta_{ab}\frac{\partial V}{\partial \phi})\,, \qquad \ldots$$

So if the system is nonlinear, i.e. $\dfrac{\partial V}{\partial \phi}$ nonlinear in ϕ, d_{h} is also nonlinear.

Intrinsic realization

Solutions: section $\sigma : X \to \mathcal{E}_X$ is called a solution if σ is parallel (covariantly constant) with respect to Cartan distribution. In coordinates: x^a, ψ^A coordinates on $\mathcal{E}_X$,

$$\frac{\partial}{\partial x^a}(\sigma^*(\psi^A) = \sigma^*(D_a\psi^A)\,,$$

Introducing Ehresman connection coefficients $\Gamma_a^B(x,\psi)$ by $D_a\psi^B = \Gamma_a^B(x,\psi)$ the condition takes the form:

$$\frac{\partial}{\partial x^a}\Psi^A(x) - \Gamma_a^A(\Psi,x) = 0$$

closely related to what physicists call "unfolded form" of the equation M. Vasiliev

The above representation determines an embedding of $\mathcal{E}_X$ into $J^1(\mathcal{E}_X)$. We call it intrinsic realization of the equation.

Variational (Lagrangian) equations

Let us get back to PDE $E_i[\phi, x] = 0$ realised in terms of jet-bundle $J^\infty(\mathcal{F}_X)$.

Def: $E_i[\phi, x]$ is called variational (or Lagrangian) if

$$E_i = \frac{\delta^{EL} L}{\delta \phi^i}\,, \qquad \frac{\delta^{EL} F[u,x]}{\delta \phi^i} \equiv \frac{\partial F}{\partial \phi^i} - D_a \frac{\partial F}{\partial \phi^i_a} + D_a D_b \frac{\partial F}{\partial \phi^i_{ab}} - \dots$$

for some local function $L = L[\phi, x]$. It is convenient to work in terms of Lagrangian density $\mathcal{L} = (dx)^n\, L$.

Here and below:

$$(dx)^n = dx^1 \dots dx^n\,, \qquad (dx)^{n-1}_a = \frac{1}{(n-1)!} \epsilon_{a b_2 \dots b_n} dx^{b_1} \dots dx^{b_n}$$

The notion of Lagrangian is explicitly based on the realization of the equation $(\mathcal{E}_X, d_{\mathsf{h}})$ in terms of a jet-bundle. For instance it's possible that $\mathcal{E} \subset \mathcal{J}_X$ is variational while $\mathcal{E} \subset \mathcal{J}'_X$ is not. Naive invariant object – the restriction of $\mathcal{L}$ to $\mathcal{E}$, does not make much sense.

Invariant version: PDE $(\mathcal{E}_X, d_{\mathsf{h}})$ is called variational if there exists $\mathcal{F}_X$ and $\mathcal{L} \in \Omega^{n,0}(J^\infty(\mathcal{F}_X))$ such that the equation manifold determined by prolongation of EL equations of $\mathcal{L}$ is isomorphic to $(\mathcal{E}_X, d_{\mathsf{h}})$.

Presymplectic structure

It is well-known that $\mathcal{L} = (dx)^n L[x, \phi]$ induce an invariant object on $\mathcal{E}$

Crnkovic, Witten, 1987, Hydon 2005, Khavkin 2012, Sharapov 2016, ...

$$d_{\mathsf{v}} \mathcal{L} = (dx)^n \, E_i + d_{\mathsf{h}} \widehat{\chi} \, ,$$

for some $(n-1,1)$-form $\widehat{\chi} = \widehat{\chi}_i d_{\mathsf{v}} \phi^i + \widehat{\chi}_{ia} d_{\mathsf{v}} \phi^i_a + \ldots$, called presymplectic potential.
Pulling back to $\mathcal{E}_X$, $\chi = i^*(\widehat{\chi})$

$$d_{\mathsf{v}} \omega = 0 = d_{\mathsf{h}} \omega \, , \qquad \omega = d_{\mathsf{v}} \chi$$

So we have conserved closed 2-form on $\mathcal{E}_X$. It's called canonical presymplectic structure. It is defined up to adding $d_{\mathsf{v}} d_{\mathsf{h}}(..)$.

As an example consider $L(\phi, \phi_a, \phi_{ab})$. One finds:

$$\widehat{\chi} = (dx)_a^{n-1}\left(\left(\frac{\partial L}{\partial \phi^a} - \partial_b^T \frac{\partial L}{\partial \phi_{ab}}\right) d_{\mathsf{V}}\phi + \frac{\partial L}{\partial \phi_{ab}} d_{\mathsf{V}}\phi_b\right)$$

In particular, for a scalar field with $L = \frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi)$

$$\chi = (dx)_a^{n-1}\ \phi^a\, d_{\mathsf{V}}\phi\,, \qquad \omega = (dx)_a^{n-1}\ d_{\mathsf{V}}\phi^a\, d_{\mathsf{V}}\phi$$

if one uses $x^a, \phi_a|_{\mathcal{E}_X}, \ldots$ as coordinates on $\mathcal{E}_X$.

More generally:

Definition: *A* $(n-1,2)$*-form* ω *on* $(\mathcal{E}_X, d_{\mathsf{h}})$ *is called compatible presymplectic structure if* $d_{\mathsf{h}}\omega = 0\,, d\omega = 0$*.*

In general, such form can be considered irrespective of any realization in terms of a jet-bundle and/or Lagrangian. We call PDE $(\mathcal{E}_X, d_{\mathsf{h}})$ equipped with a compatible presymplectic structure ω a presymplectic PDE $(\mathcal{E}_X, d_{\mathsf{h}}, \omega)$.

Symmetries and conservation laws

A well-known fact: both symmetries and conservation laws can be defined in terms of the equation manifold $(\mathcal{E}_X, d_{\mathsf{h}})$.

Recall: a vector field $\widehat{W}$ on $\mathcal{J}^\infty(\mathcal{F}_X)$ is a symmetry if it is evolutionary i.e. $[d_{\mathsf{h}}, \widehat{W}] = 0$ and tangent to $\mathcal{E}_X \subset \mathcal{J}^\infty(\mathcal{F}_X)$.

Intrinsic terms: a vector field W on $(\mathcal{E}, d_{\mathsf{h}})$ is called symmetry if $[d_{\mathsf{h}}, W] = 0$ (typically one also requires $Wx^a = 0$).

If $\mathcal{E} \subset \mathcal{J}^\infty(\mathcal{F}_X)$ is variational then variational symmetries restricted to $\mathcal{E}_X$ satisfy in addition

$$L_W \omega = d_{\mathsf{h}} d_{\mathsf{v}} \alpha$$

for some $\alpha \in \Omega^{n-1,1}(\mathcal{E}_X)$, i.e. preserves the equivalence class of ω.

Conservation law (conserved curent) is a degree $n-1$ 0-form K on $\mathcal{E}$ such that $d_{\mathsf{h}}K = 0$. K of the form $K = d_{\mathsf{h}}M$ is trivial.

Any compatible presymplectic structure determines a map from symmetries to conserved currents according to

$$d_{\mathsf{v}}K_W = i_W\omega - d_{\mathsf{h}}\alpha\,, \qquad d_{\mathsf{h}}K_W = 0$$

Trivial symmetries are mapped to trivial conserved currents. In the Lagrangian case this is the 1st Noether theorem. General case was also discussed in *Sharapov 2016*.

Suppose that $(\mathcal{E}_X, d_{\mathsf{h}}, \omega)$ is realized as $\mathcal{E} \subset J^\infty(\mathcal{F}_X)$. Then ω can be lifted to $J^\infty(\mathcal{F}_X)$ where it determines a Lagrangian $\mathcal{L}$ such that EL equations derived from $\mathcal{L}$ are consequences of those defining $\mathcal{E}$.

More precisely, if $\mathcal{E}'$ is an equation manifold defined by $\mathcal{L}$ then $\mathcal{E} \subset \mathcal{E}'$. Even if ω is canonical (derived from a Lagrangian) there is no guarantee that constructed $\mathcal{L}$ is equivalent to the starting point Lagrangian. *Khavkine 2012, Sharapov 2016, Druzhkov 2020, ...*

Intrinsic Lagrangian

Given a presymplectic PDE $(\mathcal{E}, d_{\mathsf{h}}, \omega)$ one can construct a natural Lagrangian on $J^1(\mathcal{E}_X)$, called the intrinsic Lagrangian.

It follows from $d\omega = 0$ that $\omega = d(\chi + l)$ for some $\chi \in \Omega^{(n-1,1)}(\mathcal{E}_X)$ and $l \in \Omega^{(n-1,0)}(\mathcal{E}_X)$. Indeed,

$$\omega = d_{\mathsf{v}}\chi\,, \qquad d_{\mathsf{v}}l + d_{\mathsf{h}}\chi = 0\,, \qquad d_{\mathsf{h}}l = 0$$

Note that if ω comes from a Lagrangian one can take $l = \mathcal{L}|_{\mathcal{E}_X}$.

Consider $\pi_{\mathcal{E}} : J^\infty(\mathcal{E}_X) \to \mathcal{E}_X$ and define

$$\mathcal{L}^C = H(\pi_{\mathcal{E}}^*(\chi + l))\,, \qquad H(dx^a) = dx^a\,, \quad H(df) = \mathrm{D}_{\mathsf{h}} f$$

where D_{h} denotes canonical horizontal differential on $J^\infty(\mathcal{E}_X)$.

Note that $\mathcal{L}^C$ is a pull-back froom $J^1(\mathcal{E}_X)$. Moreover, it is a local function.

Coordinate expression and AKSZ

Consider dx^a as odd coordinates. Horizontal forms $\to$ functions.

Defined generalized Hamiltonian as a conserved current associated to d_{h} seen as a symmetry of $(\mathcal{E}_X, d_{\mathsf{h}})$. Degree n function (conserved current) $\mathcal{H}$ on $\mathcal{E}$ defined by

$$d_{\mathsf{v}}\mathcal{H} = i_{d_{\mathsf{h}}}\omega\,, \qquad \text{components:} \quad \frac{\partial}{\partial\psi^A}\mathcal{H} = \omega_{AB}d_{\mathsf{h}}\psi^B$$

In the Lagrangian case

$$\mathcal{H} = \chi_A d_{\mathsf{h}}\psi^A - \mathcal{L}|_{\mathcal{E}_X}$$

E.g. in the simple case where $\mathcal{L} = (dx)^n L(\phi, \phi_a)$

$$\chi = (dx)^{n-1}_a\Big(\frac{\partial L}{\partial\phi_a}d_{\mathsf{v}}\phi\Big)\Big|_{\mathcal{E}_X}\,, \qquad \mathcal{H} = (dx)^n\Big(\frac{\partial L}{\partial\phi_a}\phi_a - L\Big)\Big|_{\mathcal{E}_X}$$

Intrinsic Lagrangian:

$$\mathcal{L}^C = i_{d_\mathsf{h}}\chi - \mathcal{H}\,, \qquad \text{components:} \quad \mathcal{L}^C = \chi_A \mathsf{D}_\mathsf{h}\psi^A - \mathcal{H}$$

Multisymplectic action *Hydon 2005*

$$S^C = \int \Big(\chi_A(\psi, x, dx)\boldsymbol{d}\psi^A(x) - \mathcal{H}(\psi, x, dx)\Big)\,, \qquad \boldsymbol{d} = dx^a \frac{\partial}{\partial x^a}$$

Particular case of presymplectic AKSZ *Alkalaev, M.G. 2013*

Equations of motion read as

$$\omega_{AB}(\mathsf{D}_\mathsf{h}\psi^B - d_\mathsf{h}\psi^B) = 0\,,$$

consequences of the original equations $\mathsf{D}_\mathsf{h}\psi^B - d_\mathsf{h}\psi^B = 0$

For a local theory $\mathcal{L}^C$ does not depend on most of the fields ψ^A. These can be treated as pure-gauge variables with algebraic (shift) gauge transformations. With this interpretation and under certain assumptions we can prove that starting point $\mathcal{L}$ and $\mathcal{L}^C$ are equivalent.

Examples: Scalar field:

Start with:

$$L = \frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi)$$

$\mathcal{E}$ is coordinatized by $x^a, \phi, \phi_a, \phi_{ab}, \dots$ with $\phi_{abc\dots}$ traceless.

$$d_{\mathsf{h}} x^a = dx^a\,, \quad d_{\mathsf{h}}\phi = dx^a\phi_a,\,, \quad d_{\mathsf{h}}\phi_a = dx^b(\phi_{ab} - \frac{1}{n}\eta_{ab}\frac{\partial V}{\partial \phi})\,, \quad \dots$$

The presymplectic potential and 2-form:

$$\chi = (dx)^{n-1}_a \phi^a d_{\mathsf{v}}\phi\,, \quad \omega = (dx)^{n-1}_a d_{\mathsf{v}}\phi^a d_{\mathsf{v}}\phi$$

The Hamiltonian:

$$\mathcal{H} = (dx)^n(\phi_a\phi^a - L|_{\mathcal{E}}) = \frac{1}{2}\phi^a\phi_a + V(\phi)$$

The intrinsic Larangian: *Schwinger*

$$\mathcal{L}^c = (dx)^n\left(\phi^a(\partial_a\phi - \frac{1}{2}\phi_a) - V(\phi)\right)$$

Polywave equation

The simplest higher derivative example is $L = \frac{1}{2}\Box\phi\Box\phi = \frac{1}{2}\phi_{aa}\phi_{bb}$ (here and below $\phi_{aa} = \eta^{ab}\phi_{ab}$). Presymplectic potential and Hamiltonian

$$\chi = (-\phi_{acc} d_{\mathsf{V}}\phi + \phi_{cc} d_{\mathsf{V}}\phi_a)(dx)^{n-1}_a$$

$$\mathcal{H} = (dx)^n(-\phi_{acc}\phi_a + \frac{1}{2}\phi_{cc}\phi_{aa})\,.$$

The intrinsic action takes the form

$$S^C = \int d^n x(-\phi_{acc}(\partial_a \phi - \phi_a) + \phi_{cc}\partial_a \phi_a - \frac{1}{2}\phi_{aa}\phi_{cc}) \,.$$

Depends on only the following variables $\phi, \phi_a, \phi_{aa}, \phi_{acc}$ but NOT on the traceless component of ϕ_{ab} and ϕ_{abc}. It is equivalent to $\int \phi_{aa}\phi_{cc}$. Indeed, varying with respect to ϕ_a and ϕ_{acc} gives $\phi_a = \partial_a \phi$ and $\phi_{acc} = \partial_a \phi_{cc}$ resulting in

$$\int d^n x(\phi_{cc}\partial_a \partial_a \phi - \frac{1}{2}\phi_{aa}\phi_{cc})$$

YM theory

The YM field is A^a taking values in a Lie algebra $\mathfrak{g}$ equipped with an invariant inner product $\langle , \rangle$. We will use notation $A_{a|b_1 \dots b_l}$ for $D_{b_1} \dots D_{b_l} A_a$. The Lagrangian:

$$L = \frac{1}{4}\langle F_{ab}, F_{ab}\rangle \,, \qquad F_{ab} := A_{a|b} - A_{b|a} + [A_a, A_b] \,.$$

Coordinates on $\mathcal{E}_X$:

$$x^a, A_a, F_{ab}, S_{ab} := A_{b|a} + A_{a|b}, A_{a|bc}, \dots$$

The one form χ is given by

$$\chi = \frac{\partial L}{\partial A_{a|b}} d_{\mathsf{V}} A^b (dx)^{n-1}_a = \langle F_{ab}, d_{\mathsf{V}} A^b\rangle (dx)^{n-1}_a$$

The Hamiltonian

$$\mathcal{H} = (\frac{\partial L}{\partial A^b_a} A^b_a - \frac{1}{4}\langle F_{ab}, F_{ab}\rangle)(dx) = \frac{1}{2}\langle F_{ab}, \frac{1}{2} F_{ab} - [A_a, A_b]\rangle$$

The intrinsic action

$$\int \frac{1}{2}\langle F_{ab}, \partial_a A_b - \partial_b A_a\rangle - \frac{1}{2}\langle F_{ab}, \frac{1}{2}F_{ab} - [A_a, A_b]\rangle =$$
$$\int \frac{1}{2}\langle F_{ab}, \partial_a A_b - \partial_b A_a + [A_a, A_b] - \frac{1}{2}F_{ab}\rangle$$

equivalent to the starting point action through the elimination of F_{ab} by its own equations of motion.

Well-known first-order action for YM.

Gravity

Gritsaenko, M.G., in proress

Jet-bundle $x^\mu, g_{\mu\nu}, g_{\mu\nu|\rho}, \ldots$. EH action

$$S_{EH} = \int d^n x \sqrt{-g} R[g]\,, \qquad R[g] = g^{\mu\nu} R_{\mu\nu}[g] \ - \ \text{scalar curvature}$$

EL equations: $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0.$

Equation manifold: $x^\mu, g_{\mu\nu}, \Gamma^\rho_{\mu\nu}, \ldots$. Presymplectic potential:

$$\chi = \sqrt{-g}(\Gamma^{\rho\mu\nu} - \frac{1}{2} g^{\rho\mu} \Gamma_\lambda{}^{\lambda\nu} - \frac{1}{2} g^{\rho\nu} \Gamma_\lambda{}^{\lambda\mu} + \frac{1}{2} g^{\mu\nu} \Gamma_\lambda{}^{\lambda\rho} - \\ - \frac{1}{2} g^{\mu\nu} \Gamma^{\rho\lambda}{}_\lambda) d_\upsilon g_{\mu\nu} (dx)^{n-1}_\rho$$

Intrinsic action:

$$S^C[g,\Gamma] = \int d^nx\sqrt{-g}g^{\mu\nu}(\partial_\lambda \Gamma^\lambda{}_{\mu\nu} - \partial_\mu \Gamma^\lambda{}_{\nu\lambda} + \Gamma^\gamma{}_{\mu\nu}\Gamma^\lambda{}_{\gamma\lambda} - \Gamma^\gamma{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\gamma}) =$$

$$= \int d^nx\sqrt{-g}g^{\mu\nu}R_{\mu\nu}[\Gamma]$$

Familiar Palatini form of GR action.

Interpretation of the intrinsic Lagrangian

The intrinsic Lagrangian does have an infinite number of algebraic gauge symmetries. EOMs are

$$\omega_{AB}(\mathsf{D}_{\mathsf{h}}\psi^B - d_{\mathsf{h}}\psi^B) = 0$$

so that any null vector of σ_{AB} gives rise to a shift gauge symmetry. If $\omega_{AB}R^B(\psi) = 0$ then $\delta\psi^A = R^A\epsilon(x)$ is a gauge symmetry of the intrinsic action. Such gauge symmetries are known as algebraic/shift or Stueckelberg.

Interpretation of the intrinsic action: all its shift gauge symmetries are to be taken into account (the respective fields are set to fixed values – i.e. gauge-fixed).

Geometrically, factoring out the kernel of ω, generated by regular commuting vertical vector fields.

Natural theories: A Lagrangian PDE is called natural if algebraic gauge symmetries are not present and by adding/eliminating auxiliary fields and local invertible change of variables the action $\int \mathcal{L}$ can be brought to the form:

$$S^{first} = \int \mathcal{L}^{first}[u] = \int d^d x (V^a_\lambda(u,x)\partial_a u^\lambda - H(u,x))$$

such that its EL equations do not imply algebraic constraints between undifferentiated fields u^λ. (i.e. u^λ reduced to $\mathcal{E}$ remain independent or, in other words, the projection of the equation manifold to 1st jets is surjective.)
Note that most of fundamental field-theoretical models are natural.

Proposition: *for a natural Lagrangian PDE the intrinsic Lagrangian is complete (reproduces the initial presymplectic PDE)*

Proof. It is enough to start with the first order Lagrangian. The respective presymplectic structure reads as

$$\chi = (dx)^{n-1}_a V^a_\lambda du^\lambda)\Big|_{\mathcal{E}} = (dx)^{n-1}_a V^a_\lambda du^\lambda$$

Hamiltonian:

$$\mathcal{H} = ((dx)^n V^a_\lambda u^\lambda_a - \mathcal{L}^{first})|_{\mathcal{E}} = (dx)^n H$$

Finally:

$$\mathcal{L}^C = (dx)^n (V^a_\lambda \partial_a u^\lambda - H(\phi))$$

and explicitly coincides with the starting point first order Lagrangian. $\square$

Missing: intrinsic characterization of natural Lagrangian PDEs.

Example of "unnatural" PDE

Massive spin-2. Lagrangian: *Fiertz-Pauli*

$$\mathcal{L} = -\frac{1}{2}\partial_c\phi^{ab}\partial^c\phi_{ab} + \partial^a\phi_{ab}\partial_c\phi^{ca} + \frac{1}{2}\partial_a\phi^b_b\partial^a\phi^c_c - \partial^c\phi_{ca}\partial^a\phi^b_b - \frac{1}{2}m^2(\phi^{ab}\phi_{ab} - \phi^a_a\phi^b_b)$$

EL equations have differential consequences of order 0. Equation manifold:

$$(\partial_c\partial^c - m^2)\phi_{ab} = 0\,, \qquad \partial^a\phi_{ab} = 0\,, \qquad \phi^c_c = 0$$

The intrinsic Lagrangian is not complete!

Can be represented in the 1st order form but at the price of extending equation manifold and presymplectic structure.

Gritsaenko, M.G. in progress

Example of a "constrained system" in the covariant Hamiltonian formalism.

BRST extension and frame-like Lagrangians

Gauge theories: PDEs with extra structures. Useful language:
Def: [M.G, Kotov 2019] Gauge PDE is a $\mathbb{Z}$-graded Q-bundle $(E_{T[1]X}, Q)$ over $(T[1]X, d_X)$. + extra conditions

An analog of presymplectic PDE's:
Def: [M.G, Kotov to appear] Presymplectic gauge PDE is a Q-bundle $(E_{T[1]X}, Q)$ over $(T[1]X, d_X)$ equipped with vertical 2-form ω of ghost degree $n-1$ satisfying

$$d\omega = 0\,, \qquad L_Q\omega = 0$$

This data determines a generalized AKSZ type Lagrangian. Can also be seen as BV-version of the above intrinsic Lagrangian.

Special case: AKSZ sigma model (locally trivial Q-bundle). Source (base) $T[1]X$ with coordinates x^μ, θ^μ and target (fiber) with coordinates ψ^A:

$$Q = Q^A(\psi)\frac{\partial}{\partial\psi^A}\,, \qquad \chi = \chi_A d\psi^A\,, \qquad \omega = d\chi$$

Presymplectic AKSZ *Alkalaev, M.G, 2013*

$$S^c(\sigma) = \int_{T[1]X} \sigma^*(\chi)(d_X) - \sigma^*(\mathcal{H}) =$$
$$= \int \chi_A(\psi(x,\theta))\boldsymbol{d}\psi^A(x,\theta) - \mathcal{H}(\psi(x,\theta))$$

where $i_Q\omega = d\mathcal{H}$.

Example: Gravity

Standard BV jet-bundle for gravity $J^\infty(\mathcal{F})$:

$$x^a, \theta^a, \quad g^{ab}_{(c)}, \quad \epsilon^a_{(b)}, \qquad +\text{antifields}$$

BRST differential:

$$s = \delta + \gamma\,, \qquad \gamma g_{ab} = \mathcal{L}_\epsilon g_{ab}\,, \quad \gamma\epsilon = \frac{1}{2}[\epsilon, \epsilon]$$

AKSZ sigma model with this target $(J^\infty(\mathcal{F}), s)$ and source $T[1]X$ is precisely AKSZ form of gravity at the level of EOMs.

Let us try to equivalently reduce the description to produce a concise form of this gauge PDE

Antifields and equations of motion form contractible pairs for s and can be eliminated.

Further equivalent reduction (disregarding global geometry) leaves us with:

$$e^a, \quad \rho^{ab}, \quad W^{cd}_{ab}, \quad W^{cd}_{ab|e}, \quad W^{cd}_{ab|e\ldots}$$

The reduced target Q-manifold (M, Q):

$$Qe^a = \rho^a{}_c\,\epsilon^c\,, \qquad Q\rho^{ab} = \rho^a{}_c\,\rho^{cb} + \frac{1}{2}e^c e^d W^{ab}_{cd}\,,$$

$$QW = eW + \rho W + \ldots$$

Minimal BRST complex (Q-manifold) for gravity. *Have been independently found by many authors. Mention F. Brandt*

Gives minimal AKSZ formulation (unfolded formulation).

A given map $\sigma : T[1]X \to M$ is parameterized by 1-forms

$$\sigma^*(e^a) = e^a_\mu(x)dx^\mu, \qquad \sigma^*(\rho^{ab}) = \rho^{ab}_\mu(x)dx^\mu$$

and 0-forms:

$$\sigma^*(W^{cd}_{ab}) = W^{cd}_{ab}(x)\,, \qquad \sigma^*(W^{cd}_{ab|e}) = W^{cd}_{ab|e}(x)\,, \qquad \ldots$$

Equations of motion:

$$de^a + \rho^a_b e^b = 0\,, \qquad d\rho^{ab} + \rho^a{}_c\,\rho^{cb} + \frac{1}{2}e^c e^d W^{ab}_{cd} = 0\,, \quad \ldots$$

This implies $Ric(e) = 0$.

Remarkably: we systematically arrived at frame-like formulation of general relativity!

Compatible presymplectic structure: *Alkalaev, M.G. 2013; M.G. 2016*

$$\chi = \frac{1}{2} d\rho^{ab} \epsilon_{abcd} e^c e^d \,, \qquad \omega = d\rho^{ab} de^c \epsilon_{abcd} e^d$$

"Hamiltonian" (terms involving $W^{cd}_{ab|e}(x)$ vanish)

$$\mathcal{H} = i_Q \chi = -\frac{1}{2} \rho^a_c \rho^{cb} \epsilon_{abcd} e^c e^d$$

Intrinsic action (frame-like GR action):

$$S^C = \int \chi_A \boldsymbol{d} \psi^A - \mathcal{H} = \int (\boldsymbol{d} \rho^{ab} + \rho^a{}_c \rho^{cb}) \epsilon_{abcd} e^c e^d$$

Familiar Cartan-Weyl action for GR. Generalization for $n > 2$ and $\Lambda \neq 0$ is straightforward.

For a wide class of gauge PDE the Lagrangian is encoded in the geometry of the equation manifold. *Alkalaev M.G. 2013, M.G. 2016*.
General theory *M.G, Kotov, to appear*

Conclusions

- Natural Lagrangian systems can be defined in terms of its equation manifold $\mathcal{E}_X$ without refereeing to any particular realization of $\mathcal{E}_X$ in terms of one or another jet-bundle. While the structure of the equation is encoded in the differential d_h the Lagrangian is encoded in d_h along with the compatible presymplectic structure ω.

- Easy to see whether Lagrangian systems are equivalent or not.

- BV-BRST extension. Intrinsic Lagrangian = Frame-like Lagrangian.Underlying supergeometry: presymplectic AKSZ sigma models Q

- The presymplectic form can be seen to originate from the odd symplectic form of the Batalin-Vilkovisky formulation Open problems:

- intrinsic characterization of natural systems
- intrinsic criteria for completeness ("non-degeneracy") of a compatible symplectic structure

Parent Lagrangian

One way to understand where do the structure of the intrinsic Lagrangian comes from is to consider "parent" action for $L = L(\phi, \phi_a, \phi_{ab})$:

$$S^P = \int (L(\phi, \phi_a, \phi_{ab}) + \pi^a(\partial_a \phi - \phi_a) + \pi^{ac}(\partial_a \phi_c - \phi_{ac}) + \ldots) \ .$$

Its equations of motion read as

$$\frac{\partial L}{\partial \phi} - \partial_a \pi^a = 0 \,,$$

$$\pi^a - \frac{\partial L}{\partial \phi_a} + \partial_c \pi^{ca} = 0 \,, \qquad \pi^{ab} - \frac{\partial L}{\partial \phi_{ab}} = 0 \,, \qquad \pi^{ab\ldots} = 0$$

$$\phi_a = \partial_a \phi \,, \qquad \phi_{ab} = \partial_{(a} \phi_{b)} \,, \qquad \ldots$$

Using the last line the derivatives in the first two lines can be replaced with the total derivatives. Using the second line the first equation becomes EL

$$\frac{\partial L}{\partial \phi} - \partial_a^T \frac{\partial L}{\partial \phi_a} + \partial_c^T \partial_a^T \frac{\partial L}{\partial \phi_{ca}} = 0 \,.$$

Introduce 1-form of degree $n-1$:

$$\bar{\chi} = (dx)_a^{n-1}(\pi^a d\phi + \pi^{ab} d\phi_b + \ldots)$$

"parent" Hamiltonian

$$\bar{\mathcal{H}} = (\pi^a \phi_a + \pi^{ab}\phi_{ab} + \ldots - L(\phi, \phi_a, \phi_{ab}))(dx)^n \,.$$

The parent action can be written as

$$S^P = \int (\bar{\chi}_A \boldsymbol{d} \Psi^A - \bar{\mathcal{H}}) \,,$$

where Ψ^A stand for all the coordinates $\phi, \phi_{\ldots}$, $\pi^{\cdots}$.

Consider the following submanifold of the space of $x^a, dx^a, \phi, \pi^{\cdots}, \phi_{\dots}$

$$\pi^a - \frac{\partial L}{\partial \phi^a} + \partial_c^T \frac{\partial L}{\partial \phi_{ca}} = 0\,, \qquad \pi^{ab} - \frac{\partial L}{\partial \phi_{ab}} = 0\,, \quad \pi^{ab\dots} = 0\,,$$
$$\partial_{a_1}^T \dots \partial_{a_k}^T (EL) = 0\,,$$

These are consequences of the parent action equations of motion.

The submanifold they single out is $\mathcal{E}$ (equation manifold of L).

$$\chi = \bar{\chi}|_{\mathcal{E}} \qquad \text{Presymplectic potential for } L$$

One can show

$$i_Q d\sigma = d\mathcal{H}\,, \qquad \mathcal{H} = \bar{\mathcal{H}}|_{\mathcal{E}}\,, \qquad \sigma = d\chi$$

Furthermore, χ and $\mathcal{H}$ determine the intrinsic action

$$S^C[\psi] = \int \Big(\chi_A(x, dx^a, \psi)\boldsymbol{d}\psi^A - \mathcal{H}(x, dx^a, \psi)\Big) \ ,$$

where x^a, ψ^A are coordinates on $\mathcal{E}$. This can be independently arrived at by eliminating auxiliary fields starting from the parent action.