

Killing compatibility complex on Kerr spacetime

arXiv:1910.08756

w/ Aksteiner, Andersson, Bäckdahl & Whiting

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Statement of the problem

- ▶ Consider a (pseudo-)Riemannian manifold (M, g) .
- ▶ ∇_a — Levi-Civita connection; R_{abcd} — Riemann tensor of ∇_a .
- ▶ $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a$ — Killing operator.
- ▶ The Killing equation $K[v]_{ab} = 0$ is an over-determined equation of finite type.
- ▶ Given g , what is the full compatibility complex of $K[v]_{ab} = 0$?

$$T^*M \xrightarrow{K} S^2 T^*M \xrightarrow{?} \dots \xrightarrow{?} \dots$$

- ▶ **Def:** C is a compatibility operator for K if $c \circ K = 0 \implies c = c' \circ C$.

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{K} & \bullet & \xrightarrow{C} & \bullet \\
 c \circ K = 0 & & & & C \circ K = 0 \\
 & & \downarrow c' \circ K \neq c & \swarrow c' & \\
 & & \bullet & &
 \end{array}$$

- ▶ In **General Relativity**: the components of C constitute a “complete set of local gauge invariant observables” for linearized gravity on the spacetime (M, g) .

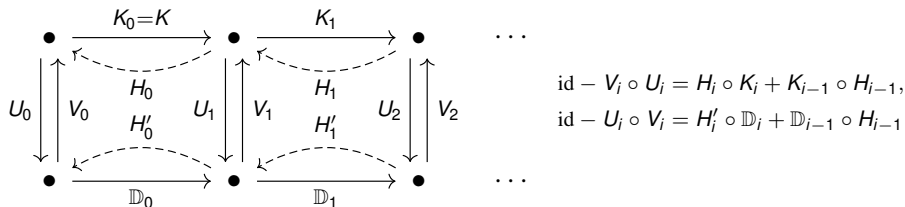
Possible approaches

How to obtain a compatibility complex of K on (M, g) ?

1. BGG machinery (representation theoretic approach)
 - ▶ Requires (M, g) to be a homogeneous space. Otherwise, does not even produce a complex!
 - ▶ Can start with a **curved BGG** machinery, but no self-contained way to complete it to a compatibility complex.
 2. Spencer-Goldschmidt theory. (Goldschmidt, Ann Math (1967) **86** 246)
 - ▶ Prolong to involution, compute Spencer cohomology.
 - ▶ Algorithmic. Implemented in computer algebra.
 - ▶ Computer algebra requires explicit coordinate components, is blind to any special geometry of (M, g) .
 - ▶ Witness: execution of the algorithm (often infeasible by hand).
 3. Reduction to canonical form. (IK [[arXiv:1805.03751](https://arxiv.org/abs/1805.03751)])
 - ▶ Canonical form: adapted to the geometry of (M, g) , but with known compatibility complex (e.g., flat connection).
 - ▶ Reduction: equivalence up to homotopy.
 - ▶ Witness: the explicit equivalence, proof for canonical form (e.g., Poincaré lemma).
- ▶ Practical applications.
- 1., 2.: only (anti-)de Sitter spacetime (maximal symmetry).
 - 3.: all other known cases; FLRW cosmology, Schwarzschild black hole [[arXiv:1805.03751](https://arxiv.org/abs/1805.03751)], now Kerr black hole [[arXiv:1910.08765](https://arxiv.org/abs/1910.08765)].

Step 1: Equivalence up to homotopy

Two complexes of differential operators, K_i and \mathbb{D}_i , are **equivalent up to homotopy** when the diagram

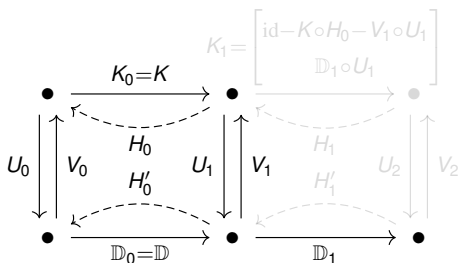


exists, where the solid arrows commute and the dashed arrows are homotopy corrections.

Lemma (homotopy equivalence as **witness**)

Consider an *equivalence up to homotopy* between complexes K_i and \mathbb{D}_i , $i \geq 0$. Then, if \mathbb{D}_i is a *full compatibility complex*, then *so is* K_i .

Step 2: Lifting compatibility operators



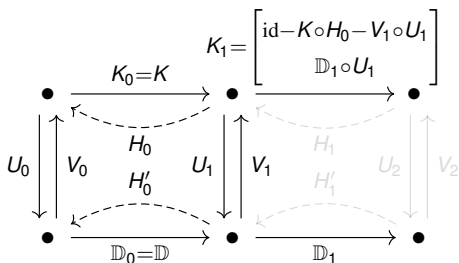
Lemma

After reduction to **canonical form**:

- (a) $\mathbb{D}_1 \circ U_1 [K[v]] = 0$
- (b) $(\text{id} - K \circ H_0 - V_1 \circ U_1) [K[v]] = 0$
- (d) \exists compatible U_2, V_2, H_1, H'_1
- (c) (a) and (b) make a **comp.op.**

- ▶ The Lemma does not depend on K_0 being of finite type, only on the equivalence between K_0 and \mathbb{D}_0 .
- ▶ Hence, we can iterate the argument (simplifying at each step!) to get a full compatibility complex for K_i , and its equivalence up to homotopy with \mathbb{D}_i .
- ▶ K_i will be of finite length when \mathbb{D}_i is of finite length.

Step 2: Lifting compatibility operators



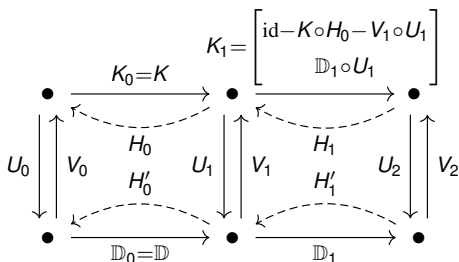
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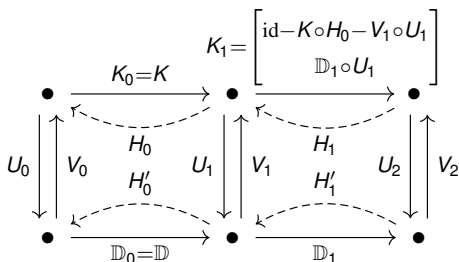
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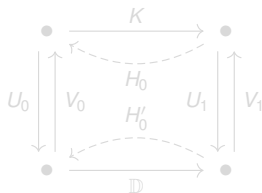
Step 3: Reduction to canonical form (non algorithmic)

Equations $K[v] = 0$ and $\mathbb{D}u = 0$ are **equivalent** only when there are local formulas $v \leftrightarrow u$ that are **bijective on solutions**:

$$\begin{aligned}K[V_0[u]] = 0 &\propto \mathbb{D}u = V_1[\mathbb{D}u], \\ \mathbb{D}U_0[v] = 0 &\propto K[v] = U_1[K[v]], \\ v - V_0[U_0[v]] = 0 &\propto K[v] = H_0[K[v]], \\ u - U_0[V_0[u]] = 0 &\propto \mathbb{D}u = H'_0[\mathbb{D}u].\end{aligned}$$

On solutions means that some 0 must become $\propto K[v]$ or $\propto \mathbb{D}u$.

The relationships between these differential operators are **visually summarized** in the following diagram:



It is here that we have the freedom to preserve the special algebraic and geometric properties of K !

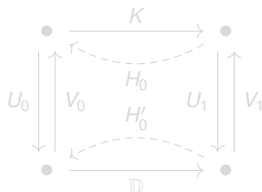
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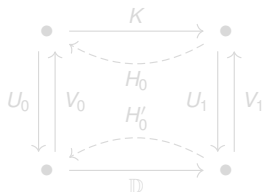
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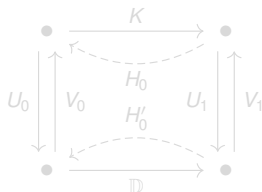
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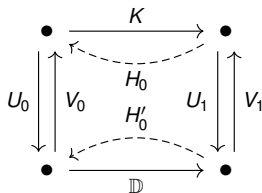
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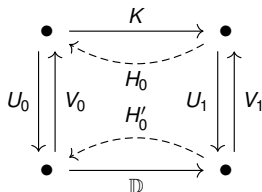
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Canonical form for PDEs of finite type

- ▶ The Killing equation $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a = 0$ is of **finite type**. There exists an $N < \infty$ such that $v(x), \partial v(x), \dots, \partial^N v(x)$ determines the solution v uniquely on a neighborhood of x .
- ▶ The **canonical form** for any PDE of finite type is $\mathbb{D}u = 0$, where \mathbb{D} is a **flat connection** on a (possibly new) set of fields u :

$$\mathbb{D}_a u^\alpha = \partial_a u^\alpha + \Gamma_{a\beta}^\alpha u^\beta = 0, \quad \text{where} \quad [\mathbb{D}_a, \mathbb{D}_b] = 0.$$

- ▶ Starting with $\mathbb{D}_0 := \mathbb{D}$, define $\mathbb{D}_\rho w = \mathbb{D} \wedge w^\alpha$ for any **vector valued ρ -form w^α** . Then $\mathbb{D}_\rho \circ \mathbb{D}_{\rho-1} = 0$ is the **de Rham complex twisted by \mathbb{D}** ; it is a full compatibility complex (Poincaré lemma).
- ▶ The number of components of u^α is the number of **independent solutions** of $K[v] = 0$ ($\leq n(n+1)/2$ in n -dim.). This number should be **locally constant!**

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Remark

The **canonical form** need not always be a **flat connection**. It need only be a PDE with a **known** compatibility complex.

But the **twisted de Rham** complex \mathbb{D}_ρ is a particularly simple construction.

Example: (anti-)de Sitter (maximal symmetry)

- ▶ The simplest example is of a **constant curvature** space, identified by

$$C[g] := R_{abcd}[g] - \alpha(g_{ac}g_{bd} - g_{ad}g_{bc}) = 0.$$

- ▶ The Calabi complex (reviewed in [\[arXiv:1409.7212\]](https://arxiv.org/abs/1409.7212))

$$K_1[h] = \dot{C}[h] = \nabla_{(a}\nabla_{c)}h_{bd} - \nabla_{(b}\nabla_{c)}h_{ad} - \nabla_{(a}\nabla_{d)}h_{bc} + \nabla_{(b}\nabla_{d)}h_{ac} \\ + \alpha(g_{ac}h_{bd} - g_{bc}h_{ad} - g_{ad}h_{bc} + g_{bd}h_{ac}),$$

$$K_2[r] = 3\nabla_{[a}r_{bc]de},$$

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$$K_i[b] = (i+1)\nabla_{[a_0}b_{a_1\dots a_i]bc} \quad (i \geq 2).$$

is already known to be a **full compatibility complex**. Our method is not necessary, but can reproduce the same result.

- ▶ These formulas work in any signature and dimension n .
- ▶ Lorentzian: **Minkowski** or **(Anti-)de Sitter** space with $\Lambda = \frac{(n-1)(n-2)}{2}\alpha$.
- ▶ Riemannian: **n -sphere** ($\alpha > 0$), **n -dimensional hyperbolic space** ($\alpha < 0$).

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Newest example: Kerr rotating black hole [\[arXiv:1910.08756\]](https://arxiv.org/abs/1910.08756)

Kerr spacetime: Lorentzian $(\mathbb{R}^2 \times S^2, g)$, outside horizon.

- ▶ 4-dimensional, asymptotically flat, Einstein vacuum ($R_{ab} = 0$)
- ▶ explicit form of g_{ab} : not important here
- ▶ stationary (time symmetry), rotating (one symmetry axis)
- ▶ $\dim \ker K = 2$ uniformly (Killing vectors)
- ▶ Killing-Yano 2-form \cong Killing 2-spinor (hidden symmetry)
- ▶ algebraically special curvature R_{abcd}
- ▶ spinor calculus adapted to geometry (!)

Other spacetimes in the same class:

- ▶ Kerr-de Sitter, Kerr-anti-de Sitter (not asymptotically flat)
- ▶ Kerr-Newman-((A)dS) (electrically charged)

Primer on spinor calculus

- ▶ Basic fact: $\mathfrak{so}(1, 3) \cong \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{so}(1, 3)_{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})'$
- ▶ fundamental irreps: $(1, 0) : \psi_A, (0, 1) : \psi_{A'}$ on \mathbb{C}^2
- ▶ invariant pairing: $\epsilon^{AB} = -\epsilon^{BA}$
- ▶ all irreducible representations: $(k, l) : \psi_{(A_1 \dots A_k)(A'_1 \dots A'_l)}$
- ▶ spinor bundle on (M, g) : $SO(g)_{\mathbb{C}}$ -tensors $\leftrightarrow \mathfrak{sl}(2, \mathbb{C})$ -spinors
- ▶ Levi-Civita connection: $\nabla_a \leftrightarrow \nabla_{AA'}$
- ▶ Translation:

	ϕ	$\psi_{AA'}$	$\Phi_{ABA'B'}$	ψ_{AB}	$\psi_{A'B'}$	Ψ_{ABCD}	$\Psi_{A'B'C'D'}$
spinor	(0, 0)	(1, 1)	(2, 2)	(2, 0)	(0, 2)	(4, 0)	(0, 4)
tensor $_{\mathbb{C}}$	\mathbb{C}	\square	$\square\square_0$	$\square + i*\square$	$\square - i*\square$	$\square\square_0 + i*\square\square_0$	$\square\square_0 - i*\square\square_0$
	ϕ	v_a	S_{ab}	\mathcal{Y}_{ab}	$\bar{\mathcal{Y}}_{ab}$	\mathcal{W}_{abcd}	$\bar{\mathcal{W}}_{abcd}$
scalar	vector	symmetric traceless 2-tensor	anti-self-dual 2-form	self-dual 2-form	anti-self-dual Weyl	self-dual Weyl	

Spinor calculus on Kerr

- ▶ Basic geometric objects:
 - ▶ Killing 2-spinor $\kappa_{AB}, \bar{\kappa}_{A'B'}$ (Killing-Yano 2-form)
 - ▶ Killing vectors: $\xi_{AA'}, \zeta_{AA'}$
 - ▶ Ricci scalar, traceless Ricci tensor: $\Lambda = 0, \Phi_{ABA'B'} = 0$
 - ▶ Weyl curvature: $\Psi_{ABCD}, \bar{\Psi}_{A'B'C'D'}$
 - ▶ Involutivity: $\nabla_{EE'}(\kappa, \xi, \zeta, \Psi) = O(\kappa, \xi, \zeta, \Psi)$
- ▶ Fundamental spin operators:

$\nabla_{AA'}$	$\nabla^{A'}_{A'}(-)_{A\dots}$	$\nabla_{(A}{}^{A'}(-)_{B\dots)}$
$\nabla_{A}{}^{A'}(-)_{A'\dots}$	\mathcal{D}	\mathcal{E}
$\nabla^A_{(A'}(-)_{B'\dots)}$	\mathcal{E}^\dagger	\mathcal{F}

- ▶ Example: 4-dimensional de Rham complex (d_0, d_1, d_2, d_3)

$$\begin{array}{ccccccc}
 [(0,0)] & \xrightarrow{[\mathcal{F}]} & [(1,1)] & \xrightarrow{\begin{bmatrix} \mathcal{E}^\dagger \\ \mathcal{E} \end{bmatrix}} & \begin{bmatrix} (0,2) \\ (2,0) \end{bmatrix} & \xrightarrow{[\mathcal{E} \quad -\mathcal{E}^\dagger]} & [(1,1)] & \xrightarrow{[\mathcal{D}]} & [(0,0)]
 \end{array}$$

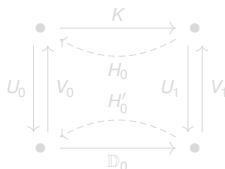
- ▶ Sought canonical form: $K \rightsquigarrow \mathbb{D}_0, \mathbb{D}_i = \begin{bmatrix} \xi \\ \zeta \end{bmatrix} \otimes d_i.$

Reduction to canonical form on Kerr

- ▶ Spinor calculus on Kerr implemented in MATHEMATICA, on top of xAct, by Aksteiner & Bäckdahl. Some packages public, some private. [\[arXiv:1601.06084\]](https://arxiv.org/abs/1601.06084)
- ▶ Geometric ingredients:
 - ▶ special spinors: κ, ξ, ζ, Ψ
 - ▶ spin operators: $\mathcal{D}, \mathcal{C}, \mathcal{C}^\dagger, \mathcal{T}$,
 - ▶ Killing vector sub-bundle $\text{span}\{\xi, \zeta\}$: inclusion, projection, orthogonal projection

We obtained **compact formulas** for all

- ▶ the operators in the reduction diagram using these ingredients.



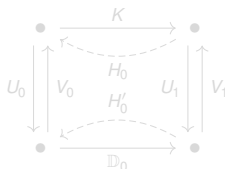
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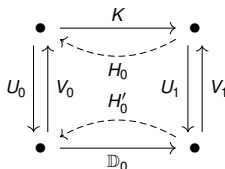
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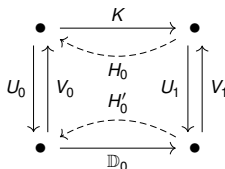
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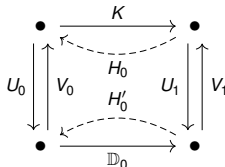
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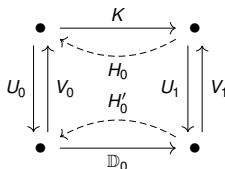
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- ▶ The core of the calculation consists of explicitly identifying integrability conditions of the **Killing equation** $K[v] = 0$ and putting it into the **canonical form**, e.g., of a **flat connection**.
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- ▶ Newest application: rotating **Kerr** black hole (together with Aksteiner, Andersson, Bäckdahl and Whiting [[arXiv:1910.08756](https://arxiv.org/abs/1910.08756)]).
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