Killing compatibility complex on Kerr spacetime

arXiv:1910.08756

w/ Aksteiner, Andersson, Bäckdahl & Whiting

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Statement of the problem

- Consider a (pseudo-)Riemannian manifold (M, g).
- ∇_a — Levi-Civita connection; R_{abcd} — Riemann tensor of ∇_a.
- $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a$ — Killing operator.
- The Killing equation $K[v]_{ab} = 0$ is an over-determined equation of finite type.
- Given g, what is the full compatibility complex of $K[v]_{ab} = 0$?

$$T^*M \xrightarrow{K} S^2 T^*M \xrightarrow{?} \cdots \xrightarrow{?} \cdots$$

- **Def:** C is a compatibility operator for K if $c \circ K = 0 \implies c = c' \circ C$.

$$c \circ K = 0 \qquad \begin{tikzcd} \bullet \arrow[r, "K"] & \bullet \arrow[r, "C"] \arrow[d, "c' \circ K = c"'] & \bullet \arrow[dl, dashed, "c'"] \\ & \bullet & \end{tikzcd} \qquad C \circ K = 0$$

- In General Relativity: the components of C constitute a "complete set of local gauge invariant observables" for linearized gravity on the spacetime (M, g).

Possible approaches

How to obtain a compatibility complex of K on (M, g)?

1. BGG machinery (representation theoretic approach)
   - Requires (M, g) to be a homogeneous space. Otherwise, does not even produce a complex!
   - Can start with a curved BGG machinery, but no self-contained way to complete it to a compatibility complex.
2. Spencer-Goldschmidt theory. (Goldschmidt, Ann Math (1967) **86** 246)
   - Prolong to involution, compute Spencer cohomology.
   - Algorithmic. Implemented in computer algebra.
   - Computer algebra requires explicit coordinate components, is blind to any special geometry of (M, g).
   - Witness: execution of the algorithm (often infeasible by hand).
3. Reduction to canonical form. (IK [arXiv:1805.03751])
   - Canonical form: adapted to the geometry of (M, g), but with known compatibility complex (e.g., flat connection).
   - Reduction: equivalence up to homotopy.
   - Witness: the explicit equivalence, proof for canonical form (e.g., Poincaré lemma).

- Practical applications.

1., 2.: only (anti-)de Sitter spacetime (maximal symmetry).
3.: all other known cases; FLRW cosmology, Schwarzschild black hole [arXiv:1805.03751], now Kerr black hole [arXiv:1910.08765].

Step 1: Equivalence up to homotopy

Two complexes of differential operators, K_i and $\mathbb{D}_i$, are **equivalent up to homotopy** when the diagram

$$\begin{tikzcd}
\bullet \arrow[r, "K_0 = K"] \arrow[d, shift right, "U_0"'] & \bullet \arrow[r, "K_1"] \arrow[l, dashed, bend left, "H_0"] \arrow[d, shift right, "U_1"'] & \bullet \arrow[l, dashed, bend left, "H_1"] \arrow[d, shift right, "U_2"'] & \cdots \\
\bullet \arrow[r, "\mathbb{D}_0"'] \arrow[u, shift right, "V_0"'] & \bullet \arrow[r, "\mathbb{D}_1"'] \arrow[l, dashed, bend right, "H'_0"'] \arrow[u, shift right, "V_1"'] & \bullet \arrow[l, dashed, bend right, "H'_1"'] \arrow[u, shift right, "V_2"'] & \cdots
\end{tikzcd}$$

$$\begin{aligned}
\mathrm{id} - V_i \circ U_i &= H_i \circ K_i + K_{i-1} \circ H_{i-1}, \\
\mathrm{id} - U_i \circ V_i &= H'_i \circ \mathbb{D}_i + \mathbb{D}_{i-1} \circ H_{i-1}
\end{aligned}$$

exists, where the solid arrows commute and the dashed arrows are homotopy corrections.

Lemma (homotopy equivalence as witness)

Consider an equivalence up to homotopy between complexes K_i and $\mathbb{D}_i$, $i \geq 0$. Then, if $\mathbb{D}_i$ is a full compatibility complex, then so is K_i.

Step 2: Lifting compatibility operators

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Lemma

After reduction to ***canonical form****:*

(a) $\mathbb{D}_1 \circ U_1[K[v]] = 0$

(b) $(\mathrm{id} - K \circ H_0 - V_1 \circ U_1)[K[v]] = 0$

(d) $\exists$ *compatible* U_2, V_2, H_1, H_1'

(c) *(a) and (b) make a* ***comp.op.***

- The Lemma **does not depend** on K_0 being of **finite type**, only on the equivalence between K_0 and $\mathbb{D}_0$.
- Hence, we can **iterate** the argument (simplifying at each step!) to get a **full compatibility complex** for K_i, and its **equivalence up to homotopy** with $\mathbb{D}_i$.
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Step 3: Reduction to canonical form (non algorithmic)

Equations $K[v] = 0$ and $\mathbb{D}u = 0$ are **equivalent** only when there are local formulas $v \leftrightarrow u$ that are **bijective on solutions**:

$$\begin{aligned} K[V_0[u]] &= 0 \propto \mathbb{D}u = V_1[\mathbb{D}u], \\ \mathbb{D}U_0[v] &= 0 \propto K[v] = U_1[K[v]], \\ v - V_0[U_0[v]] &= 0 \propto K[v] = H_0[K[v]], \\ u - U_0[V_0[u]] &= 0 \propto \mathbb{D}u = H_0'[\mathbb{D}u]. \end{aligned}$$

On solutions means that some 0 must become $\propto K[v]$ or $\propto \mathbb{D}u$.

The relationships between these differential operators are **visually summarized** in the following diagram:

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Canonical form for PDEs of finite type

- The Killing equation $K[v]_{ab} = \nabla_a v_b + \nabla_b v_a = 0$ is of **finite type**. There exists an $N < \infty$ such that $v(x), \partial v(x), \ldots, \partial^N v(x)$ determines the solution v uniquely on a neighborhood of x.
- The **canonical form** for any PDE of finite type is $\mathbb{D}u = 0$, where $\mathbb{D}$ is a **flat connection** on a (possibly new) set of fields u:

$$\mathbb{D}_a u^\alpha = \partial_a u^\alpha + \Gamma_{a\beta}^{\alpha} u^\beta = 0, \quad \text{where} \quad [\mathbb{D}_a, \mathbb{D}_b] = 0.$$

- Starting with $\mathbb{D}_0 := \mathbb{D}$, define $\mathbb{D}_p w = \mathbb{D} \wedge w^\alpha$ for any **vector valued** p**-form** w^α. Then $\mathbb{D}_p \circ \mathbb{D}_{p-1} = 0$ is the **de Rham complex twisted by** $\mathbb{D}$; it is a full compatibility complex (Poincaré lemma).
- The number of components of u^α is the number of **independent solutions** of $K[v] = 0$ ($\leq n(n+1)/2$ in n-dim.). This number should be **locally constant**!

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Remark

The **canonical form** need not always be a **flat connection**. It need only be a PDE with a **known** compatibility complex.

But the **twisted de Rham** complex $\mathbb{D}_p$ is a particularly simple construction.

Example: (anti-)de Sitter (maximal symmetry)

- The simplest example is of a **constant curvature** space, identified by

$$C[g] := R_{abcd}[g] - \alpha(g_{ac}g_{bd} - g_{ad}g_{bc}) = 0.$$

- The Calabi complex (reviewed in [arXiv:1409.7212])

$$\begin{aligned} K_1[h] = \dot{C}[h] &= \nabla_{(a}\nabla_{c)}h_{bd} - \nabla_{(b}\nabla_{c)}h_{ad} - \nabla_{(a}\nabla_{d)}h_{bc} + \nabla_{(b}\nabla_{d)}h_{ac} \\ &\quad + \alpha(g_{ac}h_{bd} - g_{bc}h_{ad} - g_{ad}h_{bc} + g_{bd}h_{ac}), \\ K_2[r] &= 3\nabla_{[a}r_{bc]de}, \\ &\cdots \\ K_i[b] &= (i+1)\nabla_{[a_0}b_{a_1\cdots a_i]bc} \quad (i \geq 2). \end{aligned}$$

  is already known to be a **full compatibility complex**. Our method is not necessary, but can reproduce the same result.

- These formulas work in any signature and dimension n.
  - Lorentzian: **Minkowski** or **(Anti-)de Sitter** space with $\Lambda = \frac{(n-1)(n-2)}{2}\alpha$.
  - Riemannian: n**-sphere** ($\alpha > 0$), n-dimensional **hyperbolic space** ($\alpha < 0$).

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Newest example: Kerr rotating black hole [arXiv:1910.08756]

Kerr spacetime: Lorentzian $(\mathbb{R}^2 \times S^2, g)$, outside horizon.

- 4-dimensional, asymptotically flat, Einstein vacuum ($R_{ab} = 0$)
- explicit form of g_{ab}: not important here
- stationary (time symmetry), rotating (one symmetry axis)
- $\dim \ker K = 2$ uniformly (Killing vectors)
- Killing-Yano 2-form $\cong$ Killing 2-spinor (hidden symmetry)
- algebraically special curvature R_{abcd}
- spinor calculus adapted to geometry (!)

Other spacetimes in the same class:

- Kerr-de Sitter, Kerr-anti-de Sitter (not asymptotically flat)
- Kerr-Newman-((A)dS) (electrically charged)

Primer on spinor calculus

- Basic fact: $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})'$
- fundamental irreps: $(1,0) : \psi_A$, $(0,1) : \psi_{A'}$ on $\mathbb{C}^2$
- invariant pairing: $\epsilon^{AB} = -\epsilon^{BA}$
- all irreducible representations: $(k,l) : \psi_{(A_1 \cdots A_k)(A'_1 \cdots A'_l)}$
- spinor bundle on (M,g): $\mathrm{SO}(g)_{\mathbb{C}}$-tensors $\leftrightsquigarrow$ $\mathfrak{sl}(2,\mathbb{C})$-spinors
- Levi-Civita connection: $\nabla_a \leftrightsquigarrow \nabla_{AA'}$
- Translation:

	ϕ	$\psi_{AA'}$	$\Phi_{ABA'B'}$	ψ_{AB}	$\psi_{A'B'}$	Ψ_{ABCD}	$\Psi_{A'B'C'D'}$
spinor	$(0,0)$	$(1,1)$	$(2,2)$	$(2,0)$	$(0,2)$	$(4,0)$	$(0,4)$
tensor$_{\mathbb{C}}$	$\mathbb{C}$	$\ydiagram{1}$	$\ydiagram{2}_0$	$\ydiagram{1,1} + i{*}\ydiagram{1,1}$	$\ydiagram{1,1} - i{*}\ydiagram{1,1}$	$\ydiagram{2,2}_0 + i{*}\ydiagram{2,2}_0$	$\ydiagram{2,2}_0 - i{*}\ydiagram{2,2}_0$
	ϕ	v_a	S_{ab}	$\mathcal{Y}_{ab}$	$\bar{\mathcal{Y}}_{ab}$	$\mathcal{W}_{abcd}$	$\bar{\mathcal{W}}_{abcd}$
	scalar	vector	symmetric traceless 2-tensor	anti-self-dual 2-form	self-dual 2-form	anti-self-dual Weyl	self-dual Weyl

Spinor calculus on Kerr

- Basic geometric objects:
  - Killing 2-spinor κ_{AB}, $\bar{\kappa}_{A'B'}$ (Killing-Yano 2-form)
  - Killing vectors: $\xi_{AA'}$, $\zeta_{AA'}$
  - Ricci scalar, traceless Ricci tensor: $\Lambda = 0$, $\Phi_{ABA'B'} = 0$
  - Weyl curvature: Ψ_{ABCD}, $\overline{\Psi}_{A'B'C'D'}$
  - Involutivity: $\nabla_{EE'}(\kappa, \xi, \zeta, \Psi) = O(\kappa, \xi, \zeta, \Psi)$
- Fundamental spin operators:

$\nabla_{AA'}$	$\nabla^{\boldsymbol{A}}{}_{A'}(-)_{\boldsymbol{A}\cdots}$	$\nabla_{(\boldsymbol{A}}{}^{A'}(-)_{\boldsymbol{B}\cdots)}$
$\nabla_A{}^{\boldsymbol{A'}}(-)_{\boldsymbol{A'}\cdots}$	$\mathscr{D}$	$\mathscr{C}$
$\nabla^A{}_{(\boldsymbol{A'}}(-)_{\boldsymbol{B'}\cdots)}$	$\mathscr{C}^\dagger$	$\mathscr{T}$

- Example: 4-dimensional de Rham complex $(\mathrm{d}_0, \mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3)$

$$[(0,0)] \xrightarrow{[\mathscr{T}]} [(1,1)] \xrightarrow{\begin{bmatrix}\mathscr{C}^\dagger \\ \mathscr{C}\end{bmatrix}} \begin{bmatrix}(0,2) \\ (2,0)\end{bmatrix} \xrightarrow{[\mathscr{C} \quad -\mathscr{C}^\dagger]} [(1,1)] \xrightarrow{[\mathscr{D}]} [(0,0)]$$

- Sought canonical form: $K \rightsquigarrow \mathbb{D}_0$, $\mathbb{D}_i = \begin{bmatrix}\xi \\ \zeta\end{bmatrix} \otimes \mathrm{d}_i$.

Reduction to canonical form on Kerr

- Spinor calculus on Kerr implemented in MATHEMATICA, on top of xAct, by Aksteiner & Bäckdahl. Some packages public, some private. [arXiv:1601.06084]
- Geometric ingredients:
  - special spinors: κ, ξ, ζ, Ψ
  - spin operators: $\mathscr{D}$, $\mathscr{C}$, $\mathscr{C}^\dagger$, $\mathscr{T}$,
  - Killing vector sub-bundle $\mathrm{span}\{\xi, \zeta\}$: inclusion, projection, orthogonal projection
- We obtained **compact formulas** for all the operators in the reduction diagram using these ingredients.

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\begin{tikzcd}
\bullet \arrow[r, "K"] \arrow[d, shift right, "U_0"'] & \bullet \arrow[l, bend left, dashed, "H_0"] \arrow[d, shift right, "U_1"'] \\
\bullet \arrow[u, shift right, "V_0"'] \arrow[r, "\mathbb{D}_0"'] & \bullet \arrow[l, bend right, dashed, "H_0'"'] \arrow[u, shift right, "V_1"']
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$$

- With U_0 and V_0, the other operators follow by **explicitly factoring** through K and $\mathbb{D}_0$. This was the most work intensive part, including some guessing/verifying.
- Following the general approach, we obtained **compact formulas** for the **full compatibility complex** K_i.
- More explicit factorizations showed the equivalence of $K_1 \leftrightsquigarrow \tilde{K}_1$ with convenient formulas previously published in [PRL (2018) **121** 051104] by Aksteiner & Bäckdahl, but not proven to be complete until now!

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Discussion

- A **full compatibility complex** K_i can be **practically** constructed for K on any spacetime of sub-maximal (but uniform) symmetry.
- The core of the calculation consists of explicitly identifying integrability conditions of the **Killing equation** $K[v] = 0$ and putting it into the **canonical form**, e.g., of a **flat connection**.
- The construction outputs a **homotopy equivalence** of K_i with a twisted de Rham complex, which **witnesses** the completeness of each K_i.
- Newest application: rotating **Kerr** black hole (together with Aksteiner, Andersson, Bäckdahl and Whiting [arXiv:1910.08756]).
- **TODO:** extend to other geometries, Myers-Perry, Kerr-Newman, ...
- **TODO:** connect with non-linear version of the question

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Thank you for your attention!