The Euler system on a space curve

Anna Duyunova, Valentin Lychagin, Sergey Tychkov

laboratory 6
Institute of Control Sciences of RAS

The Euler system

Consider flows of an inviscid medium

$$\begin{cases} \rho(u_t + uu_a) = -p_a - \rho \mathrm{g} h', \\ \rho_t + (\rho u)_a = 0, \\ \rho T\left(s_t + us_a\right) - kT_{aa} = 0, \end{cases} \tag{1}$$

on a naturally-parametrised space curve in the three-dimensional Euclidean space

$$M = \{x = f(a),\, y = g(a),\, z = h(a)\}.$$

in a field of constant gravitational force.

Thermodynamics

A thermodynamic state is a two-dimensional Legendrian manifold $L \subset \mathbb{R}^5(p, \rho, s, T, \epsilon)$, a maximal integral manifold of the differential 1-form

$$\theta = d\epsilon - Tds - p\rho^{-2}d\rho,$$

i.e. a manifold such that the first law of thermodynamics $\theta\big|_L = 0$ holds.

Following [1], we require that the quadratic differential form

$$\kappa = d(T^{-1}) \cdot d\epsilon - \rho^{-2}d(pT^{-1}) \cdot d\rho$$

on the surface L be negative definite,

$$\kappa\big|_L < 0,$$

and the entropy s satisfies the inequality $s \leq s_0$, where the constant s_0 depends on the nature of a process under consideration.

Thermodynamics

Consider the projection $\pi\colon\ (p, \rho, s, T, \epsilon) \longmapsto (p, \rho, s, T)$. The restriction of this map on the state surface L is a diffeomorphism $\bar{L} = \pi(L)$, and the surface $\bar{L} \subset \mathbb{R}^4$ is a Lagrangian manifold in the 4-dimensional symplectic space $\mathbb{R}^4$ equipped with the structure form

$$\Omega = d\theta = ds \wedge dT + \rho^{-2} d\rho \wedge dp.$$

Thus, the *thermodynamic state* is the Lagrangian submanifold $\bar{L}$ in the symplectic space $(\mathbb{R}^4, \Omega)$:

$$\begin{cases} f(p, \rho, s, T) = 0, \\ g(p, \rho, s, T) = 0, \end{cases} \quad \text{and} \quad [f, g]|_{\bar{L}} = 0, \qquad (2)$$

where $[f, g]$ is the Poisson bracket, and the symmetric differential form κ is negative definite on this surface.

Symmetry Lie algebra

We consider a Lie algebra $\mathfrak{g}$ of point symmetries of the Euler system (1).
Let $\vartheta \colon \mathfrak{g} \to \mathfrak{h}$ be the following Lie algebras homomorphism

$$\vartheta \colon X \mapsto X(\rho)\partial_\rho + X(s)\partial_s + X(p)\partial_p + X(T)\partial_T,$$

where $\mathfrak{h}$ is a Lie algebra generated by vector fields that act on the thermodynamic variables p, ρ, s and T.
The kernel of the homomorphism ϑ is an ideal $\mathfrak{g}_\mathfrak{m} \subset \mathfrak{g}$ (geometric symmetries).
Let also $\mathfrak{h}_\mathfrak{t}$ be the Lie subalgebra of the algebra $\mathfrak{h}$ that preserves thermodynamic state (2).

Theorem

A Lie algebra $\mathfrak{g}_{\mathfrak{sym}}$ of symmetries of the Euler system $\mathcal{E}$ coincides with

$$\vartheta^{-1}(\mathfrak{h}_\mathfrak{t}).$$

Symmetry Lie algebra

Let $h(a)$ be an arbitrary function.

The Lie algebra $\mathfrak{g}$ of point symmetries of the system (1) is generated by the vector fields

$$\begin{aligned} X_1 &= \partial_t, & X_4 &= T\,\partial_T, \\ X_2 &= \partial_p, & X_5 &= p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s. \\ X_3 &= \partial_s, \end{aligned}$$

The pure thermodynamic part $\mathfrak{h}$ of the system symmetry algebra is

$$\begin{aligned} Y_1 &= \partial_p, & Y_3 &= T\,\partial_T, \\ Y_2 &= \partial_s, & Y_4 &= p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s. \end{aligned}$$

Thus, in this case the system of differential equations $\mathcal{E}$ has the smallest Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}_{\mathfrak{t}})$.

Symmetry Lie algebra

- $h(a)$ is arbitrary

$$X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s, \qquad X_4 = T\,\partial_T,$$
$$X_5 = p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s$$

- $h(a) = const$

$$X_6 = \partial_a, \qquad X_7 = t\,\partial_a + \partial_u,$$
$$X_8 = t\,\partial_t + a\,\partial_a - s\,\partial_s,$$
$$X_9 = t\,\partial_t - u\,\partial_u - 2p\,\partial_p + s\,\partial_s,$$

- $h(a) = \lambda a$, $\lambda \neq 0$

$$X_6 = \partial_a, \qquad X_7 = t\,\partial_a + \partial_u,$$
$$X_8 = t\,\partial_t + 2a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho - s\,\partial_s,$$
$$X_9 = \left(\frac{t^2}{2} + \frac{a}{\lambda g}\right)\partial_a + 2\left(t + \frac{u}{\lambda g}\right)\partial_u - \frac{2\rho}{\lambda g}\,\partial_\rho$$

- $h(a) = \lambda a^2$, $\lambda \neq 0$

$$X_6 = a\partial_a + u\,\partial_u - 2\rho\,\partial_\rho,$$
$$X_7 = \sin(\sqrt{2\lambda g}\,t)\,\partial_a + \sqrt{2\lambda g}\cos(\sqrt{2\lambda g}\,t)\,\partial_u,$$
$$X_8 = \cos(\sqrt{2\lambda g}\,t)\,\partial_a - \sqrt{2\lambda g}\sin(\sqrt{2\lambda g}\,t)\,\partial_u$$

Symmetry Lie algebra

- $h(a) = \lambda_1 a^{\lambda_2}$, $\lambda_2 \neq 0, 1, 2$
  $$X_6 = t\,\partial_t - \frac{2a}{\lambda_2 - 2}\partial_a - \frac{\lambda_2 u}{\lambda_2 - 2}\partial_u + \frac{2\lambda_2 \rho}{\lambda_2 - 2}\partial_\rho - s\,\partial_s$$
- $h(a) = \lambda_1 e^{\lambda_2 a}$, $\lambda_2 \neq 0$
  $$X_6 = t\,\partial_t - \frac{2}{\lambda_2}\,\partial_a - u\,\partial_u - p\,\partial_p + \rho\,\partial_\rho$$
- $h(a) = \ln a$
  $$X_6 = t\,\partial_t + a\,\partial_a - s\,\partial_s$$

Lifting curves from the plane

Let a curve in the space be defined as a pair of a plane curve $(x(\tau), y(\tau))$ and a 'lifting' function $z(\tau)$.
Let $l(\tau) = \int\limits_0^\tau \sqrt{x_\theta^2 + y_\theta^2}\, d\theta$ – the length of the plane curve.
Then the following relation between natural parameter a and the parameter τ is valid

$$h_a = \frac{z_\tau}{\sqrt{x_\tau^2 + y_\tau^2 + z_\tau^2}}.$$

1. $h(a) = const$
The first way of lifting a plane curve is to translate the whole curve along z-axis, i.e. if $h(a) = const$ then $z(\tau) = const$.

Lifting curves from the plane

2. $h(a) = \lambda a$, $\lambda \neq 0$
The second way to lift curve is lifting proportional to the length of the plane part, i.e. if $h(a) = \lambda a$ then we have the following differential equitation on the 'lifting' function $z(\tau)$

$$\left(1 - \lambda^2\right) z_\tau^2 = \lambda^2 \left(x_\tau^2 + y_\tau^2\right),$$

solving which given $1 - \lambda^2 > 0$, we get

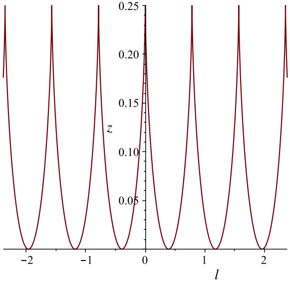
$$z(\tau) = \pm \frac{\lambda}{\sqrt{1 - \lambda^2}} \, l(\tau) + C,$$

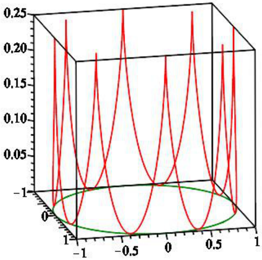
where $l(\tau)$ – is length of plane projection of curve and $C = const$.
If $\lambda = \pm 1$, then $x(t) = y(t) = const$ and we have a vertical line.

Lifting curves from the plane

3. $h(a) = \lambda a^2$, $\lambda \neq 0$
The relation between the 'lifting' function $z(\tau)$ and the length $l(\tau)$ of the plane curve is

$$\sqrt{4\lambda z(1-4\lambda z)} - \arccos(\sqrt{4\lambda z}) = \pm 4\lambda l(\tau).$$



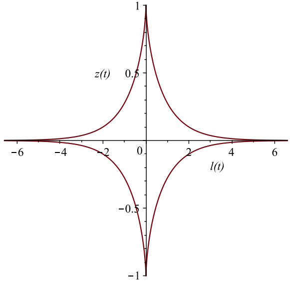


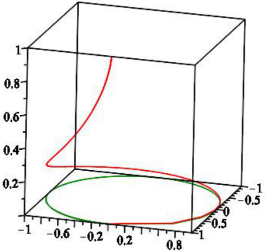
Lifting curves from the plane

5. $h(a) = \lambda_1 e^{\lambda_2 a}$

The relation between the 'lifting' function $z(\tau)$ and the length $l(\tau)$ of the plane curve is

$$\sqrt{1-\lambda_2^2 z^2} - \frac{1}{2}\ln\frac{1+\sqrt{1-\lambda_2^2 z^2}}{1-\sqrt{1-\lambda_2^2 z^2}} = \pm\lambda_2 l(\tau)$$

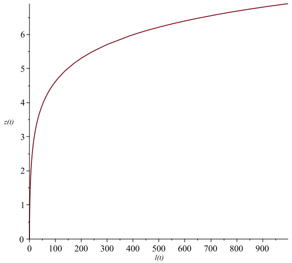


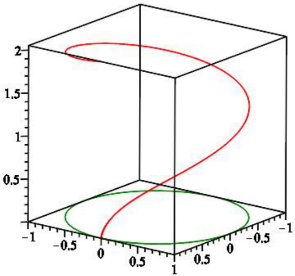


Lifting curves from the plane

6. $h(a) = \ln a$
The relation between the 'lifting' function $z(\tau)$ and the length $l(\tau)$ of the plane curve is

$$\sqrt{e^{2z} - 1} - \arctan\sqrt{e^{2z} - 1} = \pm l(\tau).$$





Thermodynamic states

Let $h(a)$ be an arbitrary function, and let the thermodynamic state admit a one-dimensional symmetry algebra

$$Z = \gamma_1 \partial_p + \gamma_2 \partial_s + \gamma_3 T\,\partial_T + \gamma_4 (p\,\partial_p + \rho\,\partial_\rho - s\,\partial_s),$$

then the Lagrangian surface $\bar{L}$ can be found from the conditions $\{\Omega|_{\bar{L}} = 0, \iota_Z \Omega|_{\bar{L}} = 0\}$, which lead to the following PDE system on the internal energy

$$\begin{cases} \gamma_4 \rho\, \epsilon_{\rho\rho} + (\gamma_2 - \gamma_4 s)\epsilon_{\rho s} + \gamma_4 \epsilon_\rho - \gamma_1 \rho^{-2} = 0, \\ (\gamma_2 - \gamma_4 s)\epsilon_{ss} + \gamma_4 \rho\, \epsilon_{\rho s} - \gamma_3\, \epsilon_s = 0. \end{cases}$$

Solving this system for the general case, we have

$$p = C_1 \rho - \frac{\gamma_1}{\gamma_4}, \quad T = C_2 (\gamma_2 - \gamma_4 s)^{-\frac{\gamma_3}{\gamma_4}},$$

where C_1, C_2 are constants. The condition $\kappa|_L < 0$ leads to the relations

$$\frac{\gamma_3}{\gamma_2 - \gamma_4 s} > 0, \quad C_1 > 0 \quad \text{for all} \quad s \in (-\infty, s_0].$$

Thermodynamic states

Theorem

The thermodynamic states admitting a one-dimensional symmetry algebra have the form

$$p = C_1\rho - \frac{\gamma_1}{\gamma_4}, \quad T = C_2(\gamma_2 - \gamma_4 s)^{-\frac{\gamma_3}{\gamma_4}},$$

where the constants defining the symmetry algebra satisfy inequalities

$$s_0 < \frac{\gamma_2}{\gamma_4}, \quad C_1 > 0, \quad \frac{\gamma_1}{\gamma_4} < 0,$$

and besides they must meet one of the following conditions:

1. *if* $\frac{\gamma_3}{\gamma_4}$ *is irrational, then* $\gamma_3 > 0,\ \gamma_4 > 0,\ C_2 > 0;$
2. *if* $\frac{\gamma_3}{\gamma_4}$ *is rational, then* $\frac{\gamma_3}{\gamma_4} > 0$ *(i.e.* $\frac{\gamma_3}{\gamma_4} = \frac{m}{k}$ *) and*
    1. *if* k *is even, then* $\gamma_4 > 0,\ C_2 > 0;$
    2. *if* k *is odd and* m *is even, then* $C_2 > 0;$
    3. *if* k *is odd and* m *is odd, then* $C_2\gamma_4 > 0.$

Kinematic invariants

We consider two group actions on the Euler system $\mathcal{E}$ – the prolonged actions of the groups generated by actions of the Lie algebras $\mathfrak{g}_{\mathfrak{m}}$ and $\mathfrak{g}_{\mathfrak{sym}}$.
A function J on the manifold $\mathcal{E}_k$ is a *kinematic differential invariant of order* $\leq k$ if

1. J is a rational function along fibers of the projection $\pi_{k,0} : \mathcal{E}_k \to \mathcal{E}_0$,
2. J is invariant with respect to the prolonged action of the Lie algebra $\mathfrak{g}_{\mathfrak{m}}$, i.e., for all $X \in \mathfrak{g}_{\mathfrak{m}}$,

$$X^{(k)}(J) = 0, \tag{3}$$

where $\mathcal{E}_k$ is the prolongation of the system $\mathcal{E}$ to k-jets, and $X^{(k)}$ is the k-th prolongation of a vector field $X \in \mathfrak{g}_{\mathfrak{m}}$.
A kinematic invariant is *an Euler invariant* if condition (3) holds for all $X \in \mathfrak{g}_{\mathfrak{sym}}$.

Kinematic invariants

Theorem

- *The kinematic invariants field is generated by first-order basis differential invariants and by basis invariant derivations. This field separates regular orbits.*
- *For the general cases of* $h(a)$*, as well as for* $h(a) = \lambda_1 a^{\lambda_2}$*,* $h(a) = \lambda_1 e^{\lambda_2 a}$ *and* $h(a) = \ln a$*, the basis differential invariants are*

$$a, \quad u, \quad \rho, \quad s, \quad u_a, \quad \rho_a, \quad s_t, \quad s_a,$$

and the basis invariant derivations are

$$\frac{\mathrm{d}}{\mathrm{d}t}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

Kinematic invariants

Theorem

- *For the cases* $h(a) = const$, $h(a) = \lambda a$ *and* $h(a) = \lambda a^2$, *the basis differential invariants are*

$$\rho, \quad s, \quad u_a, \quad \rho_a, \quad s_a, \quad s_t + u s_a,$$

and basis invariant derivations are

$$\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

- *The number of independent invariants of pure order* k *is equal to* 4 *for* $k \geq 1$.

Euler invariants

Let $h(a)$ be an arbitrary function.
If the thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field

$$\xi_1 X_2 + \xi_2 X_3 + \xi_3 X_4 + \xi_4 X_5,$$

then the field of Euler differential invariants is generated by the differential invariants

$$a, \quad \left(s - \frac{\xi_2}{\xi_4}\right)\rho, \quad u, \quad u_a, \quad \frac{\rho_a}{\rho}, \quad s_t\rho, \quad s_a\rho$$

of the first order and by the invariant derivations

$$\frac{\mathrm{d}}{\mathrm{d}t}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

Euler invariants

Let $h(a) = \lambda a^2$, $\lambda \neq 0$.
If the thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field

$$\xi_1 X_2 + \xi_2 X_3 + \xi_3 X_4 + \xi_4 X_5 + \xi_5 X_8 + \xi_6 X_9,$$

then the field of Euler differential invariants is generated by the differential invariants

$$\left(s - \frac{\xi_2}{\xi_4}\right)\frac{\rho_a}{\rho s_a}, \quad u_a, \quad \rho_a \rho^{\frac{\xi_5}{\xi_4 - 2\xi_5} - 1}, \quad \frac{s_a \rho^4}{\rho_a^3}, \quad \frac{(s_t + u s_a)\rho^3}{\rho_a^2}$$

of the first order and by the invariant derivations

$$\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}, \quad \rho^{\frac{\xi_5}{\xi_4 - 2\xi_5}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

Quotient equation

Our goal is to study a quotient equation for the case: $h(a) = \lambda a^2$, $\lambda \neq 0$.
It follows from theorem above that there are four relations (syzygies) between the following second order invariants

$$\frac{du_a}{d\rho}, \quad \frac{d\rho_a}{d\rho}, \quad \frac{dT_a}{d\rho}, \quad \frac{d(T_t + uT_a)}{d\rho},$$
$$\frac{du_a}{dT}, \quad \frac{d\rho_a}{dT}, \quad \frac{dT_a}{dT}, \quad \frac{d(T_t + uT_a)}{dT},$$

where

$$\frac{d}{d\rho} = \frac{1}{\rho_t T_a - \rho_a T_t}\left(T_a \frac{\mathrm{d}}{\mathrm{d}t} - T_t \frac{\mathrm{d}}{\mathrm{d}a}\right),$$
$$\frac{d}{dT} = \frac{1}{\rho_t T_a - \rho_a T_t}\left(-\rho_a \frac{\mathrm{d}}{\mathrm{d}t} + \rho_t \frac{\mathrm{d}}{\mathrm{d}a}\right)$$

are Tresse derivatives.

Quotient equation

Choosing ρ and T as Lie–Tresse coordinates (x, y) and $K(x, y) = u_a$, $L(x, y) = \rho_a$, $M(x, y) = T_a$, $N(x, y) = T_t + uT_a$ as unknown functions, respectively, if $u^2 + p_x \neq 0$, $xKM + LN \neq 0$, $M \neq 0$, we get the quotient equation E_q

$$
\begin{cases}
xKM_x - NM_y + LN_x + M(N_y - K) = 0, \\
k(LM_x + MM_y) + xy(xKs_x - Ns_y) = 0, \\
L\left(x(p_{xx}L^2 + 2p_{xy}LM + p_{yy}M^2) + \right. \\
\left. (xLL_x + xML_y - L^2)p_x + (xLM_x + xMM_y - LM)p_y\right) + \\
x^2\left(KNL_y - xK^2L_x + (xKM + LN)K_y + 3LK^2 + \omega^2 L\right) = 0, \\
M\left(x(p_{xx}L^2 + 2p_{xy}LM + p_{yy}M^2) + \right. \\
\left. (xLL_x + xML_y - L^2)p_x + (xLM_x + xMM_y - LM)p_y\right) - \\
x\left(N^2L_y - xKNL_x + (xKM + LN)xK_x + 2LKN - xM(K^2 + \omega^2)\right) = 0.
\end{cases}
$$

Quotient equation

The symbol of the quotient equation E_q is

$$\begin{pmatrix} 0 & 0 & k(L\xi_1 + M\xi_2) & 0 \\ 0 & 0 & xK\xi_1 - N\xi_2 & L\xi_1 + M\xi_2 \\ x^2(xKM + LN)\xi_2 & A & xp_y L(L\xi_1 + M\xi_2) & 0 \\ -x^2(xKM + LN)\xi_1 & B & xp_y M(L\xi_1 + M\xi_2) & 0 \end{pmatrix},$$

where

$$A = x(p_x L^2 - x^2 K^2)\xi_1 + x(p_x LM + xKN)\xi_2,$$
$$B = x(p_x LM + xKN)\xi_1 + x(p_x M^2 - N^2)\xi_2.$$

Its determinant has the form

$$kx^3(xKM + LN)(L\xi_1 + M\xi_2)^2\left((L\xi_1 + M\xi_2)^2 p_x - (xK\xi_1 - N\xi_2)^2\right),$$

which gives us three characteristic vector fields of E_q:

$$Z_1 = L\partial_x + M\partial_y, \qquad Z_{2,3} = xK\partial_x - N\partial_y \pm \sqrt{p_x}(L\partial_x + M\partial_y).$$

Quotient equation

We find solution of the quotient for the case of ideal gas.
Recall that the thermodynamic state is given by the Planck potential $\Phi(x, y)$, which for the case of ideal gas is the following

$$\Phi(x, y) = \frac{n}{2} \ln y - \ln x,$$

where n is a number of freedom degrees of a gas particle.
Pressure and entropy are expressed in terms of $\Phi(x, y)$:

$$p(x, y) = -Rx^2 T \Phi_x, \quad s(x, y) = R(\Phi + y\Phi_y),$$

where R is a specific gas constant.
Moreover, we consider solutions of the quotient equation E_q that are constant along the field Z_1.

Quotient equation

Let us require the functions M and L be the first integrals of the vector field Z_1. Then solving an overdetermined system $E_q \cup \{Z_1(L) = Z_1(M) = 0\}$, we obtain the following of solutions of the quotient E_q:

$$L = 0,\ M = c_1 x^{1+\frac{2}{n}},\ K = \sqrt{c_2 x^2 - \omega^2},\ N = -\frac{2y\sqrt{c_2 x^2 - \omega^2}}{n},$$

$$L = c_1 \left(\frac{x}{y}\right)^{\frac{2n}{n-2}},\ M = \frac{y}{x} L,\ K = \sqrt{c_2 \left(\frac{x}{y}\right)^{\frac{2n}{n-2}} - \omega^2},\ N = -\frac{2Ky}{n},$$

where $c_1 \neq 0$, c_2 are constants.

Euler solutions

Now we use the first solution of E_q to solve the Euler system. Recalling that x, y, K, L, M, N are invariants and adding their expressions to the Euler system, we get a finite-type system

$$\begin{cases} \rho_a = 0, \quad T_a = c_1 \rho^{1+\frac{2}{n}}, \quad u_a = \sqrt{c_2\rho^2 - \omega^2}, \\ T_t + uT_a = -\dfrac{2T}{n}\sqrt{c_2\rho^2 - \omega^2}, \\ \rho(u_t + uu_a) + R\rho T_a + \omega^2 \rho a = 0, \\ \rho_t + \rho u_a = 0, \\ 2kT_{aa} - Rn\rho(T_t + uT_a) + 2RT\rho_t = 0. \end{cases}$$

This system is overdetermined, and solving it, we get expressions for the pressure, density, temperature and velocity.

Euler solutions

$$\rho = \frac{\omega}{\sqrt{c_2}\cos(c_3 - \omega t)}, \quad u = a\,\omega \tan(c_3 - \omega t) + f(t),$$

$$T = c_1 a \rho^{1+\frac{2}{n}} - \cos^{-\frac{2}{n}}(c_3 - \omega t)\left(c_1 c_2^{-\frac{n+2}{2n}} \omega^{\frac{n+2}{n}} \int \frac{f(t)}{\cos(c_3 - \omega t)} dt + c_5\right),$$

where

$$f(t) = \frac{-R c_1 c_2^{-\frac{n+2}{2n}} \omega^{\frac{n+2}{n}}}{\cos(c_3 - \omega t)} \left(\int \cos^{-\frac{2}{n}}(c_3 - \omega t) dt + c_4\right)$$

and $c_1 \neq 0, c_2 > 0, \ldots, c_5$ are constants.

Euler solutions

Doing the same with the second solution of the quotient:

$$L = c_1 \left(\frac{x}{y}\right)^{\frac{2n}{n-2}}, \quad M = \frac{y}{x} L, \quad K = \sqrt{\frac{c_2}{c_1} L - \omega^2}, \quad N = \frac{-2Ky}{n},$$

we have

$$u = a\,\omega \tan(c_3 - \omega t) + 2f(t), \quad T = \rho \left(\frac{c_2 \cos^2(c_3 - \omega t)}{\omega^2}\right)^{\frac{n-2}{2n}},$$

$$\rho = \frac{c_1 \omega^2 a}{c_2 \cos^2(c_3 - \omega t)} - \frac{c_1 \omega^2}{c_2 \cos(c_3 - \omega t)} \left(\int \frac{2f(t)}{\cos(c_3 - \omega t)} dt + c_5\right),$$

where $f(t)$ is the same as before, and $c_1, \dots, c_5$ are constants.

Literature

V. Lychagin, *Contact Geometry, Measurement and Thermodynamics*, in: *Nonlinear PDEs, Their Geometry and Applications. Proceedings of the Wisla 18 Summer School*, Springer Nature, Switzerland, 3-54 (2019).

Thank you for attention.