Triangular decoupling of systems of differential equations, with application to separation of variables on Schwarzschild spacetime

[arXiv:1711.00585, 1801.09800, **2004.09651**]

Igor Khavkine

Institute of Mathematics
Czech Academy of Sciences (Prague)

Geometry of Differential Equations seminar
Independent University of Moscow

20 Oct 2021

Schwarzschild Scalar Wave Equation (model problem)

- **Schwarzschild:** spherically symmetric, static black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -f(\mathrm{d}t)^2 + f^{-1}(\mathrm{d}r)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right), \quad f(r) = 1 - \frac{2M}{r}.$$

- Radial mode equation of scalar wave equation (may omit ωlm):

$$z(t,r,\theta,\varphi) = \frac{\phi_{\omega lm}(r)}{r} Y^{lm}(\theta,\varphi) e^{-i\omega t}, \quad \Box_{\mathbf{g}} z = 0 \quad \Longrightarrow \quad \mathcal{D}_0\phi = 0,$$

where the spin-s Regge-Wheeler operator is ($r \in (2M,\infty)$, $l \geq s$)

$$\mathcal{D}_s\phi := \partial_r f \partial_r \phi - \underbrace{\frac{l(l+1) + (1-s^2)\frac{2M}{r}}{r^2}}_{f^{-1}(\cdots)>0}\phi + \omega^2 \underbrace{\frac{1}{f}}_{(\cdots)>0}\phi.$$

N.B.: $\mathcal{D}_s^* = \mathcal{D}_s$ is formally self-adjoint, of Sturm-Liouville type.

Schwarzschild Scalar Wave Equation (model problem)

- **Schwarzschild:** spherically symmetric, static black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -f(\mathrm{d}t)^2 + f^{-1}(\mathrm{d}r)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right), \quad f(r) = 1 - \frac{2M}{r}.$$

- Radial mode equation of scalar wave equation (may omit ωlm):

$$z(t,r,\theta,\varphi) = \frac{\phi_{\omega lm}(r)}{r} Y^{lm}(\theta,\varphi) e^{-i\omega t}, \quad \Box_{\mathbf{g}} z = 0 \quad \Longrightarrow \quad \mathcal{D}_0\phi = 0,$$

where the spin-s Regge-Wheeler operator is ($r \in (2M,\infty)$, $l \geq s$)

$$\mathcal{D}_s\phi := \partial_r f \partial_r \phi - \underbrace{\frac{l(l+1) + (1-s^2)\frac{2M}{r}}{r^2}}_{f^{-1}(\cdots)>0} \phi + \omega^2 \underbrace{\frac{1}{f}}_{(\cdots)>0} \phi.$$

N.B.: $\mathcal{D}_s^* = \mathcal{D}_s$ is formally self-adjoint, of Sturm-Liouville type.

✓ Positive self-adjoint ω^2-spectral problem for on $L^2(2M,\infty;\frac{\mathrm{d}r}{f})$.

- No complex $\nu_{\omega lm} = \sqrt{\omega^2} \in \mathbb{C} \setminus \mathbb{R}$ spectrum (growing $e^{-i\nu t}$ modes!).
- Orthogonality and completeness of generalized eigenmodes $\phi_{\omega lm}(r)$.

Schwarzschild Scalar Wave Equation (model problem)

- **Schwarzschild:** spherically symmetric, static black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -f(\mathrm{d}t)^2 + f^{-1}(\mathrm{d}r)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right), \quad f(r) = 1 - \frac{2M}{r}.$$

- Radial mode equation of scalar wave equation (may omit ωlm):

$$z(t,r,\theta,\varphi) = \frac{\phi_{\omega lm}(r)}{r} Y^{lm}(\theta,\varphi)e^{-i\omega t}, \quad \Box_{\mathbf{g}} z = 0 \quad \Longrightarrow \quad \mathcal{D}_0\phi = 0,$$

where the spin-s Regge-Wheeler operator is ($r \in (2M,\infty)$, $l \geq s$)

$$\mathcal{D}_s\phi := \partial_r f \partial_r \phi - \underbrace{\frac{l(l+1) + (1-s^2)\frac{2M}{r}}{r^2}}_{f^{-1}(\cdots)>0}\phi + \omega^2 \underbrace{\frac{1}{f}}_{(\cdots)>0}\phi.$$

N.B.: $\mathcal{D}_s^* = \mathcal{D}_s$ is formally self-adjoint, of Sturm-Liouville type.

✓ Positive self-adjoint ω^2-spectral problem for on $L^2(2M,\infty;\frac{\mathrm{d}r}{f})$.

- No complex $\nu_{\omega lm} = \sqrt{\omega^2} \in \mathbb{C}\setminus\mathbb{R}$ spectrum (growing $e^{-i\nu t}$ modes!).
- Orthogonality and completeness of generalized eigenmodes $\phi_{\omega lm}(r)$.

Schwarzschild Scalar Wave Equation (model problem)

- **Schwarzschild:** spherically symmetric, static black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -f(\mathrm{d}t)^2 + f^{-1}(\mathrm{d}r)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right), \quad f(r) = 1 - \frac{2M}{r}.$$

- Radial mode equation of scalar wave equation (may omit ωlm):

$$z(t,r,\theta,\varphi) = \frac{\phi_{\omega lm}(r)}{r} Y^{lm}(\theta,\varphi)e^{-i\omega t}, \quad \Box_{\mathbf{g}} z = 0 \quad \Longrightarrow \quad \mathcal{D}_0\phi = 0,$$

where the spin-s Regge-Wheeler operator is ($r \in (2M,\infty)$, $l \geq s$)

$$\mathcal{D}_s\phi := \partial_r f \partial_r \phi - \underbrace{\frac{l(l+1) + (1-s^2)\frac{2M}{r}}{r^2}}_{f^{-1}(\cdots)>0}\phi + \omega^2 \underbrace{\frac{1}{f}}_{(\cdots)>0}\phi.$$

N.B.: $\mathcal{D}_s^* = \mathcal{D}_s$ is formally self-adjoint, of Sturm-Liouville type.

✓ Positive self-adjoint ω^2-spectral problem for on $L^2(2M,\infty;\frac{\mathrm{d}r}{f})$.

- No complex $\nu_{\omega lm} = \sqrt{\omega^2} \in \mathbb{C}\setminus\mathbb{R}$ spectrum (growing $e^{-i\nu t}$ modes!).
- Orthogonality and completeness of generalized eigenmodes $\phi_{\omega lm}(r)$.

Schwarzschild Scalar Wave Equation (model problem)

- **Schwarzschild:** spherically symmetric, static black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -f(\mathrm{d}t)^2 + f^{-1}(\mathrm{d}r)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right), \quad f(r) = 1 - \frac{2M}{r}.$$

- Radial mode equation of scalar wave equation (may omit ωlm):

$$z(t,r,\theta,\varphi) = \frac{\phi_{\omega lm}(r)}{r} Y^{lm}(\theta,\varphi)e^{-i\omega t}, \quad \Box_{\mathbf{g}} z = 0 \quad \Longrightarrow \quad \mathcal{D}_0\phi = 0,$$

where the spin-s Regge-Wheeler operator is ($r \in (2M,\infty)$, $l \geq s$)

$$\mathcal{D}_s\phi := \partial_r f \partial_r \phi - \underbrace{\frac{l(l+1) + (1-s^2)\frac{2M}{r}}{r^2}}_{f^{-1}(\cdots)>0}\phi + \omega^2 \underbrace{\frac{1}{f}}_{(\cdots)>0}\phi.$$

N.B.: $\mathcal{D}_s^* = \mathcal{D}_s$ is formally self-adjoint, of Sturm-Liouville type.

✓ Positive self-adjoint ω^2-spectral problem for on $L^2(2M,\infty;\frac{\mathrm{d}r}{f})$.

- No complex $\nu_{\omega lm} = \sqrt{\omega^2} \in \mathbb{C}\setminus\mathbb{R}$ spectrum (growing $e^{-i\nu t}$ modes!).
- Orthogonality and completeness of generalized eigenmodes $\phi_{\omega lm}(r)$.

Schwarzschild Scalar Wave Equation (model problem)

- **Schwarzschild:** spherically symmetric, static black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -f(\mathrm{d}t)^2 + f^{-1}(\mathrm{d}r)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right), \quad f(r) = 1 - \frac{2M}{r}.$$

- Radial mode equation of scalar wave equation (may omit ωlm):

$$z(t,r,\theta,\varphi) = \frac{\phi_{\omega lm}(r)}{r} Y^{lm}(\theta,\varphi)e^{-i\omega t}, \quad \Box_{\mathbf{g}} z = 0 \quad \Longrightarrow \quad \mathcal{D}_0\phi = 0,$$

where the spin-s Regge-Wheeler operator is ($r \in (2M,\infty)$, $l \geq s$)

$$\mathcal{D}_s\phi := \partial_r f \partial_r \phi - \underbrace{\frac{l(l+1) + (1-s^2)\frac{2M}{r}}{r^2}}_{f^{-1}(\cdots)>0}\phi + \omega^2 \underbrace{\frac{1}{f}}_{(\cdots)>0}\phi.$$

N.B.: $\mathcal{D}_s^* = \mathcal{D}_s$ is formally self-adjoint, of Sturm-Liouville type.

✓ Positive self-adjoint ω^2-spectral problem for on $L^2(2M,\infty; \frac{\mathrm{d}r}{f})$.

- No complex $\nu_{\omega lm} = \sqrt{\omega^2} \in \mathbb{C} \setminus \mathbb{R}$ spectrum (growing $e^{-i\nu t}$ modes!).
- Orthogonality and completeness of generalized eigenmodes $\phi_{\omega lm}(r)$.

Schwarzschild Scalar Wave Equation (model problem)

- **Schwarzschild:** spherically symmetric, static black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -f(\mathrm{d}t)^2 + f^{-1}(\mathrm{d}r)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right), \quad f(r) = 1 - \frac{2M}{r}.$$

- Radial mode equation of scalar wave equation (may omit ωlm):

$$z(t,r,\theta,\varphi) = \frac{\phi_{\omega lm}(r)}{r} Y^{lm}(\theta,\varphi) e^{-i\omega t}, \quad \Box_{\mathbf{g}} z = 0 \quad \Longrightarrow \quad \mathcal{D}_0\phi = 0,$$

where the spin-s Regge-Wheeler operator is ($r \in (2M,\infty)$, $l \geq s$)

$$\mathcal{D}_s\phi := \partial_r f \partial_r \phi - \underbrace{\frac{l(l+1) + (1-s^2)\frac{2M}{r}}{r^2}}_{f^{-1}(\cdots)>0}\phi + \omega^2 \underbrace{\frac{1}{f}}_{(\cdots)>0}\phi.$$

N.B.: $\mathcal{D}_s^* = \mathcal{D}_s$ is formally self-adjoint, of Sturm-Liouville type.

✓ Positive self-adjoint ω^2-spectral problem for on $L^2(2M,\infty; \frac{\mathrm{d}r}{f})$.

- No complex $\nu_{\omega lm} = \sqrt{\omega^2} \in \mathbb{C} \setminus \mathbb{R}$ spectrum (growing $e^{-i\nu t}$ modes!).
- Orthogonality and completeness of generalized eigenmodes $\phi_{\omega lm}(r)$.

Schwarzschild Scalar Wave Equation (model problem)

- **Schwarzschild:** spherically symmetric, static black hole ($R_{\mu\nu} = 0$),

$$\mathbf{g} = -f(\mathrm{d}t)^2 + f^{-1}(\mathrm{d}r)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,(\mathrm{d}\varphi)^2\right), \quad f(r) = 1 - \frac{2M}{r}.$$

- Radial mode equation of scalar wave equation (may omit ωlm):

$$z(t,r,\theta,\varphi) = \frac{\phi_{\omega lm}(r)}{r} Y^{lm}(\theta,\varphi)e^{-i\omega t}, \quad \Box_{\mathbf{g}} z = 0 \quad \Longrightarrow \quad \mathcal{D}_0\phi = 0,$$

where the spin-s Regge-Wheeler operator is ($r \in (2M,\infty)$, $l \geq s$)

$$\mathcal{D}_s\phi := \partial_r f \partial_r \phi - \underbrace{\frac{l(l+1) + (1-s^2)\frac{2M}{r}}{r^2}}_{f^{-1}(\cdots)>0}\phi + \omega^2 \underbrace{\frac{1}{f}}_{(\cdots)>0}\phi.$$

N.B.: $\mathcal{D}_s^* = \mathcal{D}_s$ is formally self-adjoint, of Sturm-Liouville type.

✓ Positive self-adjoint ω^2-spectral problem for on $L^2(2M,\infty;\frac{\mathrm{d}r}{f})$.

- No complex $\nu_{\omega lm} = \sqrt{\omega^2} \in \mathbb{C}\setminus\mathbb{R}$ spectrum (growing $e^{-i\nu t}$ modes!).
- Orthogonality and completeness of generalized eigenmodes $\phi_{\omega lm}(r)$.

Tensor Wave Equations: triangular form?

Lemma (△)

$\tilde{E}_\omega$ has a positive self-adjoint ω^2-spectral problem on $L^2(2M, \infty; \frac{\mathrm{d}r}{f})^{\oplus 7}$,

$$\tilde{E}_\omega = \begin{bmatrix} \mathcal{D}_0 & & * & & * & & * \\ & \mathcal{D}_1 & & & & * & \\ & & \mathcal{D}_0 & & & & * \\ & & & \mathcal{D}_2 & & & \\ & & & & \mathcal{D}_0 & & * \\ & & & & & \mathcal{D}_1 & \\ & & & & & & \mathcal{D}_0 \end{bmatrix},$$

provided $\|\mathcal{D}_s{}^{-1}(*)\| < \infty$ (is relatively bounded). ($\tilde{E}_\omega \rightsquigarrow$ any triang.op.)

Proof: $\tilde{E}_\omega^{-1}$ is polynomial in $\mathcal{D}_s{}^{-1}$ and $\mathcal{D}_s{}^{-1}(*)$. $\square$

Q: Can we use (△) on radial mode equations of

$$(Max)\ (VW) \quad \Box_{\mathbf{g}} v_\mu - \nabla_\mu \nabla^\nu v_\nu = 0,$$

$$(Ein)\ (LW) \quad \Box_{\mathbf{g}} p_{\mu\nu} - 2\, R_\mu{}^{\lambda\kappa}{}_\nu p_{\lambda\kappa} - 2 \nabla_{(\mu} \nabla^\lambda \overline{p}_{\nu)\lambda} = 0,$$

vector wave (VW) and Lichnerowicz wave (LW) equations?

Tensor Wave Equations: triangular form?

Lemma ($\triangle$)

$\tilde{E}_\omega$ has a positive self-adjoint ω^2-spectral problem on $L^2(2M,\infty;\frac{\mathrm{d}r}{f})^{\oplus 7}$,

$$\tilde{E}_\omega = \begin{bmatrix} \mathcal{D}_0 & & * & & * & & * \\ & \mathcal{D}_1 & & & & * & \\ & & \mathcal{D}_0 & & & & * \\ & & & \mathcal{D}_2 & & & \\ & & & & \mathcal{D}_0 & & * \\ & & & & & \mathcal{D}_1 & \\ & & & & & & \mathcal{D}_0 \end{bmatrix},$$

provided $\|\mathcal{D}_s{}^{-1}(*)\| < \infty$ (is relatively bounded). ($\tilde{E}_\omega \rightsquigarrow$ any triang.op.)

Proof: $\tilde{E}_\omega^{-1}$ is polynomial in $\mathcal{D}_s{}^{-1}$ and $\mathcal{D}_s{}^{-1}(*)$. $\square$

Q: Can we use ($\triangle$) on radial mode equations of

$$(Max)\ (VW) \quad \Box_{\mathbf{g}} v_\mu - \nabla_\mu \nabla^\nu v_\nu = 0,$$

$$(Ein)\ (LW) \quad \Box_{\mathbf{g}} p_{\mu\nu} - 2\, R_\mu{}^{\lambda\kappa}{}_\nu p_{\lambda\kappa} - 2\nabla_{(\mu}\nabla^\lambda \overline{p}_{\nu)\lambda} = 0,$$

vector wave (VW) and Lichnerowicz wave (LW) equations?

Tensor Wave Equations: triangular form?

Lemma (△)

$\tilde{E}_\omega$ has a positive self-adjoint ω^2-spectral problem on $L^2(2M, \infty; \frac{\mathrm{d}r}{f})^{\oplus 7}$,

$$\tilde{E}_\omega = \begin{bmatrix} \mathcal{D}_0 & & * & & * & & * \\ & \mathcal{D}_1 & & & & * & \\ & & \mathcal{D}_0 & & & & * \\ & & & \mathcal{D}_2 & & & \\ & & & & \mathcal{D}_0 & & * \\ & & & & & \mathcal{D}_1 & \\ & & & & & & \mathcal{D}_0 \end{bmatrix},$$

provided $\|\mathcal{D}_s{}^{-1}(*)\| < \infty$ (is relatively bounded). ($\tilde{E}_\omega \rightsquigarrow$ any triang.op.)

Proof: $\tilde{E}_\omega^{-1}$ is polynomial in $\mathcal{D}_s{}^{-1}$ and $\mathcal{D}_s{}^{-1}(*)$. $\square$

Q: Can we use (△) on radial mode equations of

$$(Max)\ (VW)\quad \square_{\mathbf{g}} v_\mu - \nabla_\mu \nabla^\nu v_\nu = 0,$$

$$(Ein)\ (LW)\quad \square_{\mathbf{g}} p_{\mu\nu} - 2\, R_\mu{}^{\lambda\kappa}{}_\nu p_{\lambda\kappa} - 2\, \nabla_{(\mu} \nabla^\lambda \bar{p}_{\nu)\lambda} = 0,$$

harmonic gauge Maxwell v_μ ($s = 1$) and Einstein $p_{\mu\nu}$ ($s = 2$) pert.s?

Obstacle:

Radial Mode Equation: $VW_\omega[v] = 0$

Explicitly, $v_\mu \to v(r) = (v_t, v_r, u \mid w)$:

(odd) $$\partial_r \mathcal{B}_l r^2 f \partial_r w + \left(\omega^2 \frac{r^2}{f} - \mathcal{B}_l\right) \mathcal{B}_l w + \mathcal{B}_l \frac{2M}{r} w = 0,$$

(even) $$\begin{bmatrix} -\partial_r \frac{1}{f}\, r^2 f \partial_r\, v_t \\ \partial_r f\, r^2 f \partial_r\, v_r \\ \partial_r \mathcal{B}_l\, r^2 f \partial_r\, u \end{bmatrix} + \left(\omega^2 \frac{r^2}{f} - \mathcal{B}_l\right) \begin{bmatrix} -\frac{1}{f}\, v_t \\ f\, v_r \\ \mathcal{B}_l\, u \end{bmatrix}$$
$$+ i\omega \frac{2M}{f} \begin{bmatrix} v_r \\ -v_t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2f^2 & 2\mathcal{B}_l f \\ 0 & 2\mathcal{B}_l f & \mathcal{B}_l \frac{2M}{r} \end{bmatrix} \begin{bmatrix} v_t \\ v_r \\ u \end{bmatrix} = 0,$$

where $f(r) = 1 - \frac{2M}{r}$ and $\mathcal{B}_l = l(l+1)$.

Radial Mode Equation: $LW_\omega[p] = 0$ (odd sector)

Explicitly, $p_{\mu\nu} \to p(r) = (h_{tt}, h_{tr}, h_{rr}, j_t, j_r, K, G \mid h_t, h_r, h_2)$:

$$
\begin{bmatrix} \partial_r(-2\frac{\mathcal{B}_l}{f}\, r^2 f \partial_r)\, h_t \\ \partial_r(2\mathcal{B}_l f\, r^2 f \partial_r)\, h_r \\ \partial_r(\frac{\mathcal{A}_l}{2}\, r^2 f \partial_r)\, h_2 \end{bmatrix} - \mathcal{B}_l \begin{bmatrix} -2\frac{\mathcal{B}_l}{f}\, h_t \\ 2\mathcal{B}_l f\, h_r \\ \frac{\mathcal{A}_l}{2}\, h_2 \end{bmatrix}
+ \begin{bmatrix} -4\frac{\mathcal{B}_l}{f}\frac{2M}{r} & 0 & 0 \\ 0 & -8\mathcal{B}_l f(1-\frac{3M}{r}) & 2\mathcal{A}_l f \\ 0 & 2\mathcal{A}_l f & \mathcal{A}_l \end{bmatrix} \begin{bmatrix} h_t \\ h_r \\ h_2 \end{bmatrix}
- i\omega\frac{4M}{f} \begin{bmatrix} 0 & -\mathcal{B}_l & 0 \\ \mathcal{B}_l & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_t \\ h_r \\ h_2 \end{bmatrix} + \omega^2 \frac{r^2}{f} \begin{bmatrix} -2\frac{\mathcal{B}_l}{f}\, h_t \\ 2\mathcal{B}_l f\, h_r \\ \frac{\mathcal{A}_l}{2}\, h_2 \end{bmatrix} = 0
$$

where $f(r) = 1 - \frac{2M}{r}$, $\mathcal{A}_l = (l-1)l(l+1)(l+2)$ and $\mathcal{B}_l = l(l+1)$

Radial Mode Equation: $LW_\omega[p] = 0$ (even sector)

$$
\begin{bmatrix} \partial_r(-2\, r^2 f \partial_r)\, h_{tr} \\ \partial_r(-2\frac{\mathcal{B}_l}{f}\, r^2 f \partial_r)\, j_t \\ \partial_r(\frac{1}{f^2}\, r^2 f \partial_r)\, h_{tt} \\ \partial_r(f^2\, r^2 f \partial_r)\, h_{rr} \\ \partial_r(2\, r^2 f \partial_r)\, K \\ \partial_r(2\mathcal{B}_l f\, r^2 f \partial_r)\, j_r \\ \partial_r(\frac{\mathcal{A}_l}{2}\, r^2 f \partial_r)\, G \end{bmatrix} - \mathcal{B}_l \begin{bmatrix} -2\, h_{tr} \\ -2\frac{\mathcal{B}_l}{f}\, j_t \\ \frac{1}{f^2}\, h_{tt} \\ f^2\, h_{rr} \\ 2\, K \\ 2\mathcal{B}_l f\, j_r \\ \frac{\mathcal{A}_l}{2}\, G \end{bmatrix}
$$

$$
\begin{bmatrix} \frac{2(f^2+1)}{f} & -4\mathcal{B}_l & 0 & 0 & 0 & 0 & 0 \\ -4\mathcal{B}_l & -\frac{4\mathcal{B}_l}{f}\frac{2M}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4M^2}{2f^3r^2} & -\frac{(\frac{2M}{r}+4f)}{2f}\frac{2M}{r} & \frac{2}{f}\frac{2M}{r} & 0 & 0 \\ 0 & 0 & -\frac{(\frac{2M}{r}+4f)}{2f}\frac{2M}{e} & \frac{f(\frac{4M^2}{r^2}-8f^2)}{2} & 4f(1-\frac{3M}{r}) & 4\mathcal{B}_l f^2 & 0 \\ 0 & 0 & \frac{2}{f}\frac{2M}{r} & 4f(1-\frac{3M}{r}) & -4(1-\frac{4M}{r}) & -4\mathcal{B}_l f & 0 \\ 0 & 0 & 0 & 4\mathcal{B}_l f^2 & -4\mathcal{B}_l f & -8\mathcal{B}_l f(1-\frac{3M}{r}) & 2\mathcal{A}_l f \\ 0 & 0 & 0 & 0 & 0 & 2\mathcal{A}_l f & \mathcal{A}_l \end{bmatrix} \begin{bmatrix} h_{tr} \\ j_t \\ h_{tt} \\ h_{rr} \\ K \\ j_r \\ G \end{bmatrix}
$$

$$
-i\omega\frac{4M}{f} \begin{bmatrix} 0 & 0 & -\frac{1}{f} & -f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathcal{B}_l & 0 \\ \frac{1}{f} & 0 & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}_l & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_{tr} \\ j_t \\ h_{tt} \\ h_{rr} \\ K \\ j_r \\ G \end{bmatrix} + \omega^2 \frac{r^2}{f} \begin{bmatrix} -2\, h_{tr} \\ -2\frac{\mathcal{B}_l}{f}\, j_t \\ \frac{1}{f^2}\, h_{tt} \\ f^2\, h_{rr} \\ 2\, K \\ 2\mathcal{B}_l f\, j_r \\ \frac{\mathcal{A}_l}{2}\, G \end{bmatrix} = 0
$$

where $f(r) = 1 - \frac{2M}{r}$, $\mathcal{A}_l = (l-1)l(l+1)(l+2)$ and $\mathcal{B}_l = l(l+1)$

Final Result:

Final Reduced Decoupled Forms

- Vector wave equation [arXiv:1711.00585]:
  - $VW^{\text{odd}}_{\omega} \sim \mathcal{D}_1 \quad VW^{\text{even}}_{\omega} \sim$
- Lichnerowicz wave equation [arXiv:2004.09651]:
  - $LW^{\text{odd}}_{\omega} \sim$
  - $LW^{\text{even}}_{\omega} \sim$
- **NEW:** completes previous partial and ad hoc results. [Berndtson (PhD, 2007)]

Final Reduced Decoupled Forms

- Vector wave equation [arXiv:1711.00585]:
  - $VW_\omega^{\text{odd}} \sim \mathcal{D}_1 \quad VW_\omega^{\text{even}} \sim \begin{bmatrix} \mathcal{D}_0 & 0 & -\frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{2r}\right) \\ 0 & \mathcal{D}_1 & 0 \\ 0 & 0 & \mathcal{D}_0 \end{bmatrix}$
- Lichnerowicz wave equation [arXiv:2004.09651]:
  - $LW_\omega^{\text{odd}} \sim$
  - $LW_\omega^{\text{even}} \sim$
- **NEW:** completes previous partial and ad hoc results. [Berndtson (PhD, 2007)]

Final Reduced Decoupled Forms

- Vector wave equation [arXiv:1711.00585]:
  - $VW_\omega^{\text{odd}} \sim \mathcal{D}_1 \quad VW_\omega^{\text{even}} \sim \begin{bmatrix} \mathcal{D}_0 & 0 & -\frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{2r}\right) \\ 0 & \mathcal{D}_1 & 0 \\ 0 & 0 & \mathcal{D}_0 \end{bmatrix}$
- Lichnerowicz wave equation [arXiv:2004.09651]:
  - $LW_\omega^{\text{odd}} \sim \begin{bmatrix} \mathcal{D}_1 & 0 & \frac{2M}{r^3}\frac{\mathcal{B}_l}{3} \\ 0 & \mathcal{D}_2 & 0 \\ 0 & 0 & \mathcal{D}_1 \end{bmatrix}$
  - $LW_\omega^{\text{even}} \sim \begin{bmatrix} \mathcal{D}_0 & 0 & -\frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) & 0 & \frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) & 0 & \frac{M^2}{2r^4}(7\mathcal{B}_l + 2) \\ 0 & \mathcal{D}_1 & 0 & 0 & 0 & -\frac{2M}{r^3}\frac{5\mathcal{B}_l}{3} & 0 \\ 0 & 0 & \mathcal{D}_0 & 0 & 0 & 0 & \frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) \\ 0 & 0 & 0 & \mathcal{D}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{D}_0 & 0 & -\frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) \\ 0 & 0 & 0 & 0 & 0 & \mathcal{D}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{D}_0 \end{bmatrix}$
- **NEW:** completes previous partial and ad hoc results. [Berndtson (PhD, 2007)]

Final Reduced Decoupled Forms

- Vector wave equation [arXiv:1711.00585]:
  - $VW^{\text{odd}}_{\omega} \sim \mathcal{D}_1 \quad VW^{\text{even}}_{\omega} \sim \begin{bmatrix} \mathcal{D}_0 & 0 & -\frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{2r}\right) \\ 0 & \mathcal{D}_1 & 0 \\ 0 & 0 & \mathcal{D}_0 \end{bmatrix}$
- Lichnerowicz wave equation [arXiv:2004.09651]:
  - $LW^{\text{odd}}_{\omega} \sim \begin{bmatrix} \mathcal{D}_1 & 0 & \frac{2M}{r^3}\frac{\mathcal{B}_l}{3} \\ 0 & \mathcal{D}_2 & 0 \\ 0 & 0 & \mathcal{D}_1 \end{bmatrix}$
  - $LW^{\text{even}}_{\omega} \sim \begin{bmatrix} \mathcal{D}_0 & 0 & -\frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) & 0 & \frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) & 0 & \frac{M^2}{2r^4}(7\mathcal{B}_l + 2) \\ 0 & \mathcal{D}_1 & 0 & 0 & 0 & -\frac{2M}{r^3}\frac{5\mathcal{B}_l}{3} & 0 \\ 0 & 0 & \mathcal{D}_0 & 0 & 0 & 0 & \frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) \\ 0 & 0 & 0 & \mathcal{D}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{D}_0 & 0 & -\frac{2M}{r^3}\left(\mathcal{B}_l + \frac{M}{r}\right) \\ 0 & 0 & 0 & 0 & 0 & \mathcal{D}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{D}_0 \end{bmatrix}$
- **NEW:** completes previous partial and ad hoc results. [Berndtson (PhD, 2007)]

Solution Strategy:

Strategy: triangular decoupling and reduction

$$E_\omega = \begin{bmatrix} & & \\ & LW_\omega & \\ & & \end{bmatrix}$$

- mode hierarchy $\rightsquigarrow$ triangular form
- recursive simplification
- $\{*\} \rightsquigarrow$ sparse reduction

- On Schwarzschild, The tensor operators $\square_{\mathbf{g}}$, VW and LW are well-adapted to generalize the Euclidean identities $\Delta\partial_\mu = \partial_\mu\Delta$.
- Hierarchically simplify radial mode equations of $VW[v] = 0$ and $LW[p] = 0$.
- By $\sim$ we mean an equivalence in the category of (rational O)DEs (or D-modules).
- **N.B.:** In each triangular decoupling, the upper-right corner simplification requires a small miracle (Schwarzschild geometry).

Strategy: triangular decoupling and reduction

$$E_\omega \sim \begin{bmatrix} VW_\omega & * & * \\ & \mathcal{D}_2 & * \\ & & VW_\omega \end{bmatrix}$$

- mode hierarchy $\rightsquigarrow$ triangular form
- recursive simplification
- $\{*\} \rightsquigarrow$ sparse reduction

- On Schwarzschild, The tensor operators $\Box_{\mathbf{g}}$, VW and LW are well-adapted to generalize the Euclidean identities $\Delta\partial_\mu = \partial_\mu\Delta$.
- Hierarchically simplify radial mode equations of $VW[v] = 0$ and $LW[p] = 0$.
- By $\sim$ we mean an equivalence in the category of (rational O)DEs (or D-modules).
- **N.B.:** In each triangular decoupling, the upper-right corner simplification requires a small miracle (Schwarzschild geometry).

Strategy: triangular decoupling and reduction

$$E_\omega \sim \begin{bmatrix} \mathcal{D}_0 & * & * & * & * & * & * \\ & \mathcal{D}_1 & * & * & * & * & * \\ & & \mathcal{D}_0 & * & * & * & * \\ & & & \mathcal{D}_2 & * & * & * \\ & & & & \mathcal{D}_0 & * & * \\ & & & & & \mathcal{D}_1 & * \\ & & & & & & \mathcal{D}_0 \end{bmatrix}$$

- mode hierarchy
  $\rightsquigarrow$ triangular form
- recursive simplification
- $\{*\} \rightsquigarrow$ sparse reduction

- On Schwarzschild, The tensor operators $\Box_{\mathbf{g}}$, VW and LW are well-adapted to generalize the Euclidean identities $\Delta\partial_\mu = \partial_\mu\Delta$.
- Hierarchically simplify radial mode equations of $VW[v] = 0$ and $LW[p] = 0$.
- By $\sim$ we mean an equivalence in the category of (rational O)DEs (or D-modules).
- **N.B.:** In each triangular decoupling, the upper-right corner simplification requires a small miracle (Schwarzschild geometry).

Strategy: triangular decoupling and reduction

$$E_\omega \sim \begin{bmatrix} \mathcal{D}_0 & & * & & * & & * \\ & \mathcal{D}_1 & & & & * & \\ & & \mathcal{D}_0 & & & & * \\ & & & \mathcal{D}_2 & & & \\ & & & & \mathcal{D}_0 & & * \\ & & & & & \mathcal{D}_1 & \\ & & & & & & \mathcal{D}_0 \end{bmatrix}$$

- mode hierarchy $\rightsquigarrow$ triangular form
- recursive simplification
- $\{*\} \rightsquigarrow$ sparse reduction

- On Schwarzschild, The tensor operators $\Box_{\mathbf{g}}$, VW and LW are well-adapted to generalize the Euclidean identities $\Delta\partial_\mu = \partial_\mu\Delta$.
- Hierarchically simplify radial mode equations of $VW[v] = 0$ and $LW[p] = 0$.
- By $\sim$ we mean an equivalence in the category of (rational O)DEs (or D-modules).
- **N.B.:** In each triangular decoupling, the upper-right corner simplification requires a small miracle (Schwarzschild geometry).

Strategy: triangular decoupling and reduction

$$E_\omega \sim \begin{bmatrix} \mathcal{D}_0 & & * & & * & & * \\ & \mathcal{D}_1 & & & & * & \\ & & \mathcal{D}_0 & & & & * \\ & & & \mathcal{D}_2 & & & \\ & & & & \mathcal{D}_0 & & * \\ & & & & & \mathcal{D}_1 & \\ & & & & & & \mathcal{D}_0 \end{bmatrix}$$

- mode hierarchy
  $\rightsquigarrow$ triangular form
- recursive simplification
- $\{*\} \rightsquigarrow$ sparse reduction

- On Schwarzschild, The tensor operators $\Box_{\mathbf{g}}$, VW and LW are well-adapted to generalize the Euclidean identities $\Delta\partial_\mu = \partial_\mu\Delta$.
- Hierarchically simplify radial mode equations of $VW[v] = 0$ and $LW[p] = 0$.
- By $\sim$ we mean an equivalence in the category of (rational O)DEs (or D-modules).
- **N.B.:** In each triangular decoupling, the upper-right corner simplification requires a small miracle (Schwarzschild geometry).

Strategy: triangular decoupling and reduction

$$E_\omega \sim \begin{bmatrix} \mathcal{D}_0 & & * & & * & & * \\ & \mathcal{D}_1 & & & & * & \\ & & \mathcal{D}_0 & & & & * \\ & & & \mathcal{D}_2 & & & \\ & & & & \mathcal{D}_0 & & * \\ & & & & & \mathcal{D}_1 & \\ & & & & & & \mathcal{D}_0 \end{bmatrix}$$

- mode hierarchy $\rightsquigarrow$ triangular form
- recursive simplification
- $\{*\} \rightsquigarrow$ sparse reduction

- On Schwarzschild, The tensor operators $\Box_{\mathbf{g}}$, VW and LW are well-adapted to generalize the Euclidean identities $\Delta\partial_\mu = \partial_\mu\Delta$.
- Hierarchically simplify radial mode equations of $VW[v] = 0$ and $LW[p] = 0$.
- By $\sim$ we mean an equivalence in the category of (rational O)DEs (or D-modules).
- **N.B.:** In each triangular decoupling, the upper-right corner simplification requires a small miracle (Schwarzschild geometry).

Details in Reverse Order:

Morphisms Between (Rational O)DEs

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[r, shift left, "k"] \arrow[d, "E"] & \bullet \arrow[l, shift left, "\bar{k}"] \arrow[d, "\tilde{E}"'] \\
\bullet \arrow[u, dashed, bend left, "h"] \arrow[ur, dashed, "h"] \arrow[r, shift left, "k'"] \arrow[d, "E'"] & \bullet \arrow[u, dashed, bend right, "\tilde{h}"'] \arrow[l, shift left, "\bar{k}'"] \arrow[d, "\tilde{E}'"'] \\
\bullet \arrow[u, dashed, bend left, "h'"] \arrow[ur, dashed, "h'"] \arrow[r, shift left, "k''"] \arrow[d] & \bullet \arrow[u, dashed, bend right, "\tilde{h}'"'] \arrow[l, shift left, "\bar{k}''"] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\bullet \arrow[r, shift left, "k^{(n)}"] \arrow[d, "E^{(n)}"] & \bullet \arrow[l, shift left, "\bar{k}^{(n)}"] \arrow[d, "\tilde{E}^{(n)}"'] \\
\bullet \arrow[u, dashed, bend left, "h^{(n)}"] \arrow[ur, dashed, "h^{(n)}"] \arrow[r, shift left, "k^{(n+1)}"] \arrow[d] & \bullet \arrow[u, dashed, bend right, "\tilde{h}^{(n)}"'] \arrow[l, shift left, "\bar{k}^{(n+1)}"] \arrow[d] \\
0 & 0
\end{tikzcd}
$$

- Replace equation $E[u] = 0$ by complex of DOs E, E', $\ldots$, $E^{(n)}$. Extend by 0 when needed.
- Replace solutions by cohomology $H^{(i)}(E) = \frac{\ker E^{(i)}}{\operatorname{im} E^{(i-1)}}$. For a single operator E, $H(E) = \ker E$, $H'(E) = \operatorname{coker} E$.
- A morphism of equations is a cochain map $k, k', \ldots$ of complexes, hence induces map on cohomology $k^{(i)} \colon H^{(i)}(E) \to H^{(i)}(\tilde{E})$. For any function space.
- A homotopy k, k', $\ldots$ induces a trivial morphism $k^{(i)} = h^{(i)} \circ E^{(i)} + E^{(i-1)} \circ h^{(i-1)}$.
- Equivalent morphisms $k_1, k_1', \ldots \sim k_2, k_2', \ldots$ differ by a trivial morphism, hence induce equal maps on cohomology.
- Equivalent equations $E \sim \tilde{E}$ are related by morphisms mutually inverse up to homotopy, $k \circ \bar{k} \sim \mathrm{id}$, $\bar{k} \circ k \sim \mathrm{id}$.

Morphisms Between (Rational O)DEs

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[d, "E"] \arrow[r, shift left, "k"] & \bullet \arrow[d, "\tilde{E}"'] \arrow[l, shift left, "\bar{k}"] \\
\bullet \arrow[d, "E'"] \arrow[r, shift left, "k'"] \arrow[u, dashed, bend left, "h"] \arrow[ur, dashed, "h"] & \bullet \arrow[d, "\tilde{E}'"'] \arrow[l, shift left, "\bar{k}'"] \arrow[u, dashed, bend right, "\tilde{h}"'] \\
\bullet \arrow[d, dotted] \arrow[r, shift left, "k''"] \arrow[u, dashed, bend left, "h'"] \arrow[ur, dashed, "h'"] & \bullet \arrow[d, dotted] \arrow[l, shift left, "\bar{k}''"] \arrow[u, dashed, bend right, "\tilde{h}'"'] \\
\bullet \arrow[d, "E^{(n)}"] \arrow[r, shift left, "k^{(n)}"] & \bullet \arrow[d, "\tilde{E}^{(n)}"'] \arrow[l, shift left, "\bar{k}^{(n)}"] \\
\bullet \arrow[d] \arrow[r, shift left, "k^{(n+1)}"] \arrow[u, dashed, bend left, "h^{(n)}"] \arrow[ur, dashed, "h^{(n)}"] & \bullet \arrow[d] \arrow[l, shift left, "\bar{k}^{(n+1)}"] \arrow[u, dashed, bend right, "\tilde{h}^{(n)}"'] \\
0 & 0
\end{tikzcd}
$$

- Replace equation $E[u] = 0$ by complex of DOs E, E', $\ldots$, $E^{(n)}$. Extend by 0 when needed.
- Replace solutions by cohomology $H^{(i)}(E) = \frac{\ker E^{(i)}}{\operatorname{im} E^{(i-1)}}$. For a single operator E, $H(E) = \ker E$, $H'(E) = \operatorname{coker} E$.
- A morphism of equations is a cochain map $k, k', \ldots$ of complexes, hence induces map on cohomology $k^{(i)} \colon H^{(i)}(E) \to H^{(i)}(\tilde{E})$. For any function space.
- A homotopy k, k', $\ldots$ induces a trivial morphism $k^{(i)} = h^{(i)} \circ E^{(i)} + E^{(i-1)} \circ h^{(i-1)}$.
- Equivalent morphisms $k_1, k_1', \ldots \sim k_2, k_2', \ldots$ differ by a trivial morphism, hence induce equal maps on cohomology.
- Equivalent equations $E \sim \tilde{E}$ are related by morphisms mutually inverse up to homotopy, $k \circ \bar{k} \sim \mathrm{id}$, $\bar{k} \circ k \sim \mathrm{id}$.

Morphisms Between (Rational O)DEs

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[d, "E"] \arrow[r, shift left, "k"] & \bullet \arrow[d, "\tilde{E}"'] \arrow[l, shift left, "\bar{k}"] \\
\bullet \arrow[u, dashed, bend left, "h"] \arrow[ur, dashed, "h"] \arrow[d, "E'"] \arrow[r, shift left, "k'"] & \bullet \arrow[u, dashed, bend right, "\tilde{h}"'] \arrow[d, "\tilde{E}'"'] \arrow[l, shift left, "\bar{k}'"] \\
\bullet \arrow[u, dashed, bend left, "h'"] \arrow[ur, dashed, "h'"] \arrow[d, dotted] \arrow[r, shift left, "k''"] & \bullet \arrow[u, dashed, bend right, "\tilde{h}'"'] \arrow[d, dotted] \arrow[l, shift left, "\bar{k}''"] \\
\bullet \arrow[d, "E^{(n)}"] \arrow[r, shift left, "k^{(n)}"] & \bullet \arrow[d, "\tilde{E}^{(n)}"'] \arrow[l, shift left, "\bar{k}^{(n)}"] \\
\bullet \arrow[u, dashed, bend left, "h^{(n)}"] \arrow[ur, dashed, "h^{(n)}"] \arrow[d] \arrow[r, shift left, "k^{(n+1)}"] & \bullet \arrow[u, dashed, bend right, "\tilde{h}^{(n)}"'] \arrow[d] \arrow[l, shift left, "\bar{k}^{(n+1)}"] \\
0 & 0
\end{tikzcd}
$$

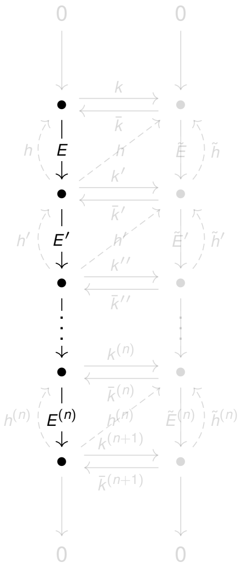
- Replace equation $E[u] = 0$ by complex of DOs E, E', $\ldots$, $E^{(n)}$. Extend by 0 when needed.
- Replace solutions by cohomology $H^{(i)}(E) = \frac{\ker E^{(i)}}{\operatorname{im} E^{(i-1)}}$. For a single operator E, $H(E) = \ker E$, $H'(E) = \operatorname{coker} E$.
- A morphism of equations is a cochain map $k, k', \ldots$ of complexes, hence induces map on cohomology $k^{(i)} \colon H^{(i)}(E) \to H^{(i)}(\tilde{E})$. For any function space.
- A homotopy k, k', $\ldots$ induces a trivial morphism $k^{(i)} = h^{(i)} \circ E^{(i)} + E^{(i-1)} \circ h^{(i-1)}$.
- Equivalent morphisms $k_1, k_1', \ldots \sim k_2, k_2', \ldots$ differ by a trivial morphism, hence induce equal maps on cohomology.
- Equivalent equations $E \sim \tilde{E}$ are related by morphisms mutually inverse up to homotopy, $k \circ \bar{k} \sim \mathrm{id}$, $\bar{k} \circ k \sim \mathrm{id}$.

Morphisms Between (Rational O)DEs

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[d, "E"] \arrow[r, shift left, "k"] & \bullet \arrow[l, shift left, "\bar{k}"] \arrow[d, "\tilde{E}"'] \\
\bullet \arrow[u, dashed, bend left, "h"] \arrow[ur, dashed, "h"] \arrow[d, "E'"] \arrow[r, shift left, "k'"] & \bullet \arrow[l, shift left, "\bar{k}'"] \arrow[u, dashed, bend right, "\tilde{h}"'] \arrow[d, "\tilde{E}'"'] \\
\bullet \arrow[u, dashed, bend left, "h'"] \arrow[ur, dashed, "h'"] \arrow[d] \arrow[r, shift left, "k''"] & \bullet \arrow[l, shift left, "\bar{k}''"] \arrow[u, dashed, bend right, "\tilde{h}'"'] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\bullet \arrow[d, "E^{(n)}"] \arrow[r, shift left, "k^{(n)}"] & \bullet \arrow[l, shift left, "\bar{k}^{(n)}"] \arrow[d, "\tilde{E}^{(n)}"'] \\
\bullet \arrow[u, dashed, bend left, "h^{(n)}"] \arrow[ur, dashed, "h^{(n)}"] \arrow[d] \arrow[r, shift left, "k^{(n+1)}"] & \bullet \arrow[l, shift left, "\bar{k}^{(n+1)}"] \arrow[u, dashed, bend right, "\tilde{h}^{(n)}"'] \arrow[d] \\
0 & 0
\end{tikzcd}$$

- Replace equation $E[u] = 0$ by complex of DOs E, E', $\ldots$, $E^{(n)}$. Extend by 0 when needed.
- Replace solutions by cohomology $H^{(i)}(E) = \frac{\ker E^{(i)}}{\operatorname{im} E^{(i-1)}}$. For a single operator E, $H(E) = \ker E$, $H'(E) = \operatorname{coker} E$.
- A morphism of equations is a cochain map $k, k', \ldots$ of complexes, hence induces map on cohomology $k^{(i)} \colon H^{(i)}(E) \to H^{(i)}(\tilde{E})$. For any function space.
- A homotopy k, k', $\ldots$ induces a trivial morphism $k^{(i)} = h^{(i)} \circ E^{(i)} + E^{(i-1)} \circ h^{(i-1)}$.
- Equivalent morphisms $k_1, k_1', \ldots \sim k_2, k_2', \ldots$ differ by a trivial morphism, hence induce equal maps on cohomology.
- Equivalent equations $E \sim \tilde{E}$ are related by morphisms mutually inverse up to homotopy, $k \circ \bar{k} \sim \mathrm{id}$, $\bar{k} \circ k \sim \mathrm{id}$.

Morphisms Between (Rational O)DEs



- Replace equation $E[u] = 0$ by complex of DOs E, E', $\dots$, $E^{(n)}$. Extend by 0 when needed.
- Replace solutions by cohomology $H^{(i)}(E) = \frac{\ker E^{(i)}}{\operatorname{im} E^{(i-1)}}$. For a single operator E, $H(E) = \ker E$, $H'(E) = \operatorname{coker} E$.
- A morphism of equations is a cochain map $k, k', \dots$ of complexes, hence induces map on cohomology $k^{(i)} \colon H^{(i)}(E) \to H^{(i)}(\tilde{E})$. For any function space.
- A homotopy k, k', $\dots$ induces a trivial morphism $k^{(i)} = h^{(i)} \circ E^{(i)} + E^{(i-1)} \circ h^{(i-1)}$.
- Equivalent morphisms $k_1, k_1', \dots \sim k_2, k_2', \dots$ differ by a trivial morphism, hence induce equal maps on cohomology.
- Equivalent equations $E \sim \tilde{E}$ are related by morphisms mutually inverse up to homotopy, $k \circ \bar{k} \sim \mathrm{id}$, $\bar{k} \circ k \sim \mathrm{id}$.

Morphisms Between (Rational O)DEs

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[d, "E"] \arrow[r, shift left, "k"] \arrow[u, dashed, bend left, "h"'] & \bullet \arrow[l, shift left, "\bar{k}"] \arrow[d, "\tilde{E}"'] \arrow[u, dashed, bend right, "\tilde{h}"] \\
\bullet \arrow[d, "E'"] \arrow[r, shift left, "k'"] \arrow[ur, dashed, "h"] \arrow[u, dashed, bend left, "h'"'] & \bullet \arrow[l, shift left, "\bar{k}'"] \arrow[d, "\tilde{E}'"'] \arrow[u, dashed, bend right, "\tilde{h}'"] \\
\bullet \arrow[d] \arrow[r, shift left, "k''"] \arrow[ur, dashed, "h'"] & \bullet \arrow[l, shift left, "\bar{k}''"] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\bullet \arrow[d, "E^{(n)}"] \arrow[r, shift left, "k^{(n)}"] \arrow[u, dashed, bend left, "h^{(n)}"'] & \bullet \arrow[l, shift left, "\bar{k}^{(n)}"] \arrow[d, "\tilde{E}^{(n)}"'] \arrow[u, dashed, bend right, "\tilde{h}^{(n)}"] \\
\bullet \arrow[d] \arrow[r, shift left, "k^{(n+1)}"] \arrow[ur, dashed, "h^{(n)}"] & \bullet \arrow[l, shift left, "\bar{k}^{(n+1)}"] \arrow[d] \\
0 & 0
\end{tikzcd}
$$

- Replace equation $E[u] = 0$ by complex of DOs E, E', $\ldots$, $E^{(n)}$. Extend by 0 when needed.
- Replace solutions by cohomology $H^{(i)}(E) = \frac{\ker E^{(i)}}{\operatorname{im} E^{(i-1)}}$. For a single operator E, $H(E) = \ker E$, $H'(E) = \operatorname{coker} E$.
- A morphism of equations is a cochain map $k, k', \ldots$ of complexes, hence induces map on cohomology $k^{(i)} \colon H^{(i)}(E) \to H^{(i)}(\tilde{E})$. For any function space.
- A homotopy k, k', $\ldots$ induces a trivial morphism $k^{(i)} = h^{(i)} \circ E^{(i)} + E^{(i-1)} \circ h^{(i-1)}$.
- Equivalent morphisms $k_1, k_1', \ldots \sim k_2, k_2', \ldots$ differ by a trivial morphism, hence induce equal maps on cohomology.
- Equivalent equations $E \sim \tilde{E}$ are related by morphisms mutually inverse up to homotopy, $k \circ \bar{k} \sim \mathrm{id}$, $\bar{k} \circ k \sim \mathrm{id}$.

Morphisms Between (Rational O)DEs

$$\begin{tikzcd}[column sep=large]
0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[r, shift left, "k"] \arrow[d, "E"'] & \bullet \arrow[l, shift left, "\bar{k}"] \arrow[d, "\tilde{E}"] \\
\bullet \arrow[r, shift left, "k'"] \arrow[d, "E'"'] \arrow[u, dashed, bend left, "h"] \arrow[ur, dashed, "h"] & \bullet \arrow[l, shift left, "\bar{k}'"] \arrow[d, "\tilde{E}'"] \arrow[u, dashed, bend right, "\tilde{h}"'] \\
\bullet \arrow[r, shift left, "k''"] \arrow[d, dotted] \arrow[u, dashed, bend left, "h'"] \arrow[ur, dashed, "h'"] & \bullet \arrow[l, shift left, "\bar{k}''"] \arrow[d, dotted] \arrow[u, dashed, bend right, "\tilde{h}'"'] \\
\bullet \arrow[r, shift left, "k^{(n)}"] \arrow[d, "E^{(n)}"'] & \bullet \arrow[l, shift left, "\bar{k}^{(n)}"] \arrow[d, "\tilde{E}^{(n)}"] \\
\bullet \arrow[r, shift left, "k^{(n+1)}"] \arrow[d] \arrow[u, dashed, bend left, "h^{(n)}"] \arrow[ur, dashed, "h^{(n)}"] & \bullet \arrow[l, shift left, "\bar{k}^{(n+1)}"] \arrow[d] \arrow[u, dashed, bend right, "\tilde{h}^{(n)}"'] \\
0 & 0
\end{tikzcd}$$

- Replace equation $E[u] = 0$ by complex of DOs E, E', $\ldots$, $E^{(n)}$. Extend by 0 when needed.
- Replace solutions by cohomology $H^{(i)}(E) = \frac{\ker E^{(i)}}{\operatorname{im} E^{(i-1)}}$. For a single operator E, $H(E) = \ker E$, $H'(E) = \operatorname{coker} E$.
- A morphism of equations is a cochain map $k, k', \ldots$ of complexes, hence induces map on cohomology $k^{(i)} \colon H^{(i)}(E) \to H^{(i)}(\tilde{E})$. For any function space.
- A homotopy k, k', $\ldots$ induces a trivial morphism $k^{(i)} = h^{(i)} \circ E^{(i)} + E^{(i-1)} \circ h^{(i-1)}$.
- Equivalent morphisms $k_1, k_1', \ldots \sim k_2, k_2', \ldots$ differ by a trivial morphism, hence induce equal maps on cohomology.
- Equivalent equations $E \sim \tilde{E}$ are related by morphisms mutually inverse up to homotopy, $k \circ \bar{k} \sim \mathrm{id}$, $\bar{k} \circ k \sim \mathrm{id}$.

Morphisms Between (Rational O)DEs

$$
\begin{tikzcd}[row sep=large, column sep=huge]
0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[r, shift left, "k"] \arrow[d, "E"] & \bullet \arrow[l, shift left, "\bar{k}"] \arrow[d, "\tilde{E}"'] \\
\bullet \arrow[r, shift left, "k'"] \arrow[d, "E'"] \arrow[ur, dashed, "h"] \arrow[u, bend left=60, "h"] & \bullet \arrow[l, shift left, "\bar{k}'"] \arrow[d, "\tilde{E}'"'] \arrow[u, bend right=60, "\bar{h}"'] \\
\bullet \arrow[r, shift left, "k''"] \arrow[d, dotted] \arrow[ur, dashed, "h'"] \arrow[u, bend left=60, "h'"] & \bullet \arrow[l, shift left, "\bar{k}''"] \arrow[d, dotted] \arrow[u, bend right=60, "\bar{h}'"'] \\
\bullet \arrow[r, shift left, "k^{(n)}"] \arrow[d, "E^{(n)}"] & \bullet \arrow[l, shift left, "\bar{k}^{(n)}"] \arrow[d, "\tilde{E}^{(n)}"'] \\
\bullet \arrow[r, shift left, "k^{(n+1)}"] \arrow[d] \arrow[ur, dashed, "h^{(n)}"] \arrow[u, bend left=60, "h^{(n)}"] & \bullet \arrow[l, shift left, "\bar{k}^{(n+1)}"] \arrow[d] \arrow[u, bend right=60, "\bar{h}^{(n)}"'] \\
0 & 0
\end{tikzcd}
$$

- Replace equation $E[u] = 0$ by complex of DOs E, E', $\ldots$, $E^{(n)}$. Extend by 0 when needed.
- Replace solutions by cohomology $H^{(i)}(E) = \frac{\ker E^{(i)}}{\operatorname{im} E^{(i-1)}}$. For a single operator E, $H(E) = \ker E$, $H'(E) = \operatorname{coker} E$.
- A morphism of equations is a cochain map $k, k', \ldots$ of complexes, hence induces map on cohomology $k^{(i)} \colon H^{(i)}(E) \to H^{(i)}(\tilde{E})$. For any function space.
- A homotopy k, k', $\ldots$ induces a trivial morphism $k^{(i)} = h^{(i)} \circ E^{(i)} + E^{(i-1)} \circ h^{(i-1)}$.
- Equivalent morphisms $k_1, k_1', \ldots \sim k_2, k_2', \ldots$ differ by a trivial morphism, hence induce equal maps on cohomology.
- Equivalent equations $E \sim \tilde{E}$ are related by morphisms mutually inverse up to homotopy, $k \circ \bar{k} \sim \mathrm{id}$, $\bar{k} \circ k \sim \mathrm{id}$.

Morphisms Between (Rational O)DEs

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[r, shift left, "k"] \arrow[d, "E"] & \bullet \arrow[l, shift left, "\bar{k}"] \arrow[d, "\tilde{E}"'] \\
\bullet \arrow[u, bend left, "h"] \arrow[ur, dashed, "h"] \arrow[r, shift left, "k'"] \arrow[d, "E'"] & \bullet \arrow[u, bend right, "\bar{h}"'] \arrow[l, shift left, "\bar{k}'"] \arrow[d, "\tilde{E}'"'] \\
\bullet \arrow[u, bend left, "h'"] \arrow[ur, dashed, "h'"] \arrow[r, shift left, "k''"] \arrow[d, dotted] & \bullet \arrow[u, bend right, "\bar{h}'"'] \arrow[l, shift left, "\bar{k}''"] \arrow[d, dotted] \\
\bullet \arrow[r, shift left, "k^{(n)}"] \arrow[d, "E^{(n)}"] & \bullet \arrow[l, shift left, "\bar{k}^{(n)}"] \arrow[d, "\tilde{E}^{(n)}"'] \\
\bullet \arrow[u, bend left, "h^{(n)}"] \arrow[ur, dashed, "h^{(n)}"] \arrow[r, shift left, "k^{(n+1)}"] \arrow[d] & \bullet \arrow[u, bend right, "\bar{h}^{(n)}"'] \arrow[l, shift left, "\bar{k}^{(n+1)}"] \arrow[d] \\
0 & 0
\end{tikzcd}
$$

- Replace equation $E[u] = 0$ by complex of DOs E, E', $\ldots$, $E^{(n)}$. Extend by 0 when needed.
- Replace solutions by cohomology $H^{(i)}(E) = \frac{\ker E^{(i)}}{\operatorname{im} E^{(i-1)}}$. For a single operator E, $H(E) = \ker E$, $H'(E) = \operatorname{coker} E$.
- A morphism of equations is a cochain map $k, k', \ldots$ of complexes, hence induces map on cohomology $k^{(i)} \colon H^{(i)}(E) \to H^{(i)}(\tilde{E})$. For any function space.
- A homotopy k, k', $\ldots$ induces a trivial morphism $k^{(i)} = h^{(i)} \circ E^{(i)} + E^{(i-1)} \circ h^{(i-1)}$.
- Equivalent morphisms $k_1, k_1', \ldots \sim k_2, k_2', \ldots$ differ by a trivial morphism, hence induce equal maps on cohomology.
- Equivalent equations $E \sim \tilde{E}$ are related by morphisms mutually inverse up to homotopy, $k \circ \bar{k} \sim \mathrm{id}$, $\bar{k} \circ k \sim \mathrm{id}$.

Morphisms Between (Rational O)DEs

$$
\begin{tikzcd}[row sep=large, column sep=huge]
0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[d, "E"'] \arrow[r, shift left, "k"] & \bullet \arrow[d, "\tilde{E}"] \arrow[l, shift left, "\bar{k}"] \\
\bullet \arrow[d, "E'"'] \arrow[r, shift left, "k'"] \arrow[u, dashed, bend left, "h"] \arrow[ur, dashed, "h"] & \bullet \arrow[d, "\tilde{E}'"] \arrow[l, shift left, "\bar{k}'"] \arrow[u, dashed, bend right, "\tilde{h}"'] \\
\bullet \arrow[d, dotted] \arrow[r, shift left, "k''"] \arrow[u, dashed, bend left, "h'"] \arrow[ur, dashed, "h'"] & \bullet \arrow[d, dotted] \arrow[l, shift left, "\bar{k}''"] \arrow[u, dashed, bend right, "\tilde{h}'"'] \\
\bullet \arrow[d, "E^{(n)}"'] \arrow[r, shift left, "k^{(n)}"] & \bullet \arrow[d, "\tilde{E}^{(n)}"] \arrow[l, shift left, "\bar{k}^{(n)}"] \\
\bullet \arrow[d] \arrow[r, shift left, "k^{(n+1)}"] \arrow[u, dashed, bend left, "h^{(n)}"] \arrow[ur, dashed, "h^{(n)}"] & \bullet \arrow[d] \arrow[l, shift left, "\bar{k}^{(n+1)}"] \arrow[u, dashed, bend right, "\tilde{h}^{(n)}"'] \\
0 & 0
\end{tikzcd}
$$

- Replace equation $E[u] = 0$ by complex of DOs E, E', $\ldots$, $E^{(n)}$. Extend by 0 when needed.
- Replace solutions by cohomology $H^{(i)}(E) = \frac{\ker E^{(i)}}{\operatorname{im} E^{(i-1)}}$. For a single operator E, $H(E) = \ker E$, $H'(E) = \operatorname{coker} E$.
- A morphism of equations is a cochain map $k, k', \ldots$ of complexes, hence induces map on cohomology $k^{(i)} \colon H^{(i)}(E) \to H^{(i)}(\tilde{E})$. For any function space.
- A homotopy k, k', $\ldots$ induces a trivial morphism $k^{(i)} = h^{(i)} \circ E^{(i)} + E^{(i-1)} \circ h^{(i-1)}$.
- Equivalent morphisms $k_1, k_1', \ldots \sim k_2, k_2', \ldots$ differ by a trivial morphism, hence induce equal maps on cohomology.
- Equivalent equations $E \sim \tilde{E}$ are related by morphisms mutually inverse up to homotopy, $k \circ \bar{k} \sim \mathrm{id}$, $\bar{k} \circ k \sim \mathrm{id}$.

Morphisms Between (Rational O)DEs

$$\begin{tikzcd}[row sep=large, column sep=huge]
0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[r, shift left, "k"] \arrow[d, "E"] & \bullet \arrow[l, shift left, "\bar{k}"] \arrow[d, "\tilde{E}"'] \\
\bullet \arrow[r, shift left, "k'"] \arrow[d, "E'"] \arrow[u, dashed, bend left, "h"] \arrow[ur, dashed, "h"] & \bullet \arrow[l, shift left, "\bar{k}'"] \arrow[d, "\tilde{E}'"'] \arrow[u, dashed, bend right, "\tilde{h}"'] \\
\bullet \arrow[r, shift left, "k''"] \arrow[d, dotted] \arrow[u, dashed, bend left, "h'"] \arrow[ur, dashed, "h'"] & \bullet \arrow[l, shift left, "\bar{k}''"] \arrow[d, dotted] \arrow[u, dashed, bend right, "\tilde{h}'"'] \\
\bullet \arrow[r, shift left, "k^{(n)}"] \arrow[d, "E^{(n)}"] & \bullet \arrow[l, shift left, "\bar{k}^{(n)}"] \arrow[d, "\tilde{E}^{(n)}"'] \\
\bullet \arrow[r, shift left, "k^{(n+1)}"] \arrow[d] \arrow[u, dashed, bend left, "h^{(n)}"] \arrow[ur, dashed, "h^{(n)}"] & \bullet \arrow[l, shift left, "\bar{k}^{(n+1)}"] \arrow[d] \arrow[u, dashed, bend right, "\tilde{h}^{(n)}"'] \\
0 & 0
\end{tikzcd}$$

- Replace equation $E[u] = 0$ by complex of DOs E, E', $\ldots$, $E^{(n)}$. Extend by 0 when needed.
- Replace solutions by cohomology $H^{(i)}(E) = \frac{\ker E^{(i)}}{\operatorname{im} E^{(i-1)}}$. For a single operator E, $H(E) = \ker E$, $H'(E) = \operatorname{coker} E$.
- A morphism of equations is a cochain map $k, k', \ldots$ of complexes, hence induces map on cohomology $k^{(i)} \colon H^{(i)}(E) \to H^{(i)}(\tilde{E})$. For any function space.
- A homotopy k, k', $\ldots$ induces a trivial morphism $k^{(i)} = h^{(i)} \circ E^{(i)} + E^{(i-1)} \circ h^{(i-1)}$.
- Equivalent morphisms $k_1, k_1', \ldots \sim k_2, k_2', \ldots$ differ by a trivial morphism, hence induce equal maps on cohomology.
- Equivalent equations $E \sim \tilde{E}$ are related by morphisms mutually inverse up to homotopy, $k \circ \bar{k} \sim \mathrm{id}$, $\bar{k} \circ k \sim \mathrm{id}$.

Triangular Reduction Strategy

- If $\exists$ diff.ops. δ, ε such that $E_0 \circ \delta = \Delta + \varepsilon \circ E_1$ $(*)$, then

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "{\begin{bmatrix} \mathrm{id} & \delta \\ 0 & \mathrm{id} \end{bmatrix}}", "\sim"'] \arrow[d, "{\begin{bmatrix} E_0 & \Delta \\ 0 & E_1 \end{bmatrix}}"'] & \bullet \arrow[d, "{\begin{bmatrix} E_0 & 0 \\ 0 & E_1 \end{bmatrix}}"] \\
\bullet \arrow[r, "{\begin{bmatrix} \mathrm{id} & \varepsilon \\ 0 & \mathrm{id} \end{bmatrix}}", "\sim"'] & \bullet
\end{tikzcd}$$

- Obvious generalization to larger triangular operator matrices.

- **Q:** How to solve the rational operator equation $(*)$?
If no solution, how to choose nice Δ (relatively bounded)?

Triangular Reduction Strategy

- If $\exists$ diff.ops. δ, ε such that $E_0 \circ \delta = \Delta + \varepsilon \circ E_1$ $(*)$, then

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, "\begin{bmatrix} \mathrm{id} & \delta \\ 0 & \mathrm{id} \end{bmatrix}", "\sim"'] \arrow[d, "\begin{bmatrix} E_0 & \Delta \\ 0 & E_1 \end{bmatrix}"'] & \bullet \arrow[d, "\begin{bmatrix} E_0 & 0 \\ 0 & E_1 \end{bmatrix}"] \\
\bullet \arrow[r, "\begin{bmatrix} \mathrm{id} & \varepsilon \\ 0 & \mathrm{id} \end{bmatrix}", "\sim"'] & \bullet
\end{tikzcd}$$

- Obvious generalization to larger triangular operator matrices.

- **Q:** How to solve the rational operator equation $(*)$?
  If no solution, how to choose nice Δ (relatively bounded)?

Triangular Reduction Strategy

- If $\exists$ diff.ops. δ, ε such that $E_0 \circ \delta = \Delta + \varepsilon \circ E_1$ $(*)$, then

$$
\begin{tikzcd}
\bullet \arrow[r, "\begin{bmatrix} \mathrm{id} & \delta \\ 0 & \mathrm{id} \end{bmatrix}", "\sim"'] \arrow[d, "\begin{bmatrix} E_0 & \Delta \\ 0 & E_1 \end{bmatrix}"'] & \bullet \arrow[d, "\begin{bmatrix} E_0 & 0 \\ 0 & E_1 \end{bmatrix}"] \\
\bullet \arrow[r, "\begin{bmatrix} \mathrm{id} & \varepsilon \\ 0 & \mathrm{id} \end{bmatrix}", "\sim"'] & \bullet
\end{tikzcd}
$$

- Obvious generalization to larger triangular operator matrices.

- **Q:** How to solve the rational operator equation $(*)$?
  If no solution, how to choose nice Δ (relatively bounded)?

Reduction for Regge-Wheeler systems

- When $E_0 = \mathcal{D}_{s_0}$, $E_1 = \mathcal{D}_{s_1}$, enough to take Δ, δ, ε of first order.
- We can parametrize (recalling $\mathcal{D}_s^* = \mathcal{D}_s$)

$$\begin{aligned} \Delta &= \frac{i\omega r}{r^2}(-\Delta_- + \{rf\Delta_+, \partial_r\}), \\ \delta &= \delta_+ - 2\partial_r(rf\delta_-) + f_1\delta_- + \{rf\delta_-, \partial_r\}, \\ \varepsilon &= \delta_+ + 2\partial_r(rf\delta_-) - f_1\delta_- + \{rf\delta_-, \partial_r\}, \end{aligned}$$

  where $\{X, Y\} = X \circ Y + Y \circ X$.
- The operator equation becomes a rational ODE ($\mathcal{R}^*_{s_0,s_1} = \mathcal{R}_{s_0,s_1}$),

$$\mathcal{D}_{s_0} \circ \delta = \Delta + \varepsilon \circ \mathcal{D}_{s_1} \quad \iff \quad \mathcal{R}_{s_0,s_1} \begin{bmatrix} \delta_+ \\ \delta_- \end{bmatrix} = \begin{bmatrix} \Delta_+ \\ \Delta_- \end{bmatrix}.$$

- **Q:** What are $\ker \mathcal{R}_{s_0,s_1}$ and $\operatorname{coker} \mathcal{R}_{s_0,s_1}$ in rational functions?

Reduction for Regge-Wheeler systems

- When $E_0 = \mathcal{D}_{s_0}$, $E_1 = \mathcal{D}_{s_1}$, enough to take Δ, δ, ε of first order.
- We can parametrize (recalling $\mathcal{D}_s^* = \mathcal{D}_s$)

$$\begin{aligned} \Delta &= \frac{i\omega r}{r^2}(-\Delta_- + \{rf\Delta_+, \partial_r\}), \\ \delta &= \delta_+ - 2\partial_r(rf\delta_-) + f_1\delta_- + \{rf\delta_-, \partial_r\}, \\ \varepsilon &= \delta_+ + 2\partial_r(rf\delta_-) - f_1\delta_- + \{rf\delta_-, \partial_r\}, \end{aligned}$$

  where $\{X, Y\} = X \circ Y + Y \circ X$.
- The operator equation becomes a rational ODE ($\mathcal{R}^*_{s_0,s_1} = \mathcal{R}_{s_0,s_1}$),

$$\mathcal{D}_{s_0} \circ \delta = \Delta + \varepsilon \circ \mathcal{D}_{s_1} \quad \iff \quad \mathcal{R}_{s_0,s_1} \begin{bmatrix} \delta_+ \\ \delta_- \end{bmatrix} = \begin{bmatrix} \Delta_+ \\ \Delta_- \end{bmatrix}.$$

- **Q:** What are $\ker\mathcal{R}_{s_0,s_1}$ and $\mathrm{coker}\mathcal{R}_{s_0,s_1}$ in rational functions?

Rational solutions of rational ODEs

- Observations:
  - Any rational $u(r)$ has a finite partial fraction decomposition.
  - The poles of $v(r)$ and the singular points of $E[u] = v$ determine the poles of $u(r)$.
  - The integer characteristic exponents (Frobenius method) determine bounds on the degrees of each pole or $u(r) = P(r, r^{-1})/d(r)$, $d(r)$—poly., $P(r)$—Laurent poly.

Lemma (IK 2018–20, Abramov *et al.* 1989–)

The dimension of $\ker_{\mathcal{U}} E < \infty$, *when* $\mathcal{U} = \mathbb{C}[r, r^{-1}]$, $\mathbb{C}[[r]][r^{-1}]$, $\mathbb{C}[r][[r^{-1}]]$, *or* $\mathbb{C}[[r, r^{-1}]]$, *for compatible* E.

- The residue pairing $\langle \alpha, v \rangle = \mathrm{Res}_{r=0}(\alpha^* v)$ satisfies $\langle \alpha, E[u] \rangle = \langle E^*[\alpha], u \rangle$ and is non-degenerate on compatible pairs

$$\left(\mathbb{C}[[r]][r^{-1}], \mathbb{C}[[r]][r^{-1}]\right), \quad \left(\mathbb{C}[r][[r^{-1}]], \mathbb{C}[r][[r^{-1}]]\right), \quad \left(\mathbb{C}[[r, r^{-1}]], \mathbb{C}[r, r^{-1}]\right).$$

Lemma (IK 2020)

The pairing $\langle \ker_{\mathcal{U}'} E^*, \operatorname{coker}_{\mathcal{U}} E \rangle$ *is non-degenerate on compatible pairs* $(\mathcal{U}', \mathcal{U})$.

Theorem (IK 2020)

$\dim \operatorname{coker}_{\mathcal{U}} \mathcal{R}_{s_0,s_1} = \dim \ker_{\mathcal{U}'} \mathcal{R}^*_{s_0,s_1} < \infty$ *(and it has relatively bounded representatives).*

Rational solutions of rational ODEs

- Observations:
  - Any rational $u(r)$ has a finite partial fraction decomposition.
  - The poles of $v(r)$ and the singular points of $E[u] = v$ determine the poles of $u(r)$.
  - The integer characteristic exponents (Frobenius method) determine bounds on the degrees of each pole or $u(r) = P(r, r^{-1})/d(r)$, $d(r)$—poly., $P(r)$—Laurent poly.

Lemma (IK 2018–20, Abramov *et al.* 1989–)

The dimension of $\ker_{\mathcal{U}} E < \infty$, *when* $\mathcal{U} = \mathbb{C}[r, r^{-1}]$, $\mathbb{C}[[r]][r^{-1}]$, $\mathbb{C}[r][[r^{-1}]]$, *or* $\mathbb{C}[[r, r^{-1}]]$, *for compatible* E.

- The residue pairing $\langle \alpha, v \rangle = \mathrm{Res}_{r=0}(\alpha^* v)$ satisfies $\langle \alpha, E[u] \rangle = \langle E^*[\alpha], u \rangle$ and is non-degenerate on compatible pairs

$$\left(\mathbb{C}[[r]][r^{-1}], \mathbb{C}[[r]][r^{-1}]\right), \quad \left(\mathbb{C}[r][[r^{-1}]], \mathbb{C}[r][[r^{-1}]]\right), \quad \left(\mathbb{C}[[r, r^{-1}]], \mathbb{C}[r, r^{-1}]\right).$$

Lemma (IK 2020)

The pairing $\langle \ker_{\mathcal{U}'} E^*, \mathrm{coker}_{\mathcal{U}} E \rangle$ *is non-degenerate on compatible pairs* $(\mathcal{U}', \mathcal{U})$.

Theorem (IK 2020)

$\dim \mathrm{coker}_{\mathcal{U}} \mathcal{R}_{s_0, s_1} = \dim \ker_{\mathcal{U}'} \mathcal{R}^*_{s_0, s_1} < \infty$ *(and it has relatively bounded representatives).*

Rational solutions of rational ODEs

- Observations:
  - Any rational $u(r)$ has a finite partial fraction decomposition.
  - The poles of $v(r)$ and the singular points of $E[u] = v$ determine the poles of $u(r)$.
  - The integer characteristic exponents (Frobenius method) determine bounds on the degrees of each pole or $u(r) = P(r, r^{-1})/d(r)$, $d(r)$—poly., $P(r)$—Laurent poly.

Lemma (IK 2018–20, Abramov *et al.* 1989–)

The dimension of $\ker_{\mathcal{U}} E < \infty$*, when* $\mathcal{U} = \mathbb{C}[r, r^{-1}]$, $\mathbb{C}[[r]][r^{-1}]$, $\mathbb{C}[r][[r^{-1}]]$*, or* $\mathbb{C}[[r, r^{-1}]]$*, for compatible* E*.*

- The residue pairing $\langle \alpha, v\rangle = \mathrm{Res}_{r=0}(\alpha^* v)$ satisfies $\langle \alpha, E[u]\rangle = \langle E^*[\alpha], u\rangle$ and is non-degenerate on compatible pairs

$$\left(\mathbb{C}[[r]][r^{-1}], \mathbb{C}[[r]][r^{-1}]\right), \quad \left(\mathbb{C}[r][[r^{-1}]], \mathbb{C}[r][[r^{-1}]]\right), \quad \left(\mathbb{C}[[r, r^{-1}]], \mathbb{C}[r, r^{-1}]\right).$$

Lemma (IK 2020)

The pairing $\langle \ker_{\mathcal{U}'} E^*, \mathrm{coker}_{\mathcal{U}} E\rangle$ *is non-degenerate on compatible pairs* $(\mathcal{U}', \mathcal{U})$*.*

Theorem (IK 2020)

$\dim \mathrm{coker}_{\mathcal{U}} \mathcal{R}_{s_0,s_1} = \dim \ker_{\mathcal{U}'} \mathcal{R}^*_{s_0,s_1} < \infty$ *(and it has relatively bounded representatives).*

Rational solutions of rational ODEs

- Observations:
  - Any rational $u(r)$ has a finite partial fraction decomposition.
  - The poles of $v(r)$ and the singular points of $E[u] = v$ determine the poles of $u(r)$.
  - The integer characteristic exponents (Frobenius method) determine bounds on the degrees of each pole or $u(r) = P(r, r^{-1})/d(r)$, $d(r)$—poly., $P(r)$—Laurent poly.

Lemma (IK 2018–20, Abramov *et al.* 1989–)

The dimension of $\ker_{\mathcal{U}} E < \infty$*, when* $\mathcal{U} = \mathbb{C}[r, r^{-1}]$, $\mathbb{C}[[r]][r^{-1}]$, $\mathbb{C}[r][[r^{-1}]]$*, or* $\mathbb{C}[[r, r^{-1}]]$*, for compatible* E*.*

- The residue pairing $\langle \alpha, v \rangle = \mathrm{Res}_{r=0}(\alpha^* v)$ satisfies $\langle \alpha, E[u] \rangle = \langle E^*[\alpha], u \rangle$ and is non-degenerate on compatible pairs

$$\left(\mathbb{C}[[r]][r^{-1}], \mathbb{C}[[r]][r^{-1}]\right), \quad \left(\mathbb{C}[r][[r^{-1}]], \mathbb{C}[r][[r^{-1}]]\right), \quad \left(\mathbb{C}[[r, r^{-1}]], \mathbb{C}[r, r^{-1}]\right).$$

Lemma (IK 2020)

The pairing $\langle \ker_{\mathcal{U}'} E^*, \mathrm{coker}_{\mathcal{U}} E \rangle$ *is non-degenerate on compatible pairs* $(\mathcal{U}', \mathcal{U})$*.*

Theorem (IK 2020)

$\dim \mathrm{coker}_{\mathcal{U}} \mathcal{R}_{s_0,s_1} = \dim \ker_{\mathcal{U}'} \mathcal{R}^*_{s_0,s_1} < \infty$ *(and it has relatively bounded representatives).*

Rational solutions of rational ODEs

- Observations:
  - Any rational $u(r)$ has a finite partial fraction decomposition.
  - The poles of $v(r)$ and the singular points of $E[u] = v$ determine the poles of $u(r)$.
  - The integer characteristic exponents (Frobenius method) determine bounds on the degrees of each pole or $u(r) = P(r, r^{-1})/d(r)$, $d(r)$—poly., $P(r)$—Laurent poly.

Lemma (IK 2018–20, Abramov *et al.* 1989–)

The dimension of $\ker_{\mathcal{U}} E < \infty$*, when* $\mathcal{U} = \mathbb{C}[r, r^{-1}]$, $\mathbb{C}[[r]][r^{-1}]$, $\mathbb{C}[r][[r^{-1}]]$*, or* $\mathbb{C}[[r, r^{-1}]]$*, for compatible* E*.*

- The residue pairing $\langle \alpha, v\rangle = \mathrm{Res}_{r=0}(\alpha^* v)$ satisfies $\langle \alpha, E[u]\rangle = \langle E^*[\alpha], u\rangle$ and is non-degenerate on compatible pairs

$$\left(\mathbb{C}[[r]][r^{-1}], \mathbb{C}[[r]][r^{-1}]\right), \quad \left(\mathbb{C}[r][[r^{-1}]], \mathbb{C}[r][[r^{-1}]]\right), \quad \left(\mathbb{C}[[r, r^{-1}]], \mathbb{C}[r, r^{-1}]\right).$$

Lemma (IK 2020)

The pairing $\langle \ker_{\mathcal{U}'} E^*, \mathrm{coker}_{\mathcal{U}} E\rangle$ *is non-degenerate on compatible pairs* $(\mathcal{U}', \mathcal{U})$*.*

Theorem (IK 2020)

$\dim \mathrm{coker}_{\mathcal{U}} \mathcal{R}_{s_0,s_1} = \dim \ker_{\mathcal{U}'} \mathcal{R}^*_{s_0,s_1} < \infty$ *(and it has relatively bounded representatives).*

Triangular Decoupling Strategy

- Start with complicated equation $E[u] = 0$. We want to find an equivalent equation in block upper triangular form.
- **Input:** equations $E[u] = 0$, $E_D[v] = 0$, $E_T[w] = 0$ with morphisms

$$
\begin{tikzcd}
\bullet \arrow[r, "T"] \arrow[d, "E"'] & \bullet \arrow[d, "E_T"] \\
\bullet \arrow[r, "T'"] & \bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\bullet \arrow[r, "D", "\sim"'] \arrow[d, "E_D"'] \arrow[dr, phantom, "\begin{bmatrix} D' \\ H_T \end{bmatrix}"] & \bullet \arrow[d, "\begin{bmatrix} E \\ T \end{bmatrix}"] \\
\bullet \arrow[r, "\sim"'] \arrow[d, "0"'] \arrow[dr, phantom, "\begin{bmatrix} -T' & E_T \end{bmatrix}"] & \bullet \arrow[d] \\
\bullet \arrow[r, "0", "\sim"'] & \bullet
\end{tikzcd}
.
$$

Output: decoupling equivalence

$$E[u] = 0 \quad \sim \quad \begin{bmatrix} E_D & \Delta_{DT} \\ 0 & E_T \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- Both $E_D[v] = 0$ and $E_T[w] = 0$ could undergo further decoupling, keeping the overall upper triangular form.
- **Q:** How to identify the input and the decoupling morphisms?

Triangular Decoupling Strategy

- Start with complicated equation $E[u] = 0$. We want to find an equivalent equation in block upper triangular form.
- **Input:** equations $E[u] = 0$, $E_D[v] = 0$, $E_T[w] = 0$ with morphisms

$$\begin{tikzcd} \bullet \arrow[r, "T"] \arrow[d, "E"'] & \bullet \arrow[d, "E_T"] \\ \bullet \arrow[r, "T'"] & \bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \bullet \arrow[r, "D", "\sim"'] \arrow[d, "E_D"'] & \bullet \arrow[d, "\begin{bmatrix} E \\ T \end{bmatrix}"] \\ \bullet \arrow[r, "\begin{bmatrix} D' \\ H_T \end{bmatrix}", "\sim"'] \arrow[d, "0"'] & \bullet \arrow[d, "\begin{bmatrix} -T' & E_T \end{bmatrix}"] \\ \bullet \arrow[r, "0", "\sim"'] & \bullet \end{tikzcd} .$$

Output: decoupling equivalence

$$E[u] = 0 \quad \sim \quad \begin{bmatrix} E_D & \Delta_{DT} \\ 0 & E_T \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- Both $E_D[v] = 0$ and $E_T[w] = 0$ could undergo further decoupling, keeping the overall upper triangular form.
- **Q:** How to identify the input and the decoupling morphisms?

Triangular Decoupling Strategy

- Start with complicated equation $E[u]=0$. We want to find an equivalent equation in block upper triangular form.
- **Input:** equations $E[u]=0$, $E_D[v]=0$, $E_T[w]=0$ with morphisms

$$
\begin{tikzcd}
\bullet \arrow[r, "T"] \arrow[d, "E"'] & \bullet \arrow[d, "E_T"] \\
\bullet \arrow[r, "T'"] & \bullet
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\bullet \arrow[r, "D", "\sim"'] \arrow[d, "E_D"'] & \bullet \arrow[d, "\begin{bmatrix} E \\ T \end{bmatrix}"] \\
\bullet \arrow[r, "\begin{bmatrix} D' \\ H_T \end{bmatrix}", "\sim"'] \arrow[d, "0"'] & \bullet \arrow[d, "\begin{bmatrix} -T' & E_T \end{bmatrix}"] \\
\bullet \arrow[r, "0", "\sim"'] & \bullet
\end{tikzcd}
.
$$

Output: decoupling equivalence

$$
E[u]=0 \quad \sim \quad \begin{bmatrix} E_D & \Delta_{DT} \\ 0 & E_T \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

- Both $E_D[v]=0$ and $E_T[w]=0$ could undergo further decoupling, keeping the overall upper triangular form.
- **Q:** How to identify the input and the decoupling morphisms?

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\bullet \arrow[r, shift left, "\begin{bmatrix} \mathrm{id} \\ T \end{bmatrix}"] \arrow[d, "E"'] & \bullet \arrow[l, shift left, "\begin{bmatrix} \mathrm{id} & 0 \end{bmatrix}"] \arrow[r, shift left, "\begin{bmatrix} \bar{D} & 0 \\ 0 & \mathrm{id} \end{bmatrix}"] \arrow[d, "\begin{bmatrix} E & 0 \\ T & -\mathrm{id} \\ 0 & E_T \end{bmatrix}" description] & \bullet \arrow[l, shift left, "\begin{bmatrix} D & \bar{T} \\ 0 & \mathrm{id} \end{bmatrix}"] \arrow[d, "\begin{bmatrix} E_D & \Delta_{DT} \\ 0 & E_T \end{bmatrix}"] \\
\bullet \arrow[u, dashed, bend right, "0"'] \arrow[r, shift right, "\begin{bmatrix} \mathrm{id} \\ 0 \\ T' \end{bmatrix}"'] \arrow[d, "0"'] & \bullet \arrow[l, shift right, "\begin{bmatrix} \mathrm{id} & 0 & 0 \end{bmatrix}"'] \arrow[u, dashed, bend left, "\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\mathrm{id} & 0 \end{bmatrix}"] \arrow[u, dashed, bend right, "\begin{bmatrix} h_E & \bar{T} & 0 \\ 0 & 0 & 0 \end{bmatrix}"'] \arrow[r, shift right, "\begin{bmatrix} \bar{D}' & -\Delta_{DT} & 0 \\ 0 & 0 & \mathrm{id} \end{bmatrix}"'] \arrow[d, "\begin{bmatrix} -T' & E_T & \mathrm{id} \end{bmatrix}" description] & \bullet \arrow[l, shift right, "\begin{bmatrix} D' & \bar{T}' \\ H_T & -h_{TT} \\ 0 & \mathrm{id} \end{bmatrix}"'] \arrow[u, dashed, bend left, "\begin{bmatrix} h_D & -\bar{H}_T \\ 0 & 0 \end{bmatrix}"] \arrow[d, "0"] \\
\bullet \arrow[r, shift right, "0"'] & \bullet \arrow[l, shift right, "0"'] \arrow[u, dashed, bend left, "\begin{bmatrix} 0 \\ 0 \\ \mathrm{id} \end{bmatrix}"] \arrow[u, dashed, bend right, "\begin{bmatrix} -\bar{T}' \\ h_{TT} \\ 0 \end{bmatrix}"'] \arrow[r, shift right, "0"'] & \bullet \arrow[l, shift right, "0"']
\end{tikzcd}
$$

Triangular Decoupling Strategy

- Start with complicated equation $E[u] = 0$. We want to find an equivalent equation in block upper triangular form.
- **Input:** equations $E[u] = 0$, $E_D[v] = 0$, $E_T[w] = 0$ with morphisms

$$\begin{tikzcd} \bullet \arrow[r, "T"] \arrow[d, "E"] & \bullet \arrow[d, "E_T"] \\ \bullet \arrow[r, "T'"] & \bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \bullet \arrow[r, "D", "\sim"'] \arrow[d, "E_D"] \arrow[dr, phantom, "\begin{bmatrix} D' \\ H_T \end{bmatrix}"] & \bullet \arrow[d, "\begin{bmatrix} E \\ T \end{bmatrix}"] \\ \bullet \arrow[r, "\sim"'] \arrow[d, "0"] & \bullet \arrow[d, "\begin{bmatrix} -T' & E_T \end{bmatrix}"] \\ \bullet \arrow[r, "0", "\sim"'] & \bullet \end{tikzcd} .$$

Output: decoupling equivalence

$$E[u] = 0 \quad \sim \quad \begin{bmatrix} E_D & \Delta_{DT} \\ 0 & E_T \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

- Both $E_D[v] = 0$ and $E_T[w] = 0$ could undergo further decoupling, keeping the overall upper triangular form.
- **Q:** How to identify the input and the decoupling morphisms?

Triangular Decoupling Strategy

- Start with complicated equation $E[u] = 0$. We want to find an equivalent equation in block upper triangular form.
- **Input:** equations $E[u] = 0$, $E_D[v] = 0$, $E_T[w] = 0$ with morphisms

$$
\begin{tikzcd}
\bullet \arrow[r, "T"] \arrow[d, "E"'] & \bullet \arrow[d, "E_T"] \\
\bullet \arrow[r, "T'"] & \bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\bullet \arrow[r, "D", "\sim"'] \arrow[d, "E_D"'] & \bullet \arrow[d, "\begin{bmatrix} E \\ T \end{bmatrix}"'] \\
\bullet \arrow[r, "\begin{bmatrix} D' \\ H_T \end{bmatrix}", "\sim"'] \arrow[d, "0"'] & \bullet \arrow[d, "\begin{bmatrix} -T' & E_T \end{bmatrix}"'] \\
\bullet \arrow[r, "0", "\sim"'] & \bullet
\end{tikzcd}
.
$$

Output: decoupling equivalence

$$
E[u] = 0 \quad \sim \quad \begin{bmatrix} E_D & \Delta_{DT} \\ 0 & E_T \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

- Both $E_D[v] = 0$ and $E_T[w] = 0$ could undergo further decoupling, keeping the overall upper triangular form.
- **Q:** How to identify the input and the decoupling morphisms?

Triangular Decoupling Strategy

- Start with complicated equation $E[u] = 0$. We want to find an equivalent equation in block upper triangular form.
- **Input:** equations $E[u] = 0$, $E_D[v] = 0$, $E_T[w] = 0$ with morphisms

$$\begin{tikzcd} \bullet \arrow[r, "T"] \arrow[d, "E"'] & \bullet \arrow[d, "E_T"] \\ \bullet \arrow[r, "T'"] & \bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \bullet \arrow[r, "D", "\sim"' red] \arrow[d, "E_D"'] & \bullet \arrow[d, "\begin{bmatrix} E \\ T \end{bmatrix}"] \\ \bullet \arrow[r, "\begin{bmatrix} D' \\ H_T \end{bmatrix}", "\sim"' red] \arrow[d, "0"'] & \bullet \arrow[d, "\begin{bmatrix} -T' & E_T \end{bmatrix}"] \\ \bullet \arrow[r, "0", "\sim"' red] & \bullet \end{tikzcd} .$$

Output: decoupling equivalence

$$E[u] = 0 \quad \sim \quad \begin{bmatrix} E_D & \Delta_{DT} \\ 0 & E_T \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

- Both $E_D[v] = 0$ and $E_T[w] = 0$ could undergo further decoupling, keeping the overall upper triangular form.
- **Q:** How to identify the input and the decoupling morphisms?

Wave Equation Identities on Schwarzschild

- **Input:** hierarchy of pure gauge, gauge invariant and constraint violating modes. The physics literature on Schwarzschild perturbations provides natural candidates, generalizing Euclidean $\mathrm{div}\,\Delta = \Delta\,\mathrm{div}$ and $\mathrm{grad}\,\Delta = \Delta\,\mathrm{grad}$.

$$LW_\omega \sim \begin{bmatrix} VW_\omega & * & * \\ 0 & \mathcal{D}_2 & * \\ 0 & 0 & VW_\omega \end{bmatrix} \begin{array}{l} \}\,\text{pure gauge} \\ \}\,\text{gauge invariant} \\ \}\,\text{constraint violating} \end{array} \qquad \text{(Einstein)}$$

- We can apply triangular decoupling again.
- Recursively apply to $LW_\omega \rightsquigarrow VW_\omega \rightsquigarrow \mathcal{D}_0$:

$$VW_\omega \sim \begin{bmatrix} \mathcal{D}_0 & * & * \\ 0 & \mathcal{D}_1 & * \\ 0 & 0 & \mathcal{D}_0 \end{bmatrix} \begin{array}{l} \}\,\text{pure gauge} \\ \}\,\text{gauge invariant} \\ \}\,\text{constraint violating} \end{array} \qquad \text{(Maxwell)}$$

- **N.B.:** minor miracle on Schwarzschild geometry [arXiv:2004.09651],

$$\text{constraint violating} = 0,\ \text{gauge invariant} = 0 \quad \sim \quad \text{pure gauge}.$$

Wave Equation Identities on Schwarzschild

- **Input:** hierarchy of pure gauge, gauge invariant and constraint violating modes. The physics literature on Schwarzschild perturbations provides natural candidates, generalizing Euclidean $\mathrm{div}\,\Delta = \Delta\,\mathrm{div}$ and $\mathrm{grad}\,\Delta = \Delta\,\mathrm{grad}$.

$$LW_\omega \sim \begin{bmatrix} VW_\omega & * & * \\ 0 & \mathcal{D}_2 & * \\ 0 & 0 & VW_\omega \end{bmatrix} \begin{array}{l} \}\text{ pure gauge} \\ \}\text{ gauge invariant} \\ \}\text{ constraint violating} \end{array} \qquad \text{(Einstein)}$$

- We can apply triangular decoupling again.
- Recursively apply to $LW_\omega \rightsquigarrow VW_\omega \rightsquigarrow \mathcal{D}_0$:

$$VW_\omega \sim \begin{bmatrix} \mathcal{D}_0 & * & * \\ 0 & \mathcal{D}_1 & * \\ 0 & 0 & \mathcal{D}_0 \end{bmatrix} \begin{array}{l} \}\text{ pure gauge} \\ \}\text{ gauge invariant} \\ \}\text{ constraint violating} \end{array} \qquad \text{(Maxwell)}$$

- **N.B.:** minor miracle on Schwarzschild geometry [arXiv:2004.09651],

$$\text{constraint violating} = 0,\ \text{gauge invariant} = 0 \quad \sim \quad \text{pure gauge.}$$

Wave Equation Identities on Schwarzschild

- **Input:** hierarchy of pure gauge, gauge invariant and constraint violating modes. The physics literature on Schwarzschild perturbations provides natural candidates, generalizing Euclidean $\operatorname{div}\Delta = \Delta\operatorname{div}$ and $\operatorname{grad}\Delta = \Delta\operatorname{grad}$.

$$LW_\omega \sim \begin{bmatrix} VW_\omega & * & * \\ 0 & \mathcal{D}_2 & * \\ 0 & 0 & VW_\omega \end{bmatrix} \begin{array}{l} \}\,\text{pure gauge} \\ \}\,\text{gauge invariant} \\ \}\,\text{constraint violating} \end{array} \qquad (\text{Einstein})$$

- We can apply triangular decoupling again.
- Recursively apply to $LW_\omega \rightsquigarrow VW_\omega \rightsquigarrow \mathcal{D}_0$:

$$VW_\omega \sim \begin{bmatrix} \mathcal{D}_0 & * & * \\ 0 & \mathcal{D}_1 & * \\ 0 & 0 & \mathcal{D}_0 \end{bmatrix} \begin{array}{l} \}\,\text{pure gauge} \\ \}\,\text{gauge invariant} \\ \}\,\text{constraint violating} \end{array} \qquad (\text{Maxwell})$$

- **N.B.:** minor miracle on Schwarzschild geometry [arXiv:2004.09651],

$$\text{constraint violating} = 0, \ \text{gauge invariant} = 0 \quad \sim \quad \text{pure gauge}.$$

Wave Equation Identities on Schwarzschild

- **Input:** hierarchy of pure gauge, gauge invariant and constraint violating modes. The physics literature on Schwarzschild perturbations provides natural candidates, generalizing Euclidean $\operatorname{div} \Delta = \Delta \operatorname{div}$ and $\operatorname{grad} \Delta = \Delta \operatorname{grad}$.

$$LW_\omega \sim \begin{bmatrix} VW_\omega & * & * \\ 0 & \mathcal{D}_2 & * \\ 0 & 0 & VW_\omega \end{bmatrix} \begin{matrix} \}\,\text{pure gauge} \\ \}\,\text{gauge invariant} \\ \}\,\text{constraint violating} \end{matrix} \qquad (\text{Einstein})$$

- We can apply triangular decoupling again.
- Recursively apply to $LW_\omega \rightsquigarrow VW_\omega \rightsquigarrow \mathcal{D}_0$:

$$VW_\omega \sim \begin{bmatrix} \mathcal{D}_0 & * & * \\ 0 & \mathcal{D}_1 & * \\ 0 & 0 & \mathcal{D}_0 \end{bmatrix} \begin{matrix} \}\,\text{pure gauge} \\ \}\,\text{gauge invariant} \\ \}\,\text{constraint violating} \end{matrix} \qquad (\text{Maxwell})$$

- **N.B.:** minor miracle on Schwarzschild geometry [arXiv:2004.09651],

$$\text{constraint violating} = 0, \ \text{gauge invariant} = 0 \quad \sim \quad \text{pure gauge}.$$

Discussion

- Abstract decoupling and reduction strategies (using ideas from homological algebra, *D*-modules, category of PDEs) allow us to reduce complicated coupled systems of vector and tensor radial mode equations to sparse upper triangular ODEs.
- Previous partial results were based on trial and error, very laborious. [Berndtson (PhD, 2007)]
- **TODO:** Construct Green function for the Lichnerowicz and vector wave equations on Schwarzschild.
- **TODO:** Generalization to rotating Kerr blackhole?

Discussion

- Abstract decoupling and reduction strategies (using ideas from homological algebra, *D*-modules, category of PDEs) allow us to reduce complicated coupled systems of vector and tensor radial mode equations to sparse upper triangular ODEs.
- Previous partial results were based on trial and error, very laborious. [Berndtson (PhD, 2007)]
- **TODO:** Construct Green function for the Lichnerowicz and vector wave equations on Schwarzschild.
- **TODO:** Generalization to rotating Kerr blackhole?

Discussion

- Abstract decoupling and reduction strategies (using ideas from homological algebra, *D*-modules, category of PDEs) allow us to reduce complicated coupled systems of vector and tensor radial mode equations to sparse upper triangular ODEs.
- Previous partial results were based on trial and error, very laborious. [Berndtson (PhD, 2007)]
- **TODO:** Construct Green function for the Lichnerowicz and vector wave equations on Schwarzschild.
- **TODO:** Generalization to rotating Kerr blackhole?

Discussion

- Abstract decoupling and reduction strategies (using ideas from homological algebra, *D*-modules, category of PDEs) allow us to reduce complicated coupled systems of vector and tensor radial mode equations to sparse upper triangular ODEs.
- Previous partial results were based on trial and error, very laborious. [Berndtson (PhD, 2007)]
- **TODO:** Construct Green function for the Lichnerowicz and vector wave equations on Schwarzschild.
- **TODO:** Generalization to rotating Kerr blackhole?

Discussion

- Abstract decoupling and reduction strategies (using ideas from homological algebra, D-modules, category of PDEs) allow us to reduce complicated coupled systems of vector and tensor radial mode equations to sparse upper triangular ODEs.
- Previous partial results were based on trial and error, very laborious. [Berndtson (PhD, 2007)]
- **TODO:** Construct Green function for the Lichnerowicz and vector wave equations on Schwarzschild.
- **TODO:** Generalization to rotating Kerr blackhole?

Thank you for your attention!

Separation of variables: 2+2 tensor formalism

- We follow the convenient formalism of [Martel & Poisson 2005].
- Schwarzschild ($\mathcal{M} \times S^2$) is spherically symmetric $f(r) = 1 - \frac{2M}{r}$:
$${}^4g_{\mu\nu} = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \to \begin{bmatrix} g_{ab} & 0 \\ 0 & r^2\Omega_{AB} \end{bmatrix}.$$
- Tensor indices $a, b, c, \ldots$ and ∇_a are for $(\mathcal{M}, g_{ab})$.
Tensor indices $A, B, C, \ldots$ and D_A are for the unit sphere (S^2, Ω_{AB}).
- Vector field $v_\mu \to \begin{bmatrix} v_a \\ v_A \end{bmatrix}$, symmetric tensor $p_{\mu\nu} \to \begin{bmatrix} p_{ab} & p_{aB} \\ p_{Ab} & p_{AB} \end{bmatrix}$.
- Connection ${}^4\nabla = (\nabla, D) + \Gamma$,
$$\Gamma^{\mu}_{\nu\lambda} = \left[\begin{bmatrix} 0 & 0 \\ 0 & -rr^a\Omega_{BC} \end{bmatrix} \quad \begin{bmatrix} 0 & \frac{r_b}{r}\delta^A_C \\ \frac{r_c}{r}\delta^A_B & 0 \end{bmatrix} \right].$$
- Formalism covariant with respect to changes of coordinates and metric on $(\mathcal{M}, g_{ab})$.

Spherical harmonics

- Spherical scalar, vector and tensor harmonics:

$$D_A D^A Y = -l(l+1) Y, \qquad Y_A = D_A Y, \qquad Y_{AB} = D_A Y_B + \frac{l(l+1)}{2}\Omega_{AB} Y,$$

$$\int_{S^2} \bar{Y}' Y \epsilon = \delta_{ll'}\delta_{mm'}, \qquad X_A = \epsilon_{BA} D^B Y, \qquad X_{AB} = D_A X_B + \frac{l(l+1)}{2}\epsilon_{AB} Y.$$

Simply normalized, orthogonal, tensor eigenfunctions of $D_A D^A$.

- Vector and Tensor decompositions

$$\begin{bmatrix} p_{ab} & p_{aB} \\ p_{Ab} & p_{AB} \end{bmatrix} = \sum_{lm} \overbrace{\begin{bmatrix} h^{lm}_{ab} Y^{lm} & r\, j^{lm}_a Y^{lm}_B \\ r\, j^{lm}_b Y^{lm}_A & r^2 (K^{lm} \Omega_{AB} Y^{lm} + G^{lm} Y^{lm}_{AB}) \end{bmatrix}}^{\text{even}} + \sum_{lm} \overbrace{\begin{bmatrix} 0 & r\, h^{lm}_a X^{lm}_B \\ r\, h^{lm}_b X^{lm}_A & r^2 h^{lm}_2 X^{lm}_{AB} \end{bmatrix}}^{\text{odd}}$$

$$\begin{bmatrix} v_a \\ v_A \end{bmatrix} = \sum_{lm} \overbrace{\begin{bmatrix} v^{lm}_a Y^{lm} \\ r\, u^{lm} Y^{lm}_A \end{bmatrix}}^{\text{even}} + \sum_{lm} \overbrace{\begin{bmatrix} 0 \\ r\, w^{lm} X^{lm}_A \end{bmatrix}}^{\text{odd}}$$

From now on, omit spherical harmonic (l, m) mode indices:

$$p = (h_{ab}, j_a, K, G \mid h_a, h_2) \quad \text{and} \quad v = (v_a, u \mid w)$$

- In static Schwarzschild (t, r) coordinates $(2M < r < \infty)$:

$$p(t,r) = p(r) e^{-i\omega t} \quad \text{and} \quad v(t,r) = v(r) e^{-i\omega t}, \quad \text{where}$$
$$p(r) = (h_{tt}, h_{tr}, h_{rr}, j_t, j_r, K, G \mid h_t, h_r, h_2), \quad v(r) = (v_t, v_r, u \mid w).$$

A toy example: equivalence up to homotopy

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[r, shift left, "\partial_r"] \arrow[d, "\partial_r^2+\omega^2"] & \bullet \arrow[l, shift left, "\frac{-\partial_r}{\omega^2}"] \arrow[d, "\partial_r^2+\omega^2"'] \\
\bullet \arrow[r, shift left, "\partial_r"] \arrow[d] \arrow[u, dashed, bend left, "\frac{1}{\omega^2}"] & \bullet \arrow[l, shift left, "\frac{-\partial_r}{\omega^2}"] \arrow[d] \arrow[u, dashed, bend right, "\frac{1}{\omega^2}"'] \\
0 & 0
\end{tikzcd},
$$

In this toy example, the morphisms satisfy

$$(\partial_r^2 + \omega^2) \circ \partial_r = \partial_r \circ (\partial_r^2 + \omega^2),$$
$$(\partial_r^2 + \omega^2) \circ \frac{-\partial_r}{\omega^2} = \frac{-\partial_r}{\omega^2} \circ (\partial_r^2 + \omega^2).$$

Intuitively, ∂_r is not invertible, but it is invertible up to homotopy:

$$\frac{-\partial_r}{\omega^2} \circ \partial_r = 1 - \frac{1}{\omega^2} \circ (\partial_r^2 + \omega^2),$$
$$\frac{-\partial_r}{\omega^2} \circ \partial_r = 1 - (\partial_r^2 + \omega^2) \circ \frac{1}{\omega^2},$$
$$\partial_r \circ \frac{-\partial_r}{\omega^2} = 1 - \frac{1}{\omega^2} \circ (\partial_r^2 + \omega^2),$$
$$\partial_r \circ \frac{-\partial_r}{\omega^2} = 1 - (\partial_r^2 + \omega^2) \circ \frac{1}{\omega^2}.$$

N.B.: ∂_r maps solutions to solutions!