Algebra and geometry of Lax representations and Bäcklund transformations for (1+1)-dimensional partial differential and differential-difference equations

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Let $\mathcal{E}$ be a manifold and $n \in \mathbb{Z}_+$. An *n-dimensional distribution* $\mathcal{C}$ on $\mathcal{E}$ is an n-dimensional subbundle of the tangent bundle $T\mathcal{E}$.

To define an n-dimensional distribution $\mathcal{C}$ on $\mathcal{E}$, we choose an n-dimensional subspace $\mathcal{C}_a \subset T_a\mathcal{E}$ for each $a \in \mathcal{E}$ such that $\mathcal{C}_a$ depends smoothly on a.

A submanifold $S \subset \mathcal{E}$ is an *integral submanifold* of the distribution $\mathcal{C}$ if S is tangent to $\mathcal{C}_a$ for each $a \in S$. That is, $T_aS \subset \mathcal{C}_a$ for each $a \in S$.

A geometric approach to partial differential equations (PDEs):

For any PDE satisfying some non-degeneracy conditions, we will define a manifold $\mathcal{E}$ and an n-dimensional distribution $\mathcal{C} \subset T\mathcal{E}$ such that solutions of the PDE correspond to n-dimensional integral submanifolds.

Here n is the number of independent variables in the PDE.

$\mathcal{E}$ and $\mathcal{C}$ are constructed by means of the theory of infinite jet bundles.

In this framework, PDEs and many structures related to PDEs (symmetries, conservation laws, Bäcklund transformations, Lax representations) can be studied geometrically in a coordinate-independent way.

Summary of the main ideas:

A PDE can be regarded as a manifold $\mathcal{E}$ with a distribution $\mathcal{C} \subset T\mathcal{E}$.
Solutions of the PDE correspond to integral submanifolds.

For any topological space X and each point $a \in X$,
one has the **fundamental group** $\pi_1(X, a)$.

Similarly, for any analytic PDE $\mathcal{E}$ and each point $a \in \mathcal{E}$,
I will define the **fundamental Lie algebra** $\pi_1(\mathcal{E}, a)$.

Fundamental Lie algebras are new geometric invariants for PDEs.

Computing these algebras, we get interesting infinite-dimensional Lie algebras.
Computations have been done in collaboration with G. Manno (Torino, Italy).

Using fundamental Lie algebras, one obtains new results on Bäcklund transformations and zero-curvature representations (Lax representations).

The fundamental group $\pi_1(X, a)$ can be defined by means of **topological coverings** of X.

The fundamental Lie algebra $\pi_1(\mathcal{E}, a)$ can be defined by means of **differential coverings** of the PDE $\mathcal{E}$.

Summary of applications of fundamental Lie algebras in the case of (1+1)-dimensional evolution PDEs:

Using fundamental Lie algebras, we obtain:

- necessary conditions for existence of a Bäcklund transformation (BT) between two given PDEs,
- necessary conditions for existence of a nontrivial zero-curvature representation (ZCR) for a given PDE.

These necessary conditions allow us to prove non-existence of BTs for some pairs of PDEs and non-existence of nontrivial ZCRs for some classes of PDEs.

We consider the widest class of BTs (not necessarily of Miura type) and the widest class of ZCRs. In particular, they may depend on derivatives of arbitrary finite order.

We find a normal form for ZCRs of a given (1+1)-dimensional evolution PDE with respect to the action of the group of local gauge transformations.

We get invariant meaning for infinite-dimensional Lie algebras and algebraic curves related to some PDEs.

Differential coverings (A. Vinogradov, I. Krasilshchik)

Example: Miura transformation from mKdV equation to KdV equation

$$\text{KdV} = \left\{ u_t = u_{xxx} + 6uu_x \right\} \quad \xleftarrow{u=v_x-v^2} \quad \text{mKdV} = \left\{ v_t = v_{xxx} - 6v^2 v_x \right\}$$

This is a map from solutions $v(x,t)$ of mKdV to solutions $u(x,t)$ of KdV.
The preimage of each solution $u(x,t)$ of KdV is a one-parameter family of solutions $v(x,t)$ of mKdV.

Informal description of differential coverings in coordinates:

$$\mathcal{E}^1 = \left\{ F\left(x_i, u(x_i), \frac{\partial u}{\partial x_i}, \ldots\right) = 0 \right\} \quad \longleftarrow \quad \mathcal{E}^2 = \left\{ G\left(y_i, v(y_i), \frac{\partial v}{\partial y_i}, \ldots\right) = 0 \right\}$$

$$u = \varphi\left(y_i, v, \frac{\partial v}{\partial y_i}, \ldots\right), \qquad x_i = \psi\left(y_s, v, \frac{\partial v}{\partial y_s}, \ldots\right)$$

This is a map from solutions $v(y_i)$ of $\mathcal{E}^2$ to solutions $u(x_i)$ of $\mathcal{E}^1$.
$u(x_i)$ and $v(y_i)$ are vector-functions.
The preimage of each solution $u(x_i)$ of $\mathcal{E}^1$ is a family of $\mathcal{E}^2$ solutions $v(y_i)$ depending on a finite number D of parameters.
D is the **dimension of fibers** of the covering.

$\mathcal{E}^1$ and $\mathcal{E}^2$ are connected by a **Bäcklund transformation** if there is $\mathcal{E}^3$ with a pair of coverings

$$\begin{tikzcd} & \mathcal{E}^3 \arrow[dl] \arrow[dr] & \\ \mathcal{E}^1 & & \mathcal{E}^2 \end{tikzcd}$$

This allows one to obtain solutions of $\mathcal{E}^2$ from solutions of $\mathcal{E}^1$:
take a solution of $\mathcal{E}^1$, find its preimage in $\mathcal{E}^3$, and project it to $\mathcal{E}^2$.

Example:

$$\begin{tikzcd} & v_t = v_{xxx} - 6v^2 v_x + 6\lambda v_x \arrow[dl, "u=v_x-v^2+\lambda"'] \arrow[dr, "u=-v_x-v^2+\lambda"] & \\ u_t = u_{xxx} + 6uu_x & & u_t = u_{xxx} + 6uu_x \end{tikzcd}$$

Trivial solution $u(x,t)=\mathrm{const}$	$\mapsto$	1-soliton solution	$\mapsto$	2-soliton solution	$\mapsto$	$\ldots$

How to define a manifold $\mathcal{E}$ and a distribution $\mathcal{C} \subset T\mathcal{E}$ for a given PDE.

Example: the infinite prolongation of KdV.

Infinite jet space $J^\infty = (x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots)$.

Total derivative operators

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \dots, \qquad D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + \dots$$

are commuting vector fields on J^∞.

Consider the submanifold $\mathcal{E} \subset J^\infty$ determined by KdV and all its differential consequences

$$u_t = u_{xxx} + 6uu_x, \quad u_{tt} = u_{xxxt} + 6u_t u_x + 6uu_{xt}, \quad u_{tx} = u_{xxxx} + 6u_x^2 + 6uu_{xx}, \ \dots$$

D_x, D_t are tangent to $\mathcal{E}$ and span a 2-dimensional distribution on $\mathcal{E}$.

Similarly, a PDE with n independent variables can be regarded as a manifold $\mathcal{E}$ with an n-dimensional distribution $\mathcal{C}$ (the **Cartan distribution**).

Solutions of the PDE correspond to integral submanifolds of this distribution.

Let $(\mathcal{E}^1, \mathcal{C}^1)$ and $(\mathcal{E}^2, \mathcal{C}^2)$ be PDEs, where $\mathcal{C}^i \subset T\mathcal{E}^i$ is the Cartan distribution.

A smooth map $\tau\colon \mathcal{E}^2 \to \mathcal{E}^1$ is a **differential covering** if
τ is a bundle with finite-dimensional fibers, $\tau_*\colon T\mathcal{E}^2 \to T\mathcal{E}^1$, $\quad \tau_*(\mathcal{C}^2) \subset \mathcal{C}^1$,

$\forall\, a \in \mathcal{E}^2 \quad \tau_*\colon \mathcal{C}^2_a \longrightarrow \mathcal{C}^1_{\tau(a)}$ is an isomorphism, $\mathcal{C}^2_a \subset T_a\mathcal{E}^2$, $\mathcal{C}^1_{\tau(a)} \subset T_{\tau(a)}\mathcal{E}^1$

If $\mathcal{C}^2_a = T_a\mathcal{E}^2$ and $\mathcal{C}^1_{\tau(a)} = T_{\tau(a)}\mathcal{E}^1$ then differential coverings are topological coverings.

Topological coverings of a manifold M are determined by actions of the fundamental group $\pi_1(M, a)$ for $a \in M$.

We need an analog of $\pi_1(M, a)$ for differential coverings of PDEs.

For any analytic PDE $\mathcal{E}$ and $a \in \mathcal{E}$, we naturally define a Lie algebra $\pi_1(\mathcal{E}, a)$.
$\pi_1(\mathcal{E}, a)$ is called the **fundamental Lie algebra** of $\mathcal{E}$ at $a \in \mathcal{E}$.
The definition of $\pi_1(\mathcal{E}, a)$ will be given later. First I describe some properties of $\pi_1(\mathcal{E}, a)$.

For any covering $\tau\colon \mathcal{E}' \to \mathcal{E}$, the algebra $\pi_1(\mathcal{E}, a)$ acts on the fiber $\tau^{-1}(a)$.
(Fibers are finite-dimensional.)

For a topological covering $\tau\colon M' \to M$,
$a' \in M', \qquad a = \tau(a') \in M, \qquad \pi_1(M', a') \hookrightarrow \pi_1(M, a).$

For a differential covering $\tau\colon \mathcal{E}' \to \mathcal{E}, \qquad a' \in \mathcal{E}', \qquad a = \tau(a') \in \mathcal{E},$
$\pi_1(\mathcal{E}', a')$ is isomorphic to a subalgebra of $\pi_1(\mathcal{E}, a)$ of finite codimension.

Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be connected by a Bäcklund transformation

$$\begin{tikzcd} & \mathcal{E}^3 \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ \mathcal{E}^1 & & \mathcal{E}^2 \end{tikzcd}$$

$$a_3 \in \mathcal{E}^3, \qquad a_1 = \tau_1(a_3) \in \mathcal{E}^1, \qquad a_2 = \tau_2(a_3) \in \mathcal{E}^2$$

$$\pi_1(\mathcal{E}^3, a_3) \hookrightarrow \pi_1(\mathcal{E}^1, a_1), \qquad \pi_1(\mathcal{E}^3, a_3) \hookrightarrow \pi_1(\mathcal{E}^2, a_2)$$

Therefore, $\pi_1(\mathcal{E}^1, a_1)$ and $\pi_1(\mathcal{E}^2, a_2)$ have a common subalgebra of finite codimension. **This is a powerful necessary condition for existence of a Bäcklund transformation between $\mathcal{E}^1$ and $\mathcal{E}^2$.**

In the considered examples (which include the KdV, nonlinear-Schrödinger, Krichever-Novikov, Landau-Lifshitz equations),
the "main part" of $\pi_1(\mathcal{E}, a)$ is an infinite-dimensional Lie algebra of certain matrix-valued functions on an algebraic curve.

Rational curve (genus $= 0$) for KdV and nonlinear-Schrödinger (NLS).
Elliptic curve for nonsingular Krichever-Novikov and Landau-Lifshitz.
The computation of $\pi_1(\mathcal{E}, a)$ was done in collaboration with G. Manno.

Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be some PDEs from these examples.
If the corresponding algebraic curves are not birationally equivalent, then there is no Bäcklund transformation between $\mathcal{E}^1$ and $\mathcal{E}^2$.

For example, there is no Bäcklund transformation between KdV and nonsingular Krichever-Novikov, between NLS and nonsingular Landau-Lifshitz.

One gets also an invariant meaning for algebraic curves related to the above-mentioned PDEs.

Similar results can be obtained for other (1+1)-dimensional evolution PDEs (e.g., the Kaup–Kupershmidt and Sawada–Kotera equations).

The fundamental group $\pi_1(M, a)$ can be defined using only (topological) coverings of M, without using loops in M. We consider all coverings τ of M.

$g \in \pi_1(M, a)$ gives a transformation $g_\tau \colon \tau^{-1}(a) \to \tau^{-1}(a)$ for each $\tau \colon M' \to M$

For any
$$\begin{tikzcd} M^1 \arrow[rr, "\varphi"] \arrow[dr, "\tau_1"'] & & M^2 \arrow[dl, "\tau_2"] \\ & M & \end{tikzcd}$$
we have $$g_{\tau_2} \circ \varphi\big|_{\tau_1^{-1}(a)} = \varphi\big|_{\tau_1^{-1}(a)} \circ g_{\tau_1} \qquad (1)$$

$g \in \pi_1(M, a)$ is uniquely determined by the collection of transformations $\left\{ g_\tau \colon \tau^{-1}(a) \to \tau^{-1}(a) \;\middle|\; \tau \text{ is a covering of } M \right\}$.

One can define an element of $\pi_1(M, a)$ as a collection of such transformations (for all coverings τ of M) satisfying (1).

To define $\pi_1(\mathcal{E}, a)$, replace transformations on fibers by vector fields on fibers.

An element of $\pi_1(\mathcal{E}, a)$ is defined as a collection of (formal) vector fields:
$\left\{ g_\tau \text{ is a vector field on } \tau^{-1}(a) \;\middle|\; \tau \text{ is a (formal) differential covering of } \mathcal{E} \right\}$,

such that for any
$$\begin{tikzcd} \mathcal{E}^1 \arrow[rr, "\varphi"] \arrow[dr, "\tau_1"'] & & \mathcal{E}^2 \arrow[dl, "\tau_2"] \\ & \mathcal{E} & \end{tikzcd}$$
we have $\varphi_*(g_{\tau_1}) = g_{\tau_2}$.

Example. Consider a (1+1)-dimensional scalar evolution equation

$$u_t = F(x, t, u_0, u_1, \dots, u_d), \qquad u = u(x,t), \qquad u_k = \frac{\partial^k u}{\partial x^k}, \quad u_0 = u. \quad (2)$$

Let V be a vector space. Then $\mathrm{gl}(V)$ is the Lie algebra of linear maps $V \to V$, and $\mathrm{GL}(V)$ is the group of invertible linear maps $V \to V$.

Let $A = A(x, t, u_0, u_1, \dots, u_p)$ and $B = B(x, t, u_0, u_1, \dots, u_{p+d-1})$ be functions with values in $\mathrm{gl}(V)$.

The functions A, B form a **zero-curvature representation** (ZCR) if

$$D_x(B) - D_t(A) + [A, B] = 0,$$

where D_x, D_t are the total derivative operators corresponding to equation (2).

A **gauge transformation** is given by a function $G = G(x, t, u_0, u_1, \dots, u_k)$ with values in $\mathrm{GL}(V)$.

The functions $\tilde{A} = GAG^{-1} - D_x(G) \cdot G^{-1}$ and $\tilde{B} = GBG^{-1} - D_t(G) \cdot G^{-1}$ satisfy $D_x(\tilde{B}) - D_t(\tilde{A}) + [\tilde{A}, \tilde{B}] = 0$, so $\tilde{A}$, $\tilde{B}$ form a ZCR.

The ZCR $\tilde{A}$, $\tilde{B}$ is **gauge equivalent** to the ZCR A, B.

$\mathcal{E}$ is the manifold with coordinates x, t, u_k. Let $a = (x{=}0,\ t{=}0,\ u_k{=}0) \in \mathcal{E}$.

After a suitable gauge transformation on a neighborhood of $a \in \mathcal{E}$, we get

$$\frac{\partial \tilde{A}}{\partial u_s}\,\bigg|_{u_k=0,\ k\geq s} = 0 \quad \forall\, s \geq 1, \qquad \tilde{A}\,\Big|_{u_k=0,\ k\geq 0} = \tilde{B}\,\Big|_{x=0,\ u_k=0,\ k\geq 0} = 0, \tag{3}$$

$$D_x(\tilde{B}) - D_t(\tilde{A}) + [\tilde{A}, \tilde{B}] = 0. \tag{4}$$

This is a normal form for zero-curvature representations (ZCRs) with respect to the action of the group of gauge transformations.

Consider the Taylor series of the functions $\tilde{A} = \tilde{A}(x, t, u_0, u_1, \dots, u_p)$ and $\tilde{B} = \tilde{B}(x, t, u_0, u_1, \dots, u_{p+d-1})$ at the point $a \in \mathcal{E}$.

So we view $\tilde{A}$ and $\tilde{B}$ as power series in the variables x, t, u_k.

We regard the coefficients of the power series $\tilde{A}$, $\tilde{B}$ as abstract symbols.

Let $F^p(\mathcal{E}, a)$ be the Lie algebra generated by these coefficients. Relations for these generators are provided by (3), (4).

Representations of $F^p(\mathcal{E}, a)$ classify (up to gauge equivalence) ZCRs of the form

$$A = A(x, t, u_0, u_1, \ldots, u_p), \;\; B = B(x, t, u_0, u_1, \ldots), \;\; D_x(B) - D_t(A) + [A, B] = 0$$

$F^p(\mathcal{E}, a)$ is defined also for (1+1)-dimensional multicomponent evolution PDEs, for any point $a \in \mathcal{E}$.

We get a sequence of surjective homomorphisms of Lie algebras

$$\cdots \to F^p(\mathcal{E}, a) \to F^{p-1}(\mathcal{E}, a) \to \cdots \to F^1(\mathcal{E}, a) \to F^0(\mathcal{E}, a).$$

The fundamental Lie algebra $\pi_1(\mathcal{E}, a)$ of the considered evolution PDE is isomorphic to the inverse limit of this sequence.

ZCRs of the form $A = A(u_0), \;\; B = B(u_0, u_1, \ldots)$ are described by representations of the Wahlquist-Estabrook prolongation Lie algebra, which does not have any coordinate-independent meaning.

The Wahlquist-Estabrook prolongation method does not use gauge equivalence.

Because of this, the Wahlquist-Estabrook prolongation method cannot classify general ZCRs $\quad A = A(x, t, u_0, u_1, \ldots, u_p), \qquad B = B(x, t, u_0, u_1, \ldots)$.

If a (1+1)-dimensional (multicomponent) evolution PDE $\mathcal{E}$ is S-integrable, then there are $p \geq 0$ and $a \in \mathcal{E}$ such that the Lie algebra $F^p(\mathcal{E}, a)$ is infinite-dimensional and is not solvable.

This gives a necessary condition for S-integrability of $\mathcal{E}$.

This approach is applicable to all S-integrable PDEs, including ones which do not possess higher local symmetries and conservation laws.

For many evolution equations $\mathcal{E}$, we can find $q \geq 0$ such that the kernel of the surjective homomorphism $F^p(\mathcal{E}, a) \to F^q(\mathcal{E}, a)$ is solvable for all $p > q$.
In this case, if $F^q(\mathcal{E}, a)$ is solvable then the equation is not S-integrable.
Example. For equations $u_t = u_5 + f(x, t, u_0, u_1, u_2, u_3)$, we have $q = 1$.
If $\frac{\partial^3 f}{\partial u_3^3} \neq 0$, then $F^p(\mathcal{E}, a)$ is nilpotent for all p, and the equation is not S-integrable. The same holds for the equation $u_t = u_5 + uu_1$.

For any PDE $\mathcal{E}$, solvable ideals of $F^p(\mathcal{E}, a)$ are not important for applications.
For the KdV, NLS, Krichever-Novikov (KN), Landau-Lifshitz (LL) equations, if we kill all solvable ideals of $F^p(\mathcal{E}, a)$, we get infinite-dimensional Lie algebras of some sl_2-valued and so_3-valued functions on algebraic curves.
Rational curve (genus $= 0$) for KdV and NLS. Elliptic curve for KN and LL.
In the computation, we used some results of H. van Eck, P. Gragert, G. Roelofs, R. Martini on Wahlquist-Estabrook prolongation algebras.

An m-component generalization of the Landau-Lifshitz equation was introduced by I. Golubchik and V. Sokolov. It possesses a ZCR parametrized by the following algebraic curve of genus $1+(m-3)2^{m-2}$

$$\mathcal{C}=\big\{(\lambda_1,\dots,\lambda_m)\in\mathbb{C}^m \;\big|\; \lambda_i^2-\lambda_j^2=r_j-r_i, \quad i,j=1,\dots,n\big\}, \tag{5}$$

where $r_1,\dots,r_m$ are constants satisfying $r_i\neq r_j$ for $i\neq j$.

For this PDE with $m\geq 4$, the Lie algebras $F^k(\mathcal{E},a)$ have the following structure (S. Ig., J. van de Leur, G. Manno, V. Trushkov):

$F^0(\mathcal{E},a)$ is isomorphic to the infinite-dimensional Lie algebra $\mathbf{L}$ of certain matrix-valued functions on the curve (5). For any $k\geq 1$, there is a surjective homomorphism $F^k(\mathcal{E},a)\to\mathbf{L}\oplus\mathfrak{so}_{m-1}(\mathbb{C})$ with solvable kernel.

(The algebra $\mathbf{L}$ was considered before by I. Golubchik, V. Sokolov, T. Skrypnyk in a different context.)

$F^0(\mathcal{E},a)$ is given by generators $p_1,\dots,p_m$ and the relations

$$\begin{aligned}
[p_i,[p_i,p_k]]-[p_j,[p_j,p_k]]&=(r_j-r_i)p_k, \quad && i\neq k,\ j\neq k,\ i,j,k=1,\dots,n. \\
[p_i,[p_j,p_k]]&=0, && i\neq j\neq k\neq i, \qquad i,j,k=1,\dots,n.
\end{aligned} \tag{6}$$

Relations (6) are very similar to equations (5). Using only the PDE, we get relations (6), which suggest to consider the curve (5).

We have methods to describe differential coverings for (1+1)-dimensional evolution PDEs, using actions of Lie algebras. Miura-type transformations (differential substitutions) are a particular class of differential coverings.

We have developed a method to construct (and to classify in some cases) Miura-type transformations (MTs) for (1+1)-dimensional evolution PDEs of the form $u_t = F(u_0, u_1, \dots, u_d)$, $u = u(x,t)$, $u_k = \frac{\partial^k u}{\partial x^k}$, using zero-curvature representations (ZCRs) and actions of Wahlquist–Estabrook prolongation Lie algebras (WE algebras). We study MTs which do not change x, t.
Some results are presented in S.Ig., *J. of Phys A* **38** (2005).

Our method is a generalization of a result of V.G. Drinfeld and V.V. Sokolov on MTs for the KdV equation: V.G. Drinfeld, V.V. Sokolov, "On equations that are related to the Korteweg–de Vries equation," *Soviet Math. Dokl.* **32** (1985).

Idea: Consider a PDE $\mathcal{E}$ of the above type, a ZCR $D_x(B) - D_t(A) + [A, B] = 0$, where A, B are functions on $\mathcal{E}$ with values in a Lie algebra $\mathfrak{g}$, and an action $\varphi\colon \mathfrak{g} \to D(W)$ on a manifold W. This gives the differential covering $\mathcal{E} \times W \to \mathcal{E}$ with the total derivatives $D_x + \varphi(A)$, $D_t + \varphi(B)$ on $\mathcal{E} \times W$.
For a scalar PDE $\mathcal{E}$, any MT is of this type with $\mathfrak{g}$ equal to the WE algebra.
Classification of MTs for some scalar PDEs. (For example, PDEs of KdV type.)
Construction of MTs for multicomponent PDEs.

$\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}, \quad \boldsymbol{\alpha} \leq \boldsymbol{\beta}$. We study differential-difference equations of the form

$$u_t = \mathbf{F}(u_{\boldsymbol{\alpha}}, u_{\boldsymbol{\alpha}+1}, \ldots, u_{\boldsymbol{\beta}}), \tag{1}$$

$u = u(n,t)$ is a vector-function of an integer variable n and a real or complex variable t. $u_t = \partial_t(u)$ and $u_i = u(n+i,t)$ for $i \in \mathbb{Z}$. $u_0 = u$.
Equation (1) must be valid for all $n \in \mathbb{Z}$, so (1) encodes an infinite sequence of differential equations

$$\partial_t\big(u(n,t)\big) = \mathbf{F}\big(u(n+\boldsymbol{\alpha},t), u(n+\boldsymbol{\alpha}+1,t), \ldots, u(n+\boldsymbol{\beta},t)\big), \qquad n \in \mathbb{Z}.$$

Consider another differential-difference equation of similar type

$$v_t = \tilde{\mathbf{F}}(v_{\tilde{\boldsymbol{\alpha}}}, v_{\tilde{\boldsymbol{\alpha}}+1}, \ldots, v_{\tilde{\boldsymbol{\beta}}}) \tag{2}$$

for a vector-function $v = v(n,t)$ and some integers $\tilde{\boldsymbol{\alpha}} \leq \tilde{\boldsymbol{\beta}}$.
A *Miura-type transformation* (MT) from equation (2) to equation (1) is determined by an expression of the form

$$u = f(v_p, v_{p+1}, \ldots, v_r), \qquad p, r \in \mathbb{Z}, \qquad p \leq r, \tag{3}$$

if v satisfies (2) then u given by (3) satisfies (1).

Let $\mathcal{S}$ be the shift operator which replaces n by $n+1$ and u_i by u_{i+1} in all considered functions.
Let $d \in \mathbb{Z}_+$. Let $M = M(u_i, \ldots, \lambda)$ and $\mathcal{U} = \mathcal{U}(u_i, \ldots, \lambda)$ be $d \times d$ matrix-functions depending on a finite number of the variables u_i and a complex parameter λ.
Suppose that the matrix M is invertible, and the equation

$$\partial_t(M) = \mathcal{S}(\mathcal{U})M - M\mathcal{U} \tag{4}$$

holds as a consequence of (1).
Such a pair $(M, \mathcal{U})$ is an analog of a zero-curvature (Lax) representation for differential-difference equations.
Such $(M, \mathcal{U})$ often arise from Darboux transformations of PDEs.
The pair $(M, \mathcal{U})$ is a *Darboux-Lax representation* (DLR) for equation (1).
This implies that the auxiliary linear system

$$\begin{aligned} \mathcal{S}(\Psi) &= M\Psi, \\ \partial_t(\Psi) &= \mathcal{U}\Psi \end{aligned} \tag{5}$$

is compatible modulo (1). $\Psi = \Psi(n, t)$ is an invertible $d \times d$ matrix-function.

We have translated some parts of the theory of coverings and MTs from the PDE case to the case of differential-difference equations.

The total derivative operators D_x, D_t in the PDE case get replaced by the operators $\mathcal{S}$, D_t in the differential-difference case.

We have developed a method to construct MTs for differential-difference (lattice) equations, using Lie group actions associated with Darboux–Lax representations of such equations. The considered examples include Volterra, Narita–Itoh–Bogoyavlensky, Toda, and Adler–Postnikov lattices.

Applying our method to these examples, we have obtained new integrable nonlinear differential-difference equations connected with these lattices by MTs.

G. Berkeley, S. Ig., "Miura-type transformations for lattice equations and Lie group actions associated with Darboux–Lax representations," *J. of Phys. A* **49** (2016).

A new MT for the Narita–Itoh–Bogoyavlensky lattice. $p \in \mathbb{Z}_+$, $c_1, c_2 \in \mathbb{C}$.

$$u_t = u\Big(\sum_{k=1}^{p} u_k - \sum_{k=1}^{p} u_{-k}\Big), \tag{6}$$

$$u = \frac{(c_1 v_{2p} - c_2)\big(\prod_{j=p}^{2p-1} v_j\big)\prod_{i=0}^{p-1}\Big(c_2\big(\prod_{j=i}^{i+p-1} v_j\big) - c_1\Big)}{\prod_{i=0}^{p}\big(-1 + \prod_{j=i}^{i+p} v_j\big)}, \tag{7}$$

$$v_t = \frac{v(c_2 - c_1 v)\big(\prod_{i=1}^{p} v_{-i} - \prod_{i=1}^{p} v_i\big)\prod_{i=0}^{p-1}\Big(c_2\big(\prod_{j=i+1-p}^{i} v_j\big) - c_1\Big)}{\prod_{i=0}^{p}\big(-1 + \prod_{j=i-p}^{i} v_j\big)} \tag{8}$$