Towards a theory of homotopy structures for differential equations: First definitions and examples -1

Volodya Rubtsov

LAREMA 6093 du CNRS, Université d'Angers
GDT in Paris and IITP RAS Moscow
volodya@univ-angers.fr

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Jean-Pierre Magnot [a b], Enrique G. Reyes [c], Vladimir Rubtsov [a d e]

Some picture files are due to:
Bruno Vallette "Algebra + Homotopy = Operad"
(arXiv: 1202.3245 v 1 [math. AT])

Introduction: DGA

$\mathcal{A}-$ *differential graded algebra* (DGA) over the field $\mathbb{K}$:

- $\mathcal{A} = \oplus_{k \geq 0} \mathcal{A}^k$, $\mathrm{deg}\, a = k$, $a \in \mathcal{A}^k$.
- associative multiplication $\wedge : \mathcal{A}^k \times \mathcal{A}^l \to \mathcal{A}^{k+l}, k, l \geq 0$.
- differential $d_{\mathcal{A}} : \mathcal{A}^k \to \mathcal{A}^{k+1}$, $k \geq 0$, $d^2 = 0$.
- $a \wedge b = (-1)^{kl} b \wedge a$, $a \in \mathcal{A}^k, b \in \mathcal{A}^l$.
- $d(a \wedge b) = da \wedge b + (-1)^k a \wedge db, a \in \mathcal{A}^k$ (the Leibniz rule).

Introduction: Homomorphisms and cohomology

- Let $(\mathcal{A}, d_{\mathcal{A}})$ and $(\mathcal{B}, d_{\mathcal{B}})$ two DGA's and $f : \mathcal{A} \to \mathcal{B} - \mathbb{K}-$linear map such that:
- $f(\mathcal{A}^k) \subset \mathcal{B}^k$, $k \geq 0$ and $f(a \wedge b) = f(a) \wedge f(b)$, $d_{\mathcal{B}}(f(a)) = f(d_{\mathcal{A}}(a))$, $f-$ a homomorphism of differential graded algebras.
- $H^k(\mathcal{A}, d_{\mathcal{A}}) := \mathrm{Ker}(d_{\mathcal{A}} : \mathcal{A}^k \to \mathcal{A}^{k+1})/\mathrm{Im}(d_{\mathcal{A}} : \mathcal{A}^{k-1} \to \mathcal{A}^k)$ and if f - DGA-homomorphism then $f^* : H^*(\mathcal{A}, d_{\mathcal{A}}) \to H^*(\mathcal{B}, d_{\mathcal{B}},\ f^*[a] = [f(a)]$.
- If $H^0(\mathcal{A}, d_{\mathcal{A}}) = \mathbb{K}$, then $\mathcal{A}-$ connected; if , in addition $H^1(\mathcal{A}, d_{\mathcal{A}}) = 0$ then $\mathcal{A}-$ simply connnected.
- If there exists a surjection $\epsilon : \mathcal{A} \to \mathbb{K}$ of DGA's where $deg(k) = 0, k \in \mathbb{K}$ and $d_{\mathbb{K}} \simeq 0$ then $\mathcal{A}-$ augmented.
- Main example: $X-$smooth manifold, $\mathcal{A} = \Omega^*(X)-$ smooth differential forms - DGA with $d_{\mathcal{A}} = d_{DR}$. $X-$ simply connected then $\Omega^*(X)$- also simply connected.

Introduction: Minimal DGA

Let $\mathbb{K} = \mathbb{R}$.

- A DGA $(\mathcal{M}, d_{\mathcal{M}})$ is *minimal* if:
- $\mathcal{M}^0 = \mathbb{R}$ and $d(\mathcal{M}^0) = 0$.
- $\mathcal{M}^+ = \oplus_{k>0}\mathcal{M}^k$ is freely generated by homogeneous elements $x_1, \ldots, x_n \ldots$ i.e. $\mathcal{M}^+ = \Lambda < x_1, x_2, \ldots >$ for each $k > 0$ there exist finitely many such generators degree k, and $\deg x_i \leq \deg x_j$, if $i \leq j$.
- differential d is *reducible*: $dx_i \in \Lambda(x_1, \ldots, x_{i-1})$, $i \geq 1$.
- $\mathcal{M}$*−simply connected* iff $\mathcal{M}^1 = 0$. In this case $\deg x_i \geq 2$ and the reducibility means $d\mathcal{M}^+ \subset \mathcal{M}^+ \wedge \mathcal{M}^+$.

Introduction: Minimal model of DGA

An algebra $(\mathcal{M}, d_{\mathcal{M}})$ is called a *minimal model* for an algebra $(\mathcal{A}, d_{\mathcal{A}})$ if

- the algebra $(\mathcal{M}, d_{\mathcal{M}})$ is minimal;
- there exists a homomorphism $h : (\mathcal{M}, d_{\mathcal{M}}) \to (\mathcal{A}, d_{\mathcal{A}})$ inducing an isomorphism of cohomology rings:

$$h^* : H^*(\mathcal{M}, d_{\mathcal{M}}) \to H^*(\mathcal{A}, d_{\mathcal{A}}).$$

(h^* — *quasi-isomorphism*)

Introduction: Existence and example

D. Sullivan (1974):

Theorem

If $(\mathcal{A}, d_{\mathcal{A}})$ is a simply connected DGA such that $\dim H^k(\mathcal{A}) < \infty$ for each $k \geq 0$, then there exists a unique (up to isomorphisms) minimal model for $\mathcal{A}$.

Example

$X-$ a simply connected compact manifold; $\mathcal{A} = \Omega^*(X)$; The minimal model $\mathcal{M}_X$ for $\mathcal{A}$ (also called the (real) minimal model for X) is isomorphic to $H^*(X, \mathbb{R})$ because

$$H^*(X, \mathbb{R}) \simeq H^*(\Omega(X)) \Rightarrow$$

$$h : (\mathcal{M}_X, d_{\mathcal{M}}) \to (\Omega^*(X), d_X) \Rightarrow h^* : H^*(\mathcal{M}_X) \simeq H^*(X, \mathbb{R}).$$

Introduction: Heisenberg group

Let G be a group of $3 \times 3-$ real upper-triangular matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R}$$

Consider Heisenberg nilmanifold $X = G/\Gamma = G/G_{\mathbb{Z}}$;with $G_{\mathbb{Z}}-$ the subgroup with $x, y, z \in \mathbb{Z}$.;

$$\omega_1 = dx, \omega_2 = dy, \omega_3 = xdy - dz.$$

The minimal model $\mathcal{M}_X$ is generated by the elements x_1, x_2, x_3 of degree 1 such that

$$dx_1 = dx_2 = 0; dx_3 = x_1 \wedge x_2.$$

Introduction: Formality

We associate with each minimal algebra $(\mathcal{M}, d)$ its cohomology ring $H^*(\mathcal{M})$, considering as a DGA with differential zero: $(H^*(\mathcal{M}), 0)$.
The minimal algebra $\mathcal{M}$ is said to be *formal* if
$\exists f : (\mathcal{M}, d) \to H^*(\mathcal{M}, 0)$ inducing an isomorphism
$f^* : H^*(\mathcal{M}) \simeq H^*(\mathcal{M})$ (in other words $(\mathcal{M}, d)-$ minimal model for its cohomology ring).
A DGA $(\mathcal{A}, d_{\mathcal{A}})-$ is formal if its minimal model $\mathcal{M}(\mathcal{A})$ is formal. It means that $H^*(\mathcal{A}, d_{\mathcal{A}}) = H^*(\mathcal{M}(\mathcal{A}), d_{\mathcal{M}})$.
If $\Omega(X)$ for smooth X is formal, then we say that X is formal.

Introduction: Example formality

Example

Let X be a Kähler simply connected compact. Then (from dd^c-lemma) X is formal:

$$d^c = I^{-1}dI; (\Omega^{*,cl}(X), d^c) \subset (\Omega^*(X), d); (\Omega^*(X), d)/(\Omega^{*,cl}(X), d^c)-$$

quotient dd^c-lemma implies two quasi-isomorphisms:

$$(\Omega^*(X), d) \leftarrow (\Omega^{*,cl}(X), d^c) \rightarrow (\Omega^*(X), d)/(\Omega^{*,cl}(X), d^c)$$

and the differential on $(\Omega^*(X), d)/(\Omega^{*,cl}(X), d^c)$ vanishes.

Introduction: Example non-formality

Example

One example of a non-formal algebra is the minimal model for the three- dimensional Heisenberg nilmanifold $X = G/G_{\mathbb{Z}}$.
Indeed, if there exists a homomorphism

$$f : (\mathcal{M}_X, d) \to (H^*(X), 0)$$

that induces an isomorphism of the cohomology rings, then

$$f(x_1) \neq 0, f(x_3) = 0 \Rightarrow f(x_1 \wedge x_3) = 0.$$

But the element $x_1 \wedge x_3$ realizes a non-trivial cohomology class.

A_∞−algebras. Formal definition

definition

Let $\mathbb{K}$ be a field and $\mathcal{A}$ a $\mathbb{Z}$−graded $\mathbb{K}$−vector space, $\mathcal{A} = \oplus_{i\in\mathbb{Z}}\mathcal{A}^i$. An A_∞−algebra structure on $\mathcal{A}$ is a family of graded linear maps $m_n : \mathcal{A}^{\otimes n} \to \mathcal{A}$, $n \geq 1$, such that the degree of m_n is $2 - n$ and the identities ("Stasheff Identities"(SI(n)):

$$\sum_{\substack{r+s+p=n \\ r,p\geq 0; s\geq 1}} (-1)^{rs+p} m_{r+1+p} \circ (id^{\otimes r} \otimes m_s \otimes id^{\otimes p}) = 0$$

hold for all $n \geq 1$.

Koszul's sign rule: if we evaluate on specific elements of a tensor product space.

$$(f \otimes g)(x \otimes y) = (-1)^{\deg(g)\deg(x)} f(x) \otimes g(y),$$

where f and g are homogeneous maps and x, y are homogeneous elements in the domains of f and g respectively.

Some consequences of the definition

- If $n = 1, r, p = 0$ and $s = 1$, m_1 is a degree 1 map, and SI(1) is simply $m_1 \circ m_1 = 0$, that is, $(\mathcal{A}, m_1)$ is a cochain differential complex.
- If $n = 2$, identity SI(2) implies that $m_2 : \mathcal{A}^{\otimes 2} \to \mathcal{A}$ is a bilinear map that behaves like a multiplication and that the differential m_1 satisfies the graded Leibniz rule with respect to m_2.
- This multiplication is not necessarily associative, as it follows from SI(3), namely

$$m_2 \circ (m_2 \otimes id) - m_2 \circ (id \otimes m_2) = \quad (1)$$

$$m_1 \circ m_3 + m_3 \circ (id^{\otimes 2} \otimes m_1 + id \otimes m_1 \otimes id + m_1 \otimes id^{\otimes 2}). \quad (2)$$

- We note that if $m_3 = 0$, then $(\mathcal{A}, m_1, m_2)$ is a DGA with a differential of degree 1.
- Every DGA is an A_∞-algebra with $m_3 = m_4 = \ldots = 0$.

$A_{\infty}-$ morphisms

- An $A_{\infty}-$algebra is minimal if $m_1 = 0$;
- Let $\mathcal{A}$ and $\mathcal{B}$ be two $A_{\infty}-$ algebras; An $A_{\infty}-$morphism $f : \mathcal{A} \to \mathcal{B}-$ a family $f_n : \mathcal{A}^{\otimes n} \to \mathcal{B}$ of degree $1-n$ linear maps, $n \geq 1$ such that

$$\sum_{\substack{r+s+p=n \\ r,p\geq 0; s\geq 1}} (-1)^{rs+p} f_{r+1+p}(id^{\otimes r} \otimes m_s \otimes id^{\otimes p}) = \tag{3}$$

$$\sum_{\substack{0\leq r\leq n \\ n=i_1+\ldots+i_r}} (-1)^s m_r(f_{i_1} \otimes \ldots \otimes f_{i_r}), \tag{4}$$

being $s = \sum_{s-1}^{k=1} k(i_{r-k} - 1)$.

- A quasi-isomorphism $f_1 : (\mathcal{A}, m_{1,\mathcal{A}}) \to (\mathcal{B}, m_{1,\mathcal{B}})$ of cochain complexes is called $A_{\infty}-$algebra quasi-isomorphism.

Informal origins of A_∞

- Let us first consider the following *homotopy data* of chain complexes:

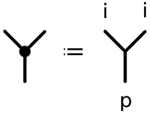
$$h \circlearrowright (\mathcal{A}, d_{\mathcal{A}}) \underset{i}{\overset{p}{\rightleftarrows}} (\mathcal{H}, d_{\mathcal{H}})$$

$$\mathrm{Id}_{\mathcal{A}} - ip = d_{\mathcal{A}} h + h d_{\mathcal{A}} \ ,$$

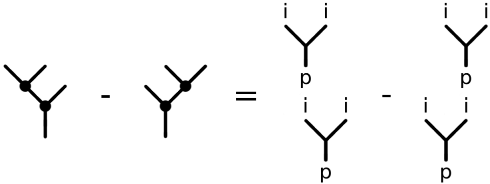
  where i and p are morphisms of chain complexes and where h is a degree $+1$ map. It is called a *homotopy retract*, when the map i is a quasi-isomorphism, i.e. when it realizes an isomorphism in homology. If moreover $pi = \mathrm{Id}_{\mathcal{H}}$, then it is called a *deformation retract*.
- Is it possible to transfer the associative algebra structure from $\mathcal{A}$ to $\mathcal{H}$ through the homotopy data in some way ?

Associativity

- $m : \mathcal{A}^{\otimes 2} \to \mathcal{A}$, such that $m(m(a \otimes b) \otimes c) = m(a \otimes m(b \otimes c))$.
- Define a binary operation $\mu : \mathcal{H}^{\otimes 2} \to \mathcal{H}$, such that $\mu := pmi^{\otimes 2}$

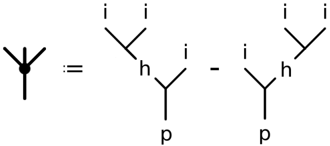


- If $I \in \mathrm{Iso}(\mathcal{H}, \mathcal{A})$ and $i^{-1} = p$ then answer: *yes*.
- But in general $ip = \mathcal{H}$ mod $h-$ homotopy. The associativity relation is equivalent to the vanishing of the associator in $\mathrm{Hom}(\mathcal{H}^{\otimes 3}, \mathcal{H})$



Associativity up to homotopy

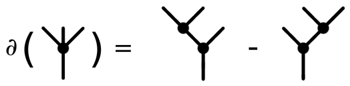
- This mapping space becomes a chain complex when equipped with the usual differential map $\partial(f) := d_{\mathcal{H}} f - (-1)^{|f|} d_{\mathcal{H}^{\otimes 3}} f$. We introduce the element μ_3 :



- $\deg(\mu_3) = 1$l, since the maps i, p, and m have degree 0 and h is a map of degree 1.

Proposition

The product μ_2 is associative up to the homotopy μ_3 in the chain complex $(\mathrm{Hom}(\mathcal{H}^{\otimes 3}, \mathcal{H}), \partial)$:



Kadeishvili transfer theorem

- The next step: to check whether μ_2 and μ_3 satisfy some relation. The answer is again yes, they satisfy one new relation but only up to yet another homotopy, which is a linear map μ_4 of degree $+2$ in $\mathrm{Hom}(\mathcal{H}^{\otimes 4}, \mathcal{H})$. And so on...
- Resume: $A_\infty-$ algebra or *an associative algebra up to homotopy*, also called *homotopy associative algebra* , is a chain complex $(\mathcal{A}, d)$ endowed with a family of operations $_n$: $\mathcal{A}^{\otimes n} \to \mathcal{A}$ of degree $n-2$ for any $n \geq 2$, satisfying the aforementioned relations.

Theorem

The operations $\{\mu_n\}, n \geq 2$ defined on $\mathcal{H}$ from the associative product m on $\mathcal{A}$ by the above formulae form an $A_\infty-$algebra structure on $\mathcal{H}$.

Merkulov's transfer theorem-assumptions

$\mathcal{A}, d, [,]-$ DGA and
$[\phi,\psi] = \phi\circ\psi - (-1)^{\mathrm{deg}\phi\mathrm{deg}\psi}\psi\circ\phi-$super-commutator;
Assumption:

- there exists a subcomplex $W \subset \mathcal{A}$, and a vector space homomorphism $Q : \mathcal{A} \to \mathcal{A}$ of degree -1, such that the image of the map $\mathrm{Id} - [d, Q] : \mathcal{A} \to \mathcal{A}$ is in W.
- Define $\lambda_n : \mathcal{A}^{\otimes n} \to \mathcal{A}$ of degree $2 - n$, $n \geq 1$, as follows:
- λ_1 is determined only by the condition $Q\lambda_1 = -\mathrm{Id}$;
-

$$\lambda_2(v \otimes w) = v \cdot w, \quad (5)$$

$$\lambda_n = \sum s + p = n; s, p \geq 1(-1)^{s+1}\lambda_2(Q\lambda_s \otimes Q\lambda_p), n \geq 2, \quad (6)$$

Merkulov's transfer theorem

Theorem (S. Merkulov)

Let $(\mathcal{A}, d)$ be a differential graded algebra and the **Assumption** *holds. Define linear maps $m_n : W^{\otimes n} \to W$, where $n \geq 1$, via*

1. $m_1 = d$,
2. $m_n = (\mathrm{Id} - [d, Q]) \circ \lambda_n$, *for* ≥ 2 , *in which λ_n are the maps constructed above. The maps m_n satisfy the identities SI(n), and therefore they determine an $A_\infty -$algebra structure on the complex W.*

Merkulov's theorem: details

Let $\mathcal{A} = \bigoplus_{p\in\mathbb{Z}} \mathcal{A}^p$ be a differential graded algebra with differential d of degree 1. B^p and Z^p - are the spaces of coboundaries and cocycles of $\mathcal{A}^p$ respectively. Then, there are subspaces H^p of Z^p and L^p of $\mathcal{A}^p$ such that

$$Z^p = B^p \oplus H^p \quad \text{and} \quad \mathcal{A}^p = Z^p \oplus L^p = B^p \oplus H^p \oplus L^p \, . \tag{7}$$

We set $W = \bigoplus_{p\in\mathbb{Z}} H^p$ and we define the map Q as follows:
$Q^p : \mathcal{A}^p \to \mathcal{A}^{p-1}$ is given by

$$Q^p|_{L^p} = Q^p|_{H^p} = 0 \, , \quad Q^p|_{B^p} = \left(d^{p-1}|_{L^{p-1}}\right)^{-1} \, .$$

We note that the map $d^{p-1}|_{L^{p-1}}$ is indeed one-to-one, since $d^{p-1}(a) = 0$ only if $a \in Z^{p-1}$, but $Z^{p-1} \cap L^{p-1} = \{0\}$.

Diagram shows how d acts:

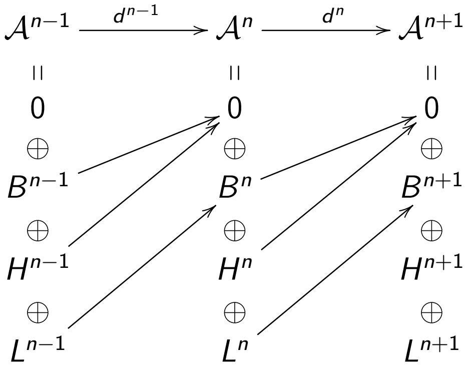
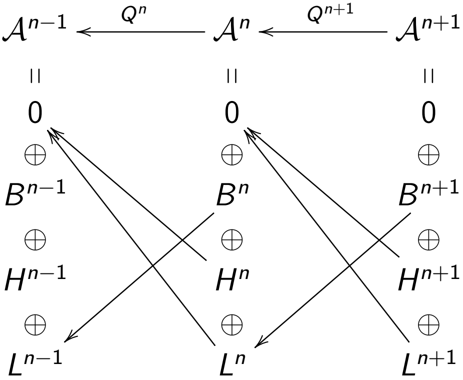


Diagram shows the action of Q:



Comments

The map Q determines an homotopy between Id and *pr*, where *pr* : $\mathcal{A} \to \mathcal{A}$ is the projection from $\mathcal{A}$ onto W: we have $\mathrm{Id} - pr = d\,Q + Q\,d$ and therefore Merkulov's **Assumption** holds with W and Q as above. Note that $d|_{H^p} = 0$, so, the operation m_1 of the Merkulov's Theorem is identically zero and therefore the operation m_2 is an *associative* multiplication on W. We identify the complex W with the cohomology of $\mathcal{A}$, and so hereafter we write $H\mathcal{A}$ instead of W.

Corollary

Theorem (4)

Consider the functions λ_n defined in (5,6).
We set $m_n = pr \circ \lambda_n : H\mathcal{A}^{\otimes n} \to H\mathcal{A}$ for $n \geq 2$. Then, $(H\mathcal{A}, 0, m_2, m_3, \dots)$ is an A_∞-algebra and $f = \{-Q\,\lambda_n\}_{n\geq 1}$ is a quasi-isomorphism of A_∞-algebras between $H\mathcal{A}$ and $\mathcal{A}$.

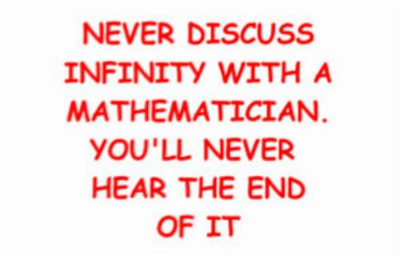
An A_∞-algebra constructed as above is called a *Merkulov model* or a *minimal model* of the DGA $\mathcal{A}$, in analogy with D. Sullivan's minimal models for DGA introduced in the context of rational homotopy theory. In the context of A_∞-algebras, being quasi-isomorphic is a *transitive* property, and therefore all Merkulov models of $\mathcal{A}$ (which obviously depend on the choice of the subspaces H^p and L^p introduced above) are quasi-isomorphic as A_∞-algebras.

Merkulov's minimal model: applications

- $M-$ compact Kähler, $\alpha, \beta \delta-$closed. But $\delta(\alpha \wedge \beta) \neq 0$
- How to cure it? - to define $\alpha \circ \beta := (\alpha \wedge \beta)|_{\mathrm{Ker}} \delta$. But $\circ-$ is no more associative.
- Merkulov: $\circ-$ *homotopy associative!*

There are various $A_\infty-$ structures on M :

1. real Hodge-de Rham - $W := \mathrm{Ker} d_c^*$, $Q := d_c \cdot G_d \cdot \Lambda$;
2. complex Hodge-Dolbeault -$W := \mathrm{Ker} \partial^*$, $Q := i\partial \cdot G_\partial \cdot \Lambda$;
3. If $M-$Calabi-Yau there is the third $A_\infty-$structure related to Barannikov-Kontsevich DGA



Happy and peaceful New Year!

Merry Xmass!

Happy Hanukkah!

Спасибо за внимание!