Coverings and multivector pseudosymmetries of differential equations

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Posing the problem: coverings over equations

$$\mathcal{E} : F(z, v, v_\sigma) = 0$$

$$v = (v^1, \dots, v^m), \qquad z = (z_1, \dots, z_n),$$
$$v_\sigma = \left(\frac{\partial^{|\sigma|} v^1}{\partial z_1^{i_1} \dots \partial z_n^{i_n}}, \dots, \frac{\partial^{|\sigma|} v^m}{\partial z_1^{i_1} \dots \partial z_n^{i_n}} \right), \quad \sigma = (i_1, \dots, i_n).$$

Posing the problem: coverings over equations

$$\tilde{\mathcal{E}} : \begin{cases} F(z, v, v_\sigma) = 0, \\ \frac{\partial w}{\partial z_i} = W_i(z, w, v, v_\sigma), \quad i = \overline{1, n}, \\ w = (w^1, \dots, w^q) \end{cases}$$

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The system $\tilde{\mathcal{E}}$ covers the system $\mathcal{E}$, ν is a covering over $\mathcal{E}$.

Posing the problem: coverings from equations

$$\mathcal{S} : G(x, u, u_\sigma) = 0, \qquad x = (x_1, \dots, x_n), \quad u = (u^1, \dots, u^{m_0})$$

$$(z, v, w) = \phi(x, u, u_\sigma) \downarrow \qquad \uparrow (x, u) = \phi^{-1}(z, v, w, v_\sigma)$$

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$$\mathcal{E} : F(z, v, v_\sigma) = 0,$$

ϕ is a $\mathcal{C}$-transformation, the system $\mathcal{S}$ covers the system $\mathcal{E}$, $\nu \circ \phi$ is a covering from $\mathcal{S}$, the fields $\partial/\partial w^i$ constitute a pseudosymmetry of $\mathcal{S}$, the system $\tilde{\mathcal{E}}$ is a decomposable form of the covering.

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Example 1: the Laplace equation

$$\mathcal{S}: \; u_{(2,0)} + u_{(0,2)} = 0, \quad u_{(2,0)}^2 + u_{(1,1)}^2 \neq 0,$$

the $\mathcal{C}$–transformation ϕ:

$$z_1 = u_{(1,0)}, \quad z_2 = u_{(0,1)}, \quad v = u, \quad w^1 = x_1, \quad w^2 = x_2,$$

the inverse transformation ϕ^{-1}: $\quad x_1 = w^1, \; x_2 = w^2, \; u = v,$
the decomposable form:

$$\tilde{\mathcal{E}}: \quad v_{(2,0)} + v_{(0,2)} = 0, \quad w^1_{(1,0)} = \frac{z_1 v_{(1,0)} - z_2 v_{(0,1)}}{z_1^2 + z_2^2} = -w^2_{(0,1)},$$

$$w^1_{(0,1)} = \frac{z_1 v_{(0,1)} + z_2 v_{(1,0)}}{z_1^2 + z_2^2} = w^2_{(1,0)},$$

the pseudosymmetry: $(\partial/\partial x_1, \partial/\partial x_2)$.

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The spaces of infinite jets

Consider an infinite-dimensional space J^∞ with coordinates

$$x_i,\ u^j,\ u^j_\sigma,\qquad i=\overline{1,n},\ j=\overline{1,m},\ |\sigma|\geq 0.$$

The total derivative with respect to x_i on J^∞:

$$D_i=\frac{\partial}{\partial x_i}+\sum_{j=1}^{m}\sum_{|\sigma|\geq 0}u^j_{\sigma+1_i}\frac{\partial}{\partial u^j_\sigma}.$$

The infinite prolongation of system $\mathcal{S}: G_\alpha(x,u,u_\sigma)=0, \alpha=\overline{1,r}$, is given by

$$\mathcal{S}^\infty:\quad D_\sigma G_\alpha=0,\qquad |\sigma|\geq 0,\quad \alpha=\overline{1,r},$$

where $D_\sigma=D_1^{i_1}\circ\ldots\circ D_n^{i_n}$ for $\sigma=(i_1,\ldots,i_n)$, $\mathcal{S}^\infty\subset J^\infty$.

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The Cartan distribution

The Cartan distribution on $\mathcal{S}^\infty$:

$$\mathcal{C}D(\mathcal{S}) = \mathrm{span}_{C^\infty(\mathcal{S}^\infty)}\{D_1\big|_{\mathcal{S}^\infty}, \ldots, D_n\big|_{\mathcal{S}^\infty}\}.$$

The Cartan forms on $\mathcal{S}^\infty$:

$$d_{\mathcal{C}}g = \mathrm{d}g - \sum_{i=1}^{n} D_i(g)\mathrm{d}x_i, \quad g \in C^\infty(\mathcal{S}^\infty).$$

$$\mathcal{C}\Lambda^1(\mathcal{S}) = \mathrm{span}_{C^\infty(\mathcal{S}^\infty)}\{d_{\mathcal{C}}g \,:\, g \in C^\infty(\mathcal{S}^\infty)\}.$$

$$\mathcal{C}^k\Lambda^k(\mathcal{S}) = \underbrace{\mathcal{C}\Lambda^1(\mathcal{S}) \wedge \cdots \wedge \mathcal{C}\Lambda^1(\mathcal{S})}_{k \text{ times}}.$$

$$\omega \in \mathcal{C}^k\Lambda^k(\mathcal{S}) \quad \Longleftrightarrow \quad D_i\big|_{\mathcal{S}^\infty} \rfloor \omega = 0, \quad i = \overline{1, n}.$$

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Mappings of diffieties

A mapping $\phi : \mathcal{S}^\infty \longrightarrow \mathcal{E}^\infty$ is $\mathcal{C}$–transformation if

- ϕ is smooth, i.e., $\phi^*\big(C^\infty(\mathcal{E}^\infty)\big)\subset C^\infty(\mathcal{S}^\infty)$, $\phi^*(g) = g \circ \phi$;
- there exists a smooth inverse mapping;
- $\phi_*(\mathcal{C}D(\mathcal{S})) = \mathcal{C}D(\mathcal{E})$.

A smooth mapping $\nu : \mathcal{S}^\infty \longrightarrow \mathcal{E}^\infty$ is a covering if

- the tangent mapping $\nu_{*,\theta}$ is a vector space epimorphism for any $\theta \in \mathcal{S}^\infty$;
- $\nu_*(\mathcal{C}D(\mathcal{S})) = \mathcal{C}D(\mathcal{E})$;
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Conditions for fibers of coverings

$$F_\beta(z, v, \dots, v_\sigma, \dots) = 0, \quad \beta = \overline{1, r_1}, \tag{1}$$

$$\frac{\partial w^j}{\partial z_i} = W_i^j(z, w, v, \dots, v_\sigma, \dots), \quad i = \overline{1, n},\ j = \overline{1, q}, \tag{2}$$

The total derivative with respect to x_i for system (1)–(2) has the form

$$D_i = \check{D}_i + \sum_{j=1}^{q} W_i^j(x, w, u, \dots, u_\sigma, \dots) \frac{\partial}{\partial w^j},$$

where $\check{D}_i$ is the total derivative w.r.t. x_i for (1).

$$[X, D_i] = A_i X, \qquad i = \overline{1, n}, \tag{3}$$

where $X = \left(\partial/\partial w^1, \dots, \partial/\partial w^q\right)^{\mathrm{T}}, A_i = \left(\partial W_i^j/\partial w^s\right)_{s,j=\overline{1,q}}.$

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Pseudosymmetries of differential equations

The invariant version of (3):

$$[X, D_i] = A_i X + B_i D, \qquad i = \overline{1, n}, \tag{4}$$

where $D = (D_1, \ldots, D_n)^{\mathrm{T}}$. A column X of vector fields satisfying (4) is a pseudosymmetry of the system $\mathcal{S}$. The relations (4) and the Jacobi identity imply:

$$\begin{aligned} D_j(A_i) - D_i(A_j) + A_j A_i - A_i A_j &= 0, \qquad \forall i, j, \\ D_j(B_i) - D_i(B_j) + A_j B_i - A_i B_j &= 0, \qquad \forall i, j. \end{aligned} \tag{5}$$

Equations (5) are the Maurer–Cartan equations.

$$(5) \quad \Longrightarrow \quad [D_i + A_i, D_j + A_j] = 0.$$

If $A_i \equiv 0$ for all i, then X is a column of higher symmetries.

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Pseudosymmetries of infinite jets spaces

Theorem 1. A q-column X is a pseudosymmetry of J^∞ iff

$$X = \ni_{\varphi,A} + MD,$$

where φ is a $q \times m$ matrix of functions on J^∞, $A = (A_1, \ldots, A_n)$ is a tuple of $q \times q$ matrices satisfying (5), M is a $q \times n$ matrix of arbitrary functions on J^∞, the term $\ni_{\varphi,A}$ has the form

$$\ni_{\varphi,A} = \sum_\sigma (D + A)^\sigma(\varphi)\frac{\partial}{\partial u_\sigma},$$

where $(D + A)^\sigma = (D_1 + A_1)^{i_1} \circ \ldots \circ (D_n + A_n)^{i_n}$, $\sigma = (i_1, \ldots, i_n)$, is a $(q \times q)$ matrix differential operator, $\dfrac{\partial}{\partial u_\sigma} = \left(\dfrac{\partial}{\partial u^1_\sigma}, \ldots, \dfrac{\partial}{\partial u^m_\sigma}\right)^{\mathrm{T}}$.

Defining equations for pseudosymmetries

Theorem 2. Let

$$\mathcal{S} : G(x, u, u_\sigma) = 0$$

be a formally integrable system. Then a column X is a pseudosymmetry of $\mathcal{S}$ iff X is the restriction of $\ni_{\varphi,A} + MD$ to $\mathcal{S}^\infty$, where the matrix $\varphi = (\varphi_{ij})$ satisfies

$$\sum_{j,\sigma} \frac{\partial G}{\partial u^j_\sigma} (D + A)^\sigma (\varphi^j)\Big|_{\mathcal{S}^\infty} = 0, \quad \varphi^j = (\varphi_{1j}, \ldots, \varphi_{qj})^{\mathrm{T}}.$$

The matrix φ is the generating matrix of the pseudosymmetry, and $\ni_{\varphi,A}$ is the evolution pseudosymmetry.

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Integrable pseudosymmetries

A column X of vector fields $X_1, \dots, X_q$ on $\mathcal{S}^\infty$ is an integrable pseudosymmetry of the system $\mathcal{S}$ if

- $X = (X_1, \dots, X_q)^{\mathrm{T}}$ is a pseudosymmetry of $\mathcal{S}$;
- $X_1, \dots, X_q$ generate an involutive distribution of dimension q;
- the finiteness condition: there exists a ring $\mathcal{K}$ such that
  (a) $\mathcal{F}_0(\mathcal{S}) \subset \mathcal{K} \subset \mathcal{F}_l(\mathcal{S})$ for some $l \geq 0$;
  (b) $X_i(\mathcal{K}) \subset \mathcal{K}$ for any $i = \overline{1, q}$.

Here $\mathcal{F}_i(\mathcal{S})$ consists of smooth functions of $t, u, \dots, u_\sigma, |\sigma| \leq i$.

Example 2.

The fields $\partial/\partial w^1, \dots, \partial/\partial w^q$ form an integrable pseudosymmetry of system (1)–(2), $\mathcal{K}$ is the ring of smooth functions of $t, w, v, \dots, v_\sigma, |\sigma| \leq s$, where s is the order of system (2).

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Properties of integrable pseudosymmetries

- The distribution generated by $X_1, \ldots, X_q$ is integrable.
- The distribution generated by $X_1, \ldots, X_q, D_1, \ldots, D_n$ is involutive, but is not integrable.
- Any finite-dimensional Lie algebra of classical symmetries of a system forms its integrable pseudosymmetry.
- Any $\mathcal{C}$-transformation maps integrable pseudosymmetries into integrable pseudosymmetries.

An integrable pseudosymmetry $X = (X_1, \ldots, X_q)^{\mathrm{T}}$ determines a covering if the fibers of the covering coincide with the maximal integral manifolds of the distribution generated by $X_1, \ldots, X_q$.

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Description of coverings

Let $\mathcal{S}$ be a formally integrable system, and let $X = (X_1, \dots, X_q)^{\mathrm{T}}$ be an integrable pseudosymmetry of $\mathcal{S}$. A point $\theta \in \mathcal{S}^\infty$ is a regular point of X if

- the fields $X_1, \dots, X_q$, $D_1, \dots, D_n$ are linearly independent at θ;
- the subspaces $\{df|_{\theta'} \in T^*_{\theta'} : f \in \mathcal{K}\}$ and $\{df|_{\theta'} \in T^*_{\theta'} : f \in D^l\mathcal{K}\}$ have constant dimension in some neighborhood of θ, where $D^l\mathcal{K}$ is the ring generated by the functions $D_\sigma(f)$, $f \in \mathcal{K}$, $|\sigma| \le l$, $\mathcal{F}_0(\mathcal{S}) \subset \mathcal{K} \subset \mathcal{F}_l(\mathcal{S})$.

Theorem 3.

- If X is an integrable pseudosymmetry of a system $\mathcal{S}$ and $\theta \in \mathcal{S}^\infty$ is a regular point of X, then X determines a covering from a neighborhood of θ.
- For any covering from a system $\mathcal{S}$ there exists an integrable pseudosymmetry of $\mathcal{S}$ determining this covering.

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Algorithm for constructing coverings

Let $X = (X_1, \ldots, X_q)^{\mathrm{T}}$ be an integrable pseudosymmetry of a system $\mathcal{S}$ and let θ be a regular point of X.

Step 1. Find an integer s such that the coefficients of matrices $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ belong to $D^s\mathcal{K}$.

Step 2. Find functions $g_1, \ldots, g_p$ such that

1. $g_1, \ldots, g_p \in D^s\mathcal{K}$;
2. $g_1, \ldots, g_p$ are common first integrals of $X_1, \ldots, X_q$;
3. $dg_1, \ldots, dg_p$ are linearly independent at each point;
4. The set $\{g_1, \ldots, g_p\}$ is a maximal set satisfying 1-3.

Step 3. Find functions $w_1, \ldots, w_q \in \mathcal{F}(\mathcal{S})$ such that the matrix $\big(X_i(w_j)(\theta)\big)$ is nonsingular.

Step 4. Choose new independent variables $x_1, \ldots, x_n$ among the functions $g_1, \ldots, g_p$ such that the matrix $\big(D_i(x_j)(\theta)\big)$ is nonsingular. By u denote the set of the functions $g_1, \ldots, g_p$ that are not $x_1, \ldots, x_n$.

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Integrable pseudosymmetries

Theorem 4. Suppose a column $X = (X_1, \ldots, X_q)^{\mathrm{T}}$ is an integrable pseudosymmetry of a system $\mathcal{S}$ and N is a nonsingular $q \times q$ matrix of functions on $\mathcal{S}^\infty$. Then

- the product NX is also an integrable pseudosymmetry of the system $\mathcal{S}$;
- the integrable pseudosymmetries X and NX determine the same covering;
- if matrices A_i and B_i correspond to X (see (4)), then the matrices $NA_iN^{-1} - D_i(N)N^{-1}$ and NB_i correspond to NX.

Methods for solving equations for pseudosymmetries are similar to methods for solving equations for higher symmetries. To find integrable pseudosymmetries, these methods should be combined with using Theorem 4 and the finiteness condition.

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Commutation of pseudosymmetries

Let X and Y be two pseudosymmetries of a system $\mathcal{S}$. Denote by $[X \cup Y]$ the set of generators of the involutive closure of the union of the sets X and Y of vector fields.

Theorem 5.

If X and Y are two pseudosymmetries of a system $\mathcal{S}$ and the set $[X \cup Y]$ is finite, then this set also forms a pseudosymmetry of the system $\mathcal{S}$.

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Multivector fields and their Schouten bracket

The r-th exterior degree of the module $\mathcal{D}(\mathcal{S}^\infty)$ of vector fields on $\mathcal{S}^\infty$:

$$\bigwedge^r \mathcal{D}(\mathcal{S}^\infty) = \operatorname{span}_{\mathcal{F}(\mathcal{S})}\{X_1 \wedge \dots \wedge X_r \,:\, X_1, \dots, X_r \in \mathcal{D}(\mathcal{S}^\infty)\}.$$

Elements of the module $\bigwedge^* \mathcal{D}(\mathcal{S}^\infty) = \bigoplus_{r>0} \bigwedge^r \mathcal{D}(\mathcal{S}^\infty)$ are multivector fields on $\mathcal{S}^\infty$. An element $X \in \bigwedge^r \mathcal{D}(\mathcal{S}^\infty)$ is a homogeneous element of degree $|X| = r$.

The Schouten bracket of $X = \bigwedge_{\alpha=1}^s X_\alpha$ and $Y = \bigwedge_{\beta=1}^k Y_\beta$ is

$$[X, Y] = \sum_{\alpha=1}^{s} \sum_{\beta=1}^{k} (-1)^{\alpha+\beta} [X_\alpha, Y_\beta] \wedge X_1 \wedge \dots \wedge \widehat{X_\alpha} \wedge \dots \wedge X_s \wedge Y_1 \wedge \dots \wedge \widehat{Y_\beta} \wedge \dots \wedge Y_k,$$

This bracket is extended to $\bigwedge^* \mathcal{D}(\mathcal{S}^\infty)$ by linearity.

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Properties of the Schouten bracket

Theorem 6. The following identities are valid for homogeneous elements X, Y, Z:

- $[X, Y] = -(-1)^{(|X|-1)(|Y|-1)}[Y, X]$ (antisymmetry of Schouten bracket);
- $(-1)^{(|X|-1)(|Z|-1)}[X, [Y, Z]] + (-1)^{(|Y|-1)(|X|-1)}[Y, [Z, X]] + (-1)^{(|Z|-1)(|Y|-1)}[Z, [X, Y]] = 0$ (Jacobi identity for Schouten bracket);
- $[X \wedge Y, Z] = (-1)^{|Y|(|Z|-1)}[X, Z] \wedge Y + X \wedge [Y, Z]$ (Poisson identity).

The Schouten bracket makes the module $\bigwedge^* \mathcal{D}(\mathcal{S}^\infty)$ into a Lie superalgebra if the degree of a homogeneous element $X \in \bigwedge^r \mathcal{D}(\mathcal{S}^\infty)$ is taken to be the integer $r - 1$.

Multivector pseudosymmetries

Theorem 7. If $(X_1, \dots, X_q)^{\mathrm{T}}$ is a pseudosymmetry of system $\mathcal{S}$, then $X = X_1 \wedge \dots \wedge X_q$ satisfies the equalities

$$[X, D_i] = a_{i,X}X + \sum_{j=1}^{n} Z_{ij,X} \wedge D_j, \qquad i = \overline{1, n}, \tag{6}$$

for some $a_{i,X} \in C^\infty(\mathcal{S}^\infty)$ and $Z_{ij,X} \in \bigwedge^{q-1} \mathcal{D}(\mathcal{S}^\infty)$.

A multivector field X on $\mathcal{S}^\infty$ satisfying the equality (6) is a multivector pseudosymmetry of system $\mathcal{S}$.

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The interior product of a multivector field $X = X_1 \wedge \cdots \wedge X_r$ and a p-form ω, $p \geq r$:

$$i_X(\omega) = (i_{X_r} \circ \cdots \circ i_{X_1})(\omega),$$

where i_{X_i} is the interior product of the vector field X_i.

Theorem 9.

$$i_X \circ Z - Z \circ i_X = i_{[X,Z]}, \quad X \in \bigwedge^* \mathcal{D}(\mathcal{S}^\infty), \quad Z \in \mathcal{D}(\mathcal{S}^\infty).$$

Theorem 10. If X is a multivector pseudosymmetry of a system $\mathcal{S}$, $q = |X|$, $\omega \in \mathcal{C}^q\Lambda^q(\mathcal{S}^\infty)$, then

$$i_X(D_l\omega) = (D_l + a_{l,X})(i_X\omega).$$

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Multivector pseudosymmetries

A scalar $\mathcal{C}$-differential operator of $\mathcal{C}^q\Lambda^q(\mathcal{S}^\infty)$ to $\mathcal{C}^q\Lambda^q(\mathcal{S}^\infty)$:

$$\omega \mapsto \sum_{|\sigma|\leq k} a_\sigma D_\sigma(\omega), \quad a_\sigma \in \mathcal{F}(\mathcal{S}).$$

The set of all scalar $\mathcal{C}$-differential operators is a noncommutative ring with respect to the composition operation, $\mathcal{C}^q\Lambda^q(\mathcal{S}^\infty)$ is a left module over this ring.

Theorem 10 $\quad\Longrightarrow\quad$ a multivector pseudosymmetry X of a system $\mathcal{S}$ is uniquely determined by its values at generators of the module $\mathcal{C}^q\Lambda^q(\mathcal{S}^\infty)$, $q = |X|$.

Example 3. For the Laplace equation $u_{x_1x_1} + u_{x_2x_2} = 0$ generators of the module $\mathcal{C}^2\Lambda^2(\mathcal{S}^\infty)$ are

$$\omega_1^k = -d_{\mathcal{C}} u_{(k,0)} \wedge d_{\mathcal{C}} u_{(k,1)}, \quad \omega_2^k = d_{\mathcal{C}} u_{(k,0)} \wedge d_{\mathcal{C}} u_{(k+1,0)}, \quad k \geq 0.$$

They are related by the equations

$$D_2(\omega_1^k) = D_1(\omega_2^k), \quad \Delta^k\big(D_1(\omega_1^0) + D_2(\omega_2^0)\big) = c_k\big(D_1(\omega_1^k) + D_2(\omega_2^k)\big),$$

where $\{c_k\}$ is some sequence of integers, $\Delta = D_1^2 + D_2^2$.
If X is the multivector pseudosymmetry such that

$$i_X(\omega_1^0) = x_2, \quad i_X(\omega_2^0) = x_1, \quad i_X(\omega_i^k) = 0, \quad a_{i,X} = 0, \quad Z_{ij,X} = 0$$

for any $k > 0$, $i, j = 1, 2$. Then

$$X = x_1 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial u_{(1,0)}} - x_2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial u_{(0,1)}} + \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial u_{(2,0)}}.$$

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Constructing coverings determined by a multivector pseudosymmetry

The Lie derivative of $g \in \mathcal{F}(\mathcal{S})$ along $X = X_1 \wedge \cdots \wedge X_q$:

$$X(g) = \sum_{i=1}^{q} (-1)^{q-i} X_i(g) X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_q.$$

A function $g \in \mathcal{F}(\mathcal{S})$ is a first integral of X if $X(g) \equiv 0$.

$$(X_1 \wedge \cdots \wedge X_q)(g) \equiv 0 \quad \Longleftrightarrow \quad X_i(g) \equiv 0, \quad i = \overline{1, q}.$$

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Example 4. For

$$X = x_1 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial u_{(1,0)}} - x_2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial u_{(0,1)}} + \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial u_{(2,0)}}$$

the corresponding decomposable form is

$$(z_2 - z_1)(z_2 v^1_{(1,0)} - z_1 v^1_{(0,1)}) = (z_2^2 + z_1^2) v^2 - (z_2 + z_1) v^1,$$
$$w^1_{(1,0)} = \frac{v^1 - z_1 w^2}{z_2 - z_1}, \qquad w^1_{(0,1)} = \frac{v^1 - z_2 w^2}{z_1 - z_2},$$

the corresponding change of variables is

$$z_1 = x_1, \quad z_2 = x_2, \quad w^1 = u, \quad w^2 = u_{(1,0)} + u_{(0,1)},$$
$$v^1 = x_2 u_{(1,0)} + x_1 u_{(0,1)}, \quad v^2 = u_{(1,0)} + u_{(0,1)} + (x_2 - x_1) u_{(2,0)}.$$

Example 5. In the situation of Example 3, but when $i_X(\omega^1_2) = -1$ and $i_X(\omega^k_i) = 0$ for other k, i, we have

$$X = \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial u_{(3,0)}} - \frac{\partial}{\partial u_{(1,0)}} \wedge \frac{\partial}{\partial u_{(2,0)}} + \frac{\partial}{\partial u_{(0,1)}} \wedge \frac{\partial}{\partial u_{(1,1)}}.$$

The decomposable form for X is

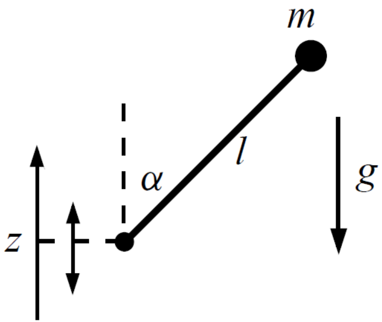
$$v^1_{(1,0)} = v^2_{(0,1)}, \quad v^1_{(0,1)} = -v^2_{(1,0)}, \quad w_{(2,1)} = v^1, \quad w_{(4,0)} = v^2.$$

The change of variables is

$$z_1 = x_1, \quad z_2 = x_2, \quad v^1 = u_{(2,1)}, \quad v^2 = u_{(4,0)}, \quad w = u.$$

Thus the dimension of a covering can be greater than the degree of the corresponding multivector pseudosymmetry.

Example 6: the Kapitsa pendulum



$$\dot{\alpha} = p + \frac{u}{l}\sin\alpha, \quad \dot{z} = u, \quad \dot{p} = \left(\frac{g}{l} - \frac{u^2}{l^2}\cos\alpha\right)\sin\alpha - \frac{u}{l}p\cos\alpha.$$

This system is equivalent to the system

$$\ddot{y} = a(\alpha)\,\dot{\alpha}^2, \quad a(\alpha) = l\,\frac{\cos\alpha}{\sin^2\alpha},$$

where

$$y = z + g\frac{t^2}{2} - l\int\frac{d\alpha}{\sin\alpha}.$$

Let us compute all the one-dimensional integrable evolution pseudosymmetries $\ni_{\varphi,A}$ of this system.

Denote by $\mathcal{F}_s$ is the ring of smooth functions depending on $t, y, \alpha, \dot{y}, \dot{\alpha}, \ldots, \alpha^{(s)}$ for $s \geq 0$. The defining equation for pseudosymmetries is

$$(D_t + A)^2(\varphi_y) = 2a\,\dot{\alpha}\,(D_t + A)(\varphi_\alpha) + a'\,\dot{\alpha}^2\,\varphi_\alpha,$$

where $\varphi = (\varphi_y, \varphi_\alpha)^T$ is a generating matrix and $A, \varphi_y, \varphi_\alpha \in \mathcal{F}_s$ for some $s \geq 0$.

The three cases are possible:

I) $\varphi_y \not\equiv 0,\ \varphi_\alpha \not\equiv 0;$

II) $\varphi_y \not\equiv 0,\ \varphi_\alpha \equiv 0;$

III) $\varphi_y \equiv 0,\ \varphi_\alpha \not\equiv 0.$

In case I) the column $(1, \psi = \varphi_\alpha/\varphi_y)^T$ is also a generating matrix of an integrable pseudosymmetry and

$$D_t(A) + A^2 = 2a(\alpha)\,\dot{\alpha}\,(D_t + A)(\psi) + a'(\alpha)\,\dot{\alpha}^2\,\psi. \tag{7}$$

$$\psi \in \mathcal{F}_s \setminus \mathcal{F}_{s-1} \quad \Longrightarrow \quad A \in \mathcal{F}_s \setminus \mathcal{F}_{s-1}.$$

$$\ni_{\psi,A} = \cdots + \sum_{k \geq 0} (D_J + A)^k(\psi) \frac{\partial}{\partial \alpha^{(k)}} \quad \Longrightarrow$$

$\ni_{\psi,A} \colon \ \mathcal{F}_k \to \mathcal{F}_{s+k} \quad \Longrightarrow \quad s = 0$ (from the finiteness condition).

Thus, ψ and A are functions of $t, y, \alpha, \dot{y}$ only.

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$$\ni_{\psi,A} \colon\ \mathcal{F}_k \to \mathcal{F}_{s+k} \quad \Longrightarrow\ s = 0 \text{ (from the finiteness condition)}.$$

Thus, ψ and A are functions of $t, y, \alpha, \dot y$ only.

Finding successively the coefficients at $\dot\alpha^3, \dot\alpha^2, \dot y^2, \dot y\dot\alpha, \dot\alpha$, and $\dot y$ in (7), we get

$$\psi = \frac{b + c_1}{2\sqrt{a}(y + c_2)},\ A = \frac{\dot y}{y + c_2},\ b = \int \sqrt{a}d\alpha,$$

where c_1, c_2 are arbitrary constants.

Similarly, in the cases II) and III) we get the following pseudosymmetries:

II) $\varphi_y = 1,\ \varphi_\alpha = 0,\ A = \frac{1}{t+c_3},\ c_3 = \mathrm{const}$, or
$\varphi_y = 1,\ \varphi_\alpha = 0,\ A = 0;$

III) $\varphi_y = 0,\ \varphi_\alpha = 1,\ A = -\frac{a'\,\dot\alpha}{2a}.$

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These pseudosymmetries define the following decomposable forms:

1) $\begin{cases} \dot v_2 = (\dot v_1^2 + \frac{1}{2} v_1 v_2)^2 - {v_2}^2 \\ \dot w = v_2(w + c_2), \end{cases}$ $\quad v_1 = \frac{\beta}{\sqrt{y+c_2}}, \quad v_2 = \frac{\dot y}{y+c_2}, \quad w = y;$

2) $\begin{cases} \dot v_1 = -(t + c_3)a(v_2)\dot v_2^2 \\ \dot w = \frac{w - v_1}{t + c_3}, \end{cases}$ $\quad v_1 = y - \dot y(t + c_3), \quad v_2 = \alpha, \quad w = y;$

$\begin{cases} \dot v_1 = a(v_2)\dot v_2^2 \\ \dot w = v_1, \end{cases}$ $\quad v_1 = \dot y, \quad v_2 = \alpha, \quad w = y;$

3) $\dot w = \sqrt{\frac{\ddot v}{a(w)}},$ $\quad v = y, \quad w = \alpha.$

Commutation of the obtained pseudosymmetries

Let $\ni_1$ and $\ni_3$ be the pseudosymmetries corresponding to cases (1) and (3), and let $\ni_2$ and $\ni_4$ be the pseudosymmetry corresponding to case (2). Then

$$[\ni_1, \ni_2] = \frac{1}{y + c_2}(\ni_1 - \ni_2), \quad [\ni_1, \ni_3] = \frac{h(\alpha)}{y + c_2}\ni_3,$$
$$[\ni_1, \ni_4] = \frac{1}{y + c_2}(\ni_1 - \ni_4), \quad [\ni_2, \ni_3] = 0, \quad [\ni_2, \ni_4] = 0,$$
$$[\ni_3, \ni_4] = 0, \quad h(\alpha) = \frac{(b + c_1)a'}{4a^{3/2}} - \frac{1}{2}.$$

Thus, any nonempty subset of the tuple $\{\ni_1, \ni_2, \ni_3, \ni_4\}$ generates an integrable pseudosymmetry of the system.

We obtain 6 two-dimensional, 4 three-dimensional, and 1 four-dimensional integrable pseudosymmetries and 11 coverings from the system. The decomposable forms corresponding to the pseudosymmetries $(\ni_1, \ni_3)$, $(\ni_1, \ni_2, \ni_3)$, and $(\ni_1, \ni_2, \ni_3, \ni_4)$ are:

$$\begin{cases} \dot{w}_1 = v(w_1 + c_2) \\ \dot{w}_2 = \sqrt{\frac{(\dot{v}+v^2)(w_1+c_2)}{a(w_2)}}, \end{cases} \qquad v = \frac{\dot{y}}{y+c_2}, \quad w_1 = y, \quad w_2 = \alpha;$$

$$\begin{cases} \ddot{w}_1 = \frac{\dot{w}_1(t+c_3)-w_1-c_2}{v} \\ \dot{w}_2 = \sqrt{\frac{\dot{w}_1(t+c_3)-w_1-c_2}{va(w_2)}}, \end{cases} \qquad v = \frac{\dot{y}(t+c_3)-y-c_2}{a(\alpha)\,\dot{\alpha}^2}, \quad w_1 = y, \quad w_2 = \alpha;$$

$$\begin{cases} \dddot{w}_1 = v\ddot{w}_1 \\ \dot{w}_2 = \sqrt{\frac{\ddot{w}_1}{a(w_2)}}, \end{cases} \qquad v = \frac{\dddot{y}}{\ddot{y}}, \quad w_1 = y, \quad w_2 = \alpha.$$

THANK YOU
FOR YOUR ATTENTION