New perspectives for generalized Kadomtsev-Petviashvili hierarchies

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Presentation of the talk

Plan of the talk:

1. From differential diffeological algebras to Mulase groups
   - Diffeologies and differential algebras
   - $\Psi DO(A)$ and its diffeology, and more...
   - Mulase groups revisited.
2. Aspects of the generalized KP hierarchy and its solution
   - Statement of well-posedness
   - On the Sato operator
   - On some Hamiltonian formulation
3. Deducing the generalized KP equation from the generalized KP hierarchy.
   - The Zakharov-Shabat equations
   - The beginning of a long list of generalized KP equations...

A primer on diffeologies (1)

Definition

Let X be a set.

• A **p-parametrization** of dimension p on X is a map from an open subset O of $\mathbb{R}^p$ to X.

• A **diffeology** on X is a set $\mathcal{P}$ of parametrizations on X such that:

- For each $p \in \mathbb{N}$, any constant map $\mathbb{R}^p \to X$ is in $\mathcal{P}$;

- For each arbitrary set of indexes I and family $\{f_i : O_i \to X\}_{i \in I}$ of compatible maps that extend to a map $f : \bigcup_{i \in I} O_i \to X$, if $\{f_i : O_i \to X\}_{i \in I} \subset \mathcal{P}$, then $f \in \mathcal{P}$.

- For each $f \in \mathcal{P}$, $f : O \subset \mathbb{R}^p \to X$, and $g : O' \subset \mathbb{R}^q \to O$, in which g is a smooth map (in the usual sense) from an open set $O' \subset \mathbb{R}^q$ to O, we have $f \circ g \in \mathcal{P}$.

A pair $(X, \mathcal{P})$ is called diffeological space.

A primer on diffeologies (2)

If $(X, \mathcal{P})$ and $(X', \mathcal{P}')$ are two diffeological spaces, a map $f : X \to X'$ is **smooth** if and only if $f \circ \mathcal{P} \subset \mathcal{P}'$.

- Let $f : X \to (Y, \mathcal{P})$. the biggest diffeology on X for which f is smooth is the pull-back diffeology $f^*(\mathcal{P})$.
- Let $X = \prod_{i \in I} X_i$ with canonical projections π_i. If $\{(X_i, p_i) \mid i \in I\}$ are diffeological spaces, $\bigcap_{i \in I} \pi_i^*(\mathcal{P}_i)$ is the product diffeology
- let $i : Y \hookrightarrow (X, \mathcal{P})$ be a set inclusion map. $i^*(\mathcal{P})$ is the subset diffeology on Y.
- Any algebraic operation can be defined as smooth on diffeological spaces, which defines: diffeological monoids, diffeological groups, diffeological fields and rings, diffeological vector spaces, diffeological algebras, **diffeological differential algebras**...

Examples of useful diffeological spaces

- locally convex (maybe infinite dimensional) smooth manifolds
- non commutative torus
- algebraic varieties
- diffieties
- CW complexes
- groups of diffeomorphisms of any (maybe infinite dimensional) locally convex smooth manifold
- spaces of functions with low regularity e.g. $H^{1/2}(S^1, N)$

As an important class of examples, **Frölicher spaces** form a subcategory of the category of diffeological spaces.

The diffeological algebra $\Psi DO(A)$

Let $\mathbb{K}$ be a characteristic zero diffeological field and let $(A, +, *, \partial)$ be a $\mathbb{K}-$differential diffeological unital associative algebra.

Definition

$$\Psi DO(A) = \left\{ \sum_{n \in \mathbb{Z}} a_n \xi^n : a_n \in A; a_n = 0 \text{ for } n >> 0 \right\}$$

where ξ is a formal variable.

$\Psi DO(A) \subset A^{\mathbb{Z}}$ is a diffeological space and the classical splitting $\Psi DO(A) = DO(A) \oplus IO(A)$ is smooth.

The diffeological algebra $\Psi DO(A)$ (2)

Proposition

Let us define addition and multiplication on $\Psi DO(A)$. Let $a(\xi) = \sum a_n \xi^n$ and $b(\xi) = \sum b_n \xi^n$.

$$(a(\xi), b(\xi)) \mapsto \sum_{p \in \mathbb{Z}} (a_p + b_p)\xi^p \text{ and}$$

$$(a(\xi), b(\xi)) \mapsto \sum_{m,n} \sum_{\alpha \in \mathbb{N}} \frac{1}{\alpha!} (a_n \partial^\alpha b_m)(D_\xi^\alpha \xi^n)\xi^m$$

are smooth.

Let us recall that inversion is smooth in A^*, which implies:

Theorem

$(\Psi DO(A), +, *)$ *is a diffeological unital (associative)* $\mathbb{K}-$*algebra.*

On Mulase groups of PDOs (1)

Let $T = (t_1, t_2, \ldots)$ be a family of (independent) formal variables, for which we set $val_t(t_n) = n$. We set $\widehat{A} = A[[T]]$. This is also a (unital) differential diffeological algebra and $\Psi DO(\widehat{A})$ is a unital diffeological algebra.

Theorem

The algebras
$\widehat{\Psi} =$
$\left\{\sum_{\alpha \in \mathbb{N}} a_\alpha \partial^\alpha : \exists (C, N) \in \mathbb{R} \times \mathbb{N} \text{ so that } val_t(a_\alpha) > C\alpha - N \; \forall \; \alpha \gg 0\right\}$
and
$\widehat{\mathcal{D}} = \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha : P \in \widehat{\Psi} \text{ and } a_\alpha = 0 \text{ for } \alpha < 0 \right\}$ *are diffeological*

On Mulase groups of PDOs (2)

Theorem

Setting $G = 1 + \Psi DO^{-1}(\widehat{A})$, G *is a diffeological Lie group with Lie algebra* $\Psi DO^{-1}(\widehat{A})$.
Moreover, the groups
$\widehat{\Psi}^* = \left\{ \sum_{\alpha \in \mathbb{N}} a_\alpha \, \partial^\alpha \in \widehat{\Psi} \,|\, a_0 \in G \right\}$
and
$\widehat{\mathcal{D}}^* = \left\{ \sum_{\alpha \in \mathbb{N}} a_\alpha \, \partial^\alpha \in \widehat{\mathcal{D}} \,|\, a_0 = 1 \right\}$ *are diffeological groups and there is a decomposition into a diffeological bicross product* $\widehat{\Psi}^* = G \bowtie \widehat{\mathcal{D}}^*$:

$$U = S^{-1} Y$$

(the map $U \mapsto S^{-1} Y$ *is smooth)*

On Mulase groups of PDOs (3)

Digression: A diffeological group may not have a well-defined Lie algebra, in the kinematic approach, that is, considering germs of smooth paths.
If they have one, they are called diffeological Lie groups.
Actually, $\widehat{\Psi}^*$ is a good candidate to be the first diffeological group to be proven to be a diffeological (non-Lie) group.

Precision: Here, the existence of the exponential map is not assumed for the definition of the Lie algebra, while another approach, via formal exponential maps, leads to define a Lie algebra on a group of PDOs by log-PDOs.

The KP hierarchy

$L \in \partial + \Psi DO^{-1}(\widehat{A})$ is a solution of the t_k-KP equation iff:

$$\frac{dL}{dt_k} = \left[(L^k)_+, L\right] = -\left[(L^k)_-, L\right] , \qquad k \geq 1 , \qquad (1)$$

with initial condition $L(0) \in \partial + \Psi DO^{-1}(A)$.

Brief outline of the algebraic construction of L:

$U = exp(\sum_{k \in \mathbb{N}^*} t_k L_0^k) \in \widehat{\Psi}^*$ enables the decomposition

$$U = S^{-1} Y$$

which is the key point to prove that

$$L = Y L_0 Y^{-1} = S L_0 S^{-1}.$$

Hint: All these algebraic operations are smooth.

Statement of the integration of the KP hierarchy

Theorem

Consider the KP hierarchy in the family of variables T with initial condition $L(0) = L_0$. Then,

1. *There exists a pair $(S, Y) \in G \times \widehat{\mathcal{D}}^*$ such that the solution to the KP hierarchy with $L(0) = L_0$ is*
$$L(t) = Y\, L_0\, Y^{-1} = SL_0S^{-1} .$$
2. *The pair (S, Y) is uniquely determined by the decomposition problem $\exp\left(\sum t_k L_0^k\right) = S^{-1}Y$.*
3. *the map $L_0 \in \partial + \Psi DO^{-1}(A) \mapsto L \in \partial + \Psi DO^{-1}(\widehat{A})$ is smooth.*

Precisions on the Sato operator

Usually, when $A = C^\infty(S^1, \mathbb{R})$, $\mathcal{S}(\mathbb{R}, \mathbb{R})$ or $C^\infty(\mathbb{R}, \mathbb{R})$, with $\partial = \frac{d}{dx}$, one assumes that

$$L_0 = S_0 \partial S_0^{-1}$$

where S_0 is a Sato operator in $1 + \Psi DO^{-1}(A)$. This is a restrictive condition on the possible initial condition L_0 and moreover, the Sato operator is not uniquely determined.

Analyzing the proof, this is not necessary for the construction of L.

This is even more an accurate remark while changing A into any differential diffeological algebra.

On Hamiltonian formalism

However, we feel that the Hamiltional approach formulation can be extended to more genral contexts, in a generalized Hamiltonian formulation. This is a work in progress but we can already state a published example of such a generalization.

Example

$A = C^\infty(S^1, \mathbb{R})[[z]]$ with natural addition, multiplication and differential. Then the Adler residue extends to a "trace"

$$Tr : \Psi DO(A) \rightarrow \mathbb{R}[[z]]$$

that defines an adequate "generalized $\mathbb{R}[[z]]-$valued pairing" $\langle ., . \rangle$ and Hamitonian functions $H_k(\mu) = \frac{1}{k} Tr(L^{k+1})$ for $\mu = \langle L, . \rangle$ and Poisson bracket obtained from the r-matrix already defined.

Zakharov-Shabat equations (1)

The (zero curvature) Zakharov-Shabat-equation for the couple (t_k, t_l) is

$$\frac{dL_+^k}{dt_l} - \frac{dL_+^l}{dt_k} = \left[L_+^l, L_+^k\right] \tag{2}$$

remains valid for the our generalized KP hierarchies. We expand

$$L = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots$$

where $\forall k,\ u_k \in \widehat{A}$. For the (t_1, t_2) and the (t_1, t_3) variables, we deduce from (2) $\frac{du_{-1}}{dt_1} = \partial u_{-1}$ and $\frac{du_{-2}}{dt_1} = \partial u_{-2}$ which here only formally integrates to

$$\begin{cases} u_{-1}(t_1, t_2, \cdots) & = & exp(t_1\partial)u_{-1}(0, t_2, \cdots) \\ u_{-2}(t_1, t_2, \cdots) & = & exp(t_1\partial)u_{-2}(0, t_2, \cdots) \end{cases}$$

They are trivially well-posed.

Zakharov-Shabbat equations (2)

For the (t_2, t_3) variables, we deduce from (2) the following system:

$$\begin{cases} \frac{du_{-1}}{dt_2} = \partial^2 u_{-1} + 2\partial u_{-2} \\ 3\frac{du_{-2}}{dt_2} - 2\frac{du_{-1}}{dt_3} = [u_{-1}, u_{-2}] - 6\partial u_{-1} u_{-1} - 2\partial^3 u_{-1} - 3\partial^2 u_{-2} \end{cases} \quad (3)$$

Theorem

The KP hierarchy (1) with initial condition $L_0 = \partial + u_{-1}(0)\partial^{-1} + u_{-2}(0)\partial^{-2}$ has a solution $L = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots$ such that

$$(u_{-1}(0, t_2, t_3, 0, \cdots), u_{-2}(0, t_2, t_3, 0, \cdots))$$

is the solution of (3) . Moreover, the problem is well-posed.

Comments on the classical KP-II equation

If $A = C^\infty(S^1, \mathbb{R})$, with $\partial = \frac{d}{dx}$, one can deduce from (3) the classical KP-II equation

$$\frac{1}{2}u_{yy} = \left(-\frac{1}{6}u_{xxx} - uu_x + \frac{2}{3}u_t\right)_x$$

BUT

- the KP hierarchy has functions in the variable x as initial values
- while the KP-II equation has functions in the variables (x, y) as initial values.

This is the reason why the classical KP hierarchy only produce a class of solutions for KP-II equations.

Examples of equations (1): on commutative algebras

If A is commutative, $[u_1, u_2] = 0$ in (3).

- If $\partial = \nabla$ is the gradient on $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, with coordinatewise multiplication, one consequence of (3) is

$$-2\frac{du}{dt} + 6u\nabla u + \nu\Delta u = \nabla h,$$

  while **the constraint** $div(u) = 0$ **does not seem to be a consequence of (3)**. Adding this constraint, we get the Navier-Stokes equation.

- If $A = C^\infty(S^1, \mathbb{R}) + i\epsilon(D)C^\infty(S^1, \mathbb{R})$ where $\epsilon(D)$ is the sign of the Dirac operator with symbol $\frac{\xi}{|\xi|}$ give also a KP-like equation, related to the KP hierarchy on a complexification of $\Psi DO(S^1, \mathbb{R})$. Its full study and formulation is under progress.

Examples of equations (2): on non-commutative algebras

We have well-posedness of all these sample equations for a family of initial conditions:

- The case $A = C^\infty(\mathbb{R}, \mathbb{H})$ is already described in the literature, but we state well-posedness.
- If $\partial^2 = 0$, we get $4\partial_t\partial u - 3\partial_y^2 u - \partial_y[4\partial_y^{-1}\partial u, 3u] = 0$
- If $(A, \star)$ is the algebra $C^\infty(\mathbb{R}^n)[[h]]$ equipped with Moyal product $\star$, we recover Hamanaka's Moyal KP equation

$$3u_{yy} = \partial\left(4u_t - 3(\partial u \star u + u \star \partial u) - \partial^3 u\right) + 3\partial\left[u, \partial^{-1}u_y\right] .$$

- If $A = \Psi DO(\mathbb{R}[[X]])$, $\partial = [X, .]$ is a derivation on A and the equation (3) reads as

$$\begin{cases} \frac{du_{-1}}{dt_2} = [X,[X,u_{-1}]] - 2[X,u_{-2}] \\ 3\frac{du_{-2}}{dt_2} - 2\frac{du_{-1}}{dt_3} = [u_{-1},u_{-2}] + 6[X,u_{-1}]u_{-1} + 2[X,[X,[X,u_{-1}]]] \end{cases} \quad (4)$$

Perspectives

- Applications to diffieties and to solutions of PDEs (partially accepted, a refined study on the elementary diffiety is a part of a more important work to be pre-published in November 2022, $23+\epsilon$)
- Full study of generalized equations, maybe with generalized initial values and/or production of the same results in more extended "algebras"
- KP hierarchy on non-formal PDOs.
  Already available as a preprint: the action of $Diff_+(S^1)$ on non-formal Sato operators is in relation with a well-posed (rescaled) non-formal KP hierarchy
- and more...

Thank you for your attention!

Main references:

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