Geometric structures on solutions of equation of adiabatic gas motions

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1. Equation of adiabatic gas motion*

$x = (x^0, x^1, \dots, x^n) \in \mathbb{R}^{1+n}$, n=1,2,3,

$t = x^0$ is time, $v = (\, v^1, \dots, v^n \,)$ is velocity,

p is pressure, and ρ is density.

$$\mathcal{E} \begin{cases} v^i_t + v^1 v^i_{x^1} + \ldots + v^n v^i_{x^n} + p_{x^i}/\rho = 0, \; i = 1, \dots, n, \\ \rho_t + v^1 \rho_{x^1} + \ldots + v^n \rho_{x^n} + \rho(v^1_{x^1} + \ldots + v^n_{x^n}) = 0, \\ p_t + v^1 p_{x^1} + \ldots + v^n p_{x^n} + A(v^1_{x^1} + \ldots + v^n_{x^n}) = 0, \end{cases} \qquad (1)$$

$$A = A(\rho, p) > 0.$$

A solution of equation (1) is a vector-function

$$(\, v(x), \rho(x), p(x) \,)$$

satisfying to this equation.

2. Differential operator associated with equation (1)

$$u = (u^1, \dots, u^{n+2}) = (v^1, \dots, v^n, \rho, p) \in \mathbb{R}^{n+2}.$$

$$\zeta : \mathbb{R}^{n+1} \times \mathbb{R}^{n+2} \longrightarrow \mathbb{R}^{n+1}, \;\; \zeta : (x, u) \mapsto x.$$

$$\zeta_k : J^k\zeta \longrightarrow \mathbb{R}^{n+1}, \quad \zeta_k : j^k_x u \mapsto x, \;\; k = 0, 1, 2, \dots$$

$$\mathcal{E} \subset J^1\zeta_1.$$

Let $F^1, \dots, F^{n+2}$ are left parts of system (1). Then the quasilinear differential operator

$$\Delta : \Gamma(\zeta) \longrightarrow \Gamma(\zeta),$$

$$\Delta(u) = \begin{pmatrix} F^1(j_1 u) \\ \cdots \\ F^{n+2}(j_1 u) \end{pmatrix} = \Big(\frac{\partial F^i}{\partial u^j_k}(u)\, \partial_{x^k} \Big) u^T.$$

is associated with equation (1).

3. Symbols of the operator Δ

$x_1 \in \mathcal{E} \subset J^1\zeta, \quad x = \zeta_1(x_1),$

$\xi = \xi_0 dt + \ldots + \xi_n dx^n \in T_x^*(\mathbb{R}^{n+1})$

ζ_x is the fiber of ζ at point x.

$w = (w^1, \ldots, w^{n+2}) \in \zeta_x$.

The symbol of the operator Δ at point x_1:

$$\sigma(\Delta)_{x_1} : T_x^*(\mathbb{R}^{n+1}) \otimes \zeta_x \longrightarrow \zeta_x,$$

$$\sigma(\Delta)_{x_1} : (\xi \otimes w) \mapsto \Big(\frac{\partial F^i}{\partial u_k^j}(u)\, \xi_k\Big) w^T.$$

The value of $\sigma(\Delta)_{x_1}$ at a fixed covector ξ:

$$\sigma_\xi(\Delta)_{x_1} = \sigma(\Delta)_{x_1}\Big|_{\xi \otimes \zeta_x} : \zeta_x \longrightarrow \zeta_x.$$

4. Characteristic set

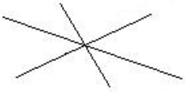
A covector $\xi \in T_x^*(\mathbb{R}^{n+1})$ is *characteristic w.r.t.* $x_1 \in \mathcal{E}$ *if the linear map* $\sigma_\xi(\Delta)_{x_1}$ *is degenerate*, i.e.

$$\det\Big(\frac{\partial F^i}{\partial u_k^j}(u)\,\xi_k\Big)\Big|_{x_1} = 0.$$

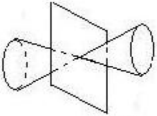
Calculating this determinant, we get:

$$(\xi_0 + v^1\xi_1 + \ldots + v^n\xi_n)^3$$
$$\times\Big((\xi_0 + v^1\xi_1 + v^2\xi_2 + v^3\xi_n)^2 - \frac{A}{\rho}(\xi_1^2 + \xi_2^2 + \xi_3^2)\Big) = 0.$$

The characteristic set in $T_x^*(\mathbb{R}^{n+1})$:



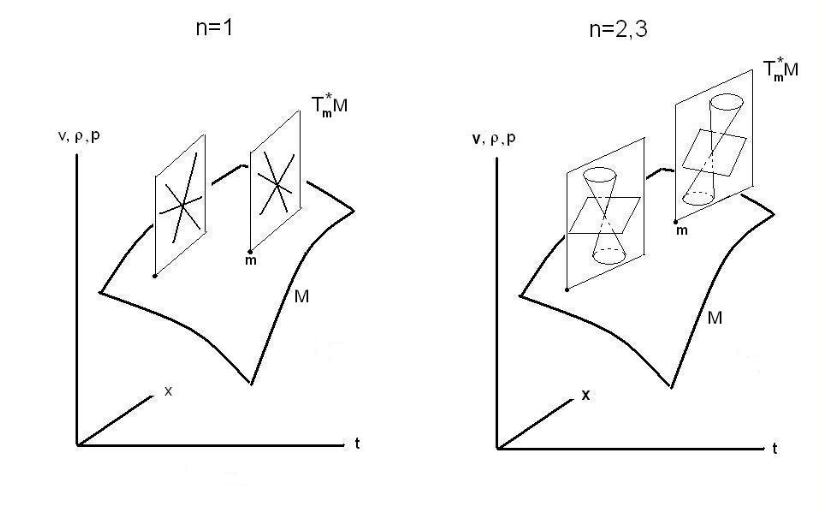
n=1



n=2,3

5. Geometric structures on solutions

Let (v, ρ, p) be a solution of system (1) and let M be its graph. Characteristic sets of equation (1) define the geometric structure S on M:



6. Differential invariants of 0-order on solutions

The field of cones is identified with the natural conformal ctructure on M defined by metric

$$g^{-1} = (\partial_{x^0} + v^1\partial_{x^1} + \ldots + v^n\partial_{x^n})^2 - \frac{A}{\rho}\left(\partial_{x^1}\cdot\partial_{x^1} + \ldots + \partial_{x^n}\cdot\partial_{x^n}\right).$$

The natural conformal metric on M is defined by formula

$$\begin{aligned} g = \frac{u^{n+1}}{A}\Big(\big(\frac{A}{u^{n+1}} - (u^1)^2 - \ldots - (u^n)^2 \big)(dx^0)^2 \\ + 2u^1 dx^0 dx^1 + \ldots + 2u^n dx^0 dx^n \\ - (dx^1)^2 - \ldots - (dx^n)^2 \Big). \end{aligned}$$

Its signature is $(1,3)$.

7. The natural bundle of planes and cones

The geometric structure S on M is defined by the equations:

$$\xi_0 + v^1\xi_1 + \ldots + v^n\xi_n = 0,$$
$$(\xi_0 + v^1\xi_1 + \ldots + v^n\xi_n)^2 - \frac{A(p,\rho)}{\rho}(\xi_1^2 + \ldots + \xi_n^2) = 0.$$

$$[1 : v^1 : \ldots : v^n] \in \mathbb{R}\mathrm{P}^{\mathrm{n}},$$
$$[\,1 : v^1 : \ldots : v^n : (v^1)^2 - \frac{A}{\rho} : \ldots : (v^n)^2 - \frac{A}{\rho}\,] \in \mathbb{R}\mathrm{P}^{\frac{(\mathrm{n}+1)(\mathrm{n}+2)}{2}-1}.$$

The structure S is identifies with the section

$$S : x \mapsto \big([1 : v^1(x) : \ldots : v^n(x)], [1 : v^1(x) : \ldots : v^n(x)^2 - \frac{A(p(x),\rho(x))}{\rho(x)}]\big)$$

of the natural bundle of planes and cones

$$\pi : E(\pi) = \big(M \times (\mathbb{R}\mathrm{P}^{\mathrm{n}} \times \mathbb{R}\mathrm{P}^{\frac{(\mathrm{n}+1)(\mathrm{n}+2)}{2}-1})\big) \longrightarrow \mathrm{M},$$

8. Spaces $\mathcal{A}_{x_1}$ and algebras $\mathfrak{g}_{x_0}$

Let $x \in M$, $x_0 = S(x)$, and $x_1 = j^1_x S$.

Let X be a vector field in M defined in a neighborhood of x and let $X^{(0)}$ be its natural lifting in $E(\pi)$.

The value $X^{(0)}_{x_0}$ of $X^{(0)}$ at x_0, is defined by the 1-jet $j^1_x X$.

The jet $x_1 = j^1_x S$ is identified with the tangent space $\mathcal{K}_{x_1}$ to image of section S at point $x_0 = S(x)$.

Consider all vector fields X in neighborhoods of x.

$$\mathcal{A}_{x_1} = \{\, j^1_x X \mid X^{(0)}_{x_0} \in \mathcal{K}_{x_1} \,\}.$$

$$\mathfrak{g}_{x_0} = \{\, j^1_x X \mid X^{(0)}_{x_0} = 0 \,\} \subset T_x(M) \otimes T^*_x(M).$$

$$\mathfrak{g}_{x_0} \subset \mathcal{A}_{x_1}.$$

*

9. Horizontal subspaces and exterior forms

A subspace $H \subset \mathcal{A}_{x_1}$ is *horizontal*, if the projection

$$\tau_{1,0}|_H : H \to T_x(M), \quad \tau_{1,0} : j^1_x X \mapsto X_x \,,$$

is an isomorphism.

$$\mathcal{A}_{x_1} = H \oplus \mathfrak{g}_{x_0}.$$

Every horizontal subspace $H \subset \mathcal{A}_{x_1}$ define the exterior 2-form ω_H on $T_x(M)$ with values in $T_x(M)$:

$$\omega_H(X_x, Y_x) = [\,(\tau_{1,0}\big|_H)^{-1}(X_x),\, (\tau_{1,0}\big|_H)^{-1}(Y_x)\,]_x, \qquad (2)$$

$$[\,j^1_x X,\, j^1_x Y\,] \overset{df}{=} [\,X,\, Y\,]_x.$$

Back

$$\omega_H \in T_x(M) \otimes (T^*_x(M) \wedge T^*_x(M))\,.$$

10. Spencer cohomologies

$$\mathfrak{g}_{x_0} \subset T_x(M) \otimes T_x^*(M)\,.$$

The Spencer complex

$$0 \to (\mathfrak{g}_{x_0})^{(1)} \hookrightarrow \mathfrak{g}_{x_0} \otimes T_x^*(M) \xrightarrow{\partial_{1,1}} T_x(M) \otimes \big(T_x^*(M) \wedge T_x^*(M)\big) \to 0\,,$$

where

$$(\mathfrak{g}_{x_0})^{(1)} = (\,\mathfrak{g}_{x_0} \otimes T_x^*(M)\,) \cap \big(\,T_x(M) \otimes (T_x^*(M) \odot T_x^*(M))\,\big)\,,$$

$$\partial_{1,1}(h)(X_x, Y_x) = h(X_x)(Y_x) - h(Y_x)(X_x)\,, \quad \forall\, X_x, Y_x \in T_x(M)\,.$$

Theorem. *The cohomology class*

$$\omega_{x_1} = \omega_H + \partial_{1,1}\big(\mathfrak{g}_{x_0} \otimes T_x^*(M)\big)$$

is independent of choice of horizontal subspace $H \subset \mathcal{A}_{x_1}$.

11. Differential invariants of 1-order on solutions

The function

$$\omega_S : x_1 \longmapsto \omega_{x_1}, \quad x_1 \in L_S^{(1)} \subset J^1\pi$$

is a 1-order differential invariant of structure S.

The natural conformal metric g on M generates the natural conformal metric $\tilde{g}$ on the tensor space

$$T(M) \otimes (T^*(M) \wedge T^*(M)).$$

$$T_x(M) \otimes \left(T_x^*(M) \wedge T_x^*(M)\right) = \left(\partial_{1,1}\left(g_{x_0} \otimes T_x^*(M)\right)\right)_{\tilde{g}}^{\perp} \oplus \partial_{1,1}\left(g_{x_0} \otimes T_x^*(M)\right).$$

The invariant ω_S * can be considered now as a natural tensor field on M.

12. Natural linear connections on solutions

Theorem. *Let $x \in M$ and $x_1 = j^1_x S$. Then there is a unique horizontal subspace $H_x \subset \mathcal{A}_{x_1}$ defining $\omega_S|_x$ by formula (2), that is*

$$\omega_S|_x = \omega_{H_x}.$$

Consider the bundle

$$\tau_{1,0} : J^1\tau \longrightarrow J^0\tau$$

The section of $\tau_{1,0}$

$$\Gamma : J^0\tau \longrightarrow J^1\tau,$$
$$\forall x \in M \quad \Gamma(X_x) = (\tau_{1,0}|_{H_x})^{-1}(X_x),$$

is a natural linear connection on M.

13. The natural linear connection on solutions, $n = 1$.

Let (v, ρ, p) be a solution of equation (1) for $n = 1$ and let Γ^i_{jk} be components of the natural linear connection on M. Then

$$\Gamma^0_{00} = \frac{v\alpha_x - v_x\alpha + \alpha_t}{\alpha}, \quad \Gamma^0_{01} = \Gamma^0_{10} = 0, \quad \Gamma^0_{11} = 0,$$
$$\Gamma^1_{00} = \frac{v\alpha_t - v_t\alpha}{\alpha}, \quad \Gamma^1_{01} = \Gamma^1_{10} = \frac{v\alpha_x - v_x\alpha}{\alpha}, \quad \Gamma^1_{11} = -\frac{\alpha_x}{\alpha},$$

where $\alpha = \sqrt{A(\rho, p)/\rho}$.

Let R_S be a curvature tensor of this connection. Then

$$\mathrm{R}_S = \frac{1}{\alpha^2}\big(-\alpha\alpha_{tx} - v\alpha\alpha_{xx} + v_{xx}\alpha^2 + \alpha_t\alpha_x + v\alpha_x^2 - v_x\alpha\alpha_x \big)\, dt \wedge dx.$$

14.1. Explicit solutions, $n = 1$.

$$A(\rho, p) = \rho.$$

In this case

$$\mathrm{R}_S = v_{xx}.$$

The following solutions of system (1) have the linear connection with zero curvature tensor.

$$v(t,x) = \frac{x + c_3}{t + c_1} - \frac{t + c_1}{2} - c_2\,,$$

$$\rho(t,x) = c_2(t + c_1)^{c_2^2 - 1}\exp\Big(\frac{c_2(x + c_3)}{t + c_1} + \frac{c_2}{2}t + c_4\Big)\,,$$

$$p(t,x) = \Big(\frac{c_2}{t + c_1} + 1\Big)(t + c_1)^{c_2^2}\exp\Big(\frac{c_2(x + c_3)}{t + c_1} + \frac{c_2}{2}t + c_4\Big)\,,$$

where c_1, c_2, c_3 и c_4 are constants.

14.2. Explicit solutions, $n = 2$.

$$A(\rho, p) = \gamma p,$$

where γ is a positive constant.

$$\omega_S = 0 \iff v^1_x = v^2_y \quad \text{and} \quad v^1_y = -v^2_x.$$

$$v^1(t,x,y) = \frac{t+k_1}{(t+k_1)^2+k_2^2}\,x + \frac{k_2}{(t+k_1)^2+k_2^2}\,y + c(t)\,,\ k_2 \neq 0\,,$$

$$v^2(t,x,y) = \frac{k_2}{(t+k_1)^2+k_2^2}\,x + \frac{t+k_1}{(t+k_1)^2-k_2^2}\,y + d(t)\,,$$

$$\rho(t,x,y) = \frac{e^{-2k_3+k_4}}{(t+k_1)^2+k_2^2}\,\exp(f)\,,$$

$$p(t,x,y) = \frac{e^{-2\gamma k_3+k_4}}{\left((t+k_1)^2+k_2^2\right)^{\gamma}}\,\exp(f)\,,$$

where $\Downarrow$

14.2.1. Explicit solutions, $n = 2$.

$$f = \frac{e^{-k_3}}{\big((t+k_1)^2 + k_2^2\big)^{1/2}}\big(x(\cos\beta - \sin\beta) + y(\cos\beta + \sin\beta) \big)$$

$$-\int \frac{e^{-k_3}}{\big((t+k_1)^2 + k_2^2\big)^{1/2}}\big(c(t)(\cos\beta - \sin\beta) + d(t)(\cos\beta + \sin\beta) \big) dt\,,$$

$$\beta = \operatorname{arctg}\Big(\frac{t+k_1}{k_2}\Big) + k_5\,,$$

$$c(t) = \Big(\big(e^{(1-2\gamma)k_3}\Gamma_c + k_6\big)\cos\beta - \big(e^{(1-2\gamma)k_3}\Gamma_d + k_7\big)\sin\beta\Big)\big((t+k_1)^2 + k_2^2\big)^{\frac{1}{2}},$$

$$d(t) = \Big(\big(e^{(1-2\gamma)k_3}\Gamma_c + k_6\big)\sin\beta + \big(e^{(1-2\gamma)k_3}\Gamma_d + k_7\big)\cos\beta\Big)\big((t+k_1)^2 + k_2^2\big)^{\frac{1}{2}},$$

$$\Gamma_c = \frac{1}{\delta^2+1}\int \frac{1}{\big((t+k_1)^2 + k_2^2\big)^{\gamma+1}}\Big((\delta^2 - 2\delta - 1)(t+k_1)^2 + 2k_1(-\delta^2 - 2\delta + 1)(t+k_1) + k_1^2(-\delta^2 + 2\delta + 1)\Big) dt\,,$$

$\Downarrow$

14.2.2. Explicit solutions, $n = 2$.

$$\Gamma_d = \frac{1}{\delta^2+1}\int \frac{1}{\big((t+k_1)^2+k_2^2\big)^{\gamma+1}}\big((\delta^2+2\delta-1)(t+k_1)^2$$
$$+\,2k_1(\delta^2-2\delta-1)(t+k_1)+k_1^2(-\delta^2-2\delta+1)\big)\,dt\,,$$

$\delta = \mathrm{tg}(k_5)$ и $k_1, \ldots, k_7$ are arbitrary constants.

14.3. Explicit solutions, $n = 3$.

$$\omega_S = 0 \iff \begin{cases} v^2_x + v^1_y = 0, \\ v^3_x + v^1_z = 0, \\ v^3_y + v^2_z = 0, \\ v^3_z + v^2_y - 2v^1_x = 0, \\ v^3_z - 2v^2_y + v^1_x = 0, \\ 2v^3_z - v^2_y - v^1_x = 0. \end{cases}$$

14.3.1. Polytropic gas motion, $\gamma \neq 4/3$ and $\gamma \neq 1$

$A(\rho, p) = \gamma p.$

$$
\begin{aligned}
v^1 &= \frac{x + k_3}{t + k_1} - \frac{e^{(2-3\gamma)k_2}}{4 - 3\gamma}(t + k_1)^{2-3\gamma}|t + k_1|, \\
v^2 &= \frac{y + k_4}{t + k_1} - \frac{e^{(2-3\gamma)k_2}}{4 - 3\gamma}(t + k_1)^{2-3\gamma}|t + k_1|, \\
v^3 &= \frac{z + k_5}{t + k_1} - \frac{e^{(2-3\gamma)k_2}}{4 - 3\gamma}(t + k_1)^{2-3\gamma}|t + k_1|, \\
\rho &= \frac{e^{-3k_2 + k_6}}{|t + k_1|^3}\exp(f), \\
p &= \frac{e^{-3k_2\gamma + k_6}}{|t + k_1|(t + k_1)^{3\gamma - 1}}\exp(f),
\end{aligned}
$$

where $k_1, \ldots, k_6$ are arbitrary constants and

$$
f = e^{-k_2}\frac{x + y + z + k_3 + k_4 + k_5}{|t + k_1|} + e^{(1-3\gamma)k_2}\frac{3(t + k_1)^{3-3\gamma}}{(3 - 3\gamma)(4 - 3\gamma)}.
$$

14.3.2. Polytropic gas motion, $\gamma = 4/3$

$$
\begin{aligned}
v^1 &= \frac{x + k_3}{t + k_1} - e^{-2k_2} \frac{\ln|t + k_1|}{t + k_1}, \\
v^2 &= \frac{y + k_4}{t + k_1} - e^{-2k_2} \frac{\ln|t + k_1|}{t + k_1}, \\
v^3 &= \frac{z + k_5}{t + k_1} - e^{-2k_2} \frac{\ln|t + k_1|}{t + k_1}, \\
\rho &= \frac{e^{-3k_2}}{|t + k_1|^3} \exp(f), \\
p &= \frac{e^{-4k_2}}{(t + k_1)^4} \exp(f),
\end{aligned}
$$

where

$$
f = e^{-k_2} \frac{x + y + z + k_3 + k_4 + k_5}{|t + k_1|} - 3e^{-3k_2} \frac{\ln|t + k_1| + 1}{|t + k_1|} + k_6
$$

and $k_1, \ldots, k_6$ are arbitrary constants.

14.3.3. Polytropic gas motion, $\gamma = 1$

$$v^1 = \frac{x + k_3}{t + k_1} - e^{-k_2} \frac{t}{|t + k_1|},$$
$$v^2 = \frac{y + k_4}{t + k_1} - e^{-k_2} \frac{t}{|t + k_1|},$$
$$v^3 = \frac{z + k_5}{t + k_1} - e^{-k_2} \frac{t}{|t + k_1|},$$
$$\rho = \frac{e^{-3k_2}}{|t + k_1|^3} \exp(f),$$
$$p = \frac{e^{-3k_2}}{|t + k_1|^3} \exp(f),$$

where

$$f = e^{-k_2} \frac{x + y + z + k_3 + k_4 + k_5}{|t + k_1|} + 3e^{-2k_2} \left(\ln|t + k_1| + \frac{k_1}{t + k_1}\right) + k_6$$

and $k_1, \ldots, k_6$ are arbitrary constants.

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Initial equations of adiabatic gas motion

$$\rho\, d_t v + \operatorname{grad} p = 0, \quad d_t\rho + \rho \operatorname{div} v = 0, \quad d_t S(p,\rho) = 0,$$

where $d_t = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}$, $S(p,\rho)$ is entropy, and $\partial S/\partial p \neq 0$.

Introducing

$$A(p,\rho) = -\rho \frac{\partial S}{\partial \rho} \Big/ \frac{\partial S}{\partial p},$$

the initial equations can be rewritten to the form (1)

Back

Algebras $\mathfrak{g}_{x_0}$

$$n=1,\quad \mathfrak{g}_{\theta_0}=\{\, a\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ |\ a\in\mathbb{R}\,\},$$

$$n=2,\quad \mathfrak{g}_{\theta_0}=\{\begin{pmatrix} a & 0 & 0\\ -b\,u^2(x) & a & b\\ b\,u^1(x) & -b & a\end{pmatrix}\,|\,a,b\in\mathbb{R}\},$$

$$n=3,\quad \mathfrak{g}_{\theta_0}=\{\begin{pmatrix} a & 0 & 0 & 0\\ -b\,u^2(x)-c\,u^3(x) & a & b & c\\ b\,u^1(x)-d\,u^3(x) & -b & a & d\\ c\,u^1(x)+d\,u^2(x) & -c & -d & a\end{pmatrix}\,|\,a,b,c,d\in\mathbb{R}\}.$$

Back

The tensor ω_S, $n = 1, 2$.

1. If $n = 1$, then
$$\omega_S = 0.$$
2. If $n = 2$, then

$$\omega_S = \frac{1}{2}(u^1_1 - u^2_2)\partial_{x^1} \otimes dx^0 \wedge dx^1 + \frac{1}{2}(u^1_2 + u^2_1)\partial_{x^1} \otimes dx^0 \wedge dx^2$$
$$+ \frac{1}{2}(u^1_2 + u^2_1)\partial_{x^2} \otimes dx^0 \wedge dx^1 - \frac{1}{2}(u^1_1 - u^2_2)\partial_{x^2} \otimes dx^0 \wedge dx^2.$$

$$\Downarrow$$

The tensor ω_S, $n = 3$.

1. If $n = 3$, then

$$\omega_S = \frac{1}{3}(-u^3_3 - u^2_2 + 2u^1_1)\,\partial_{x^1} \otimes dx^0 \wedge dx^1 + \frac{1}{2}(u^2_1 + u^1_2)\,\partial_{x^1} \otimes dx^0 \wedge dx^2$$
$$+ \frac{1}{2}(u^3_1 + u^1_3)\,\partial_{x^1} \otimes dx^0 \wedge dx^3$$
$$+ \frac{1}{2}(u^2_1 + u^1_2)\,\partial_{x^2} \otimes dx^0 \wedge dx^1 + \frac{1}{3}(-u^3_3 + 2u^2_2 - u^1_1)\,\partial_{x^2} \otimes dx^0 \wedge dx^2$$
$$+ \frac{1}{2}(u^3_2 + u^2_3)\,\partial_{x^2} \otimes dx^0 \wedge dx^3$$
$$+ \frac{1}{2}(u^3_1 + u^1_3)\,\partial_{x^3} \otimes dx^0 \wedge dx^1 + \frac{1}{2}(u^3_2 + u^2_3)\,\partial_{x^3} \otimes dx^0 \wedge dx^2$$
$$+ \frac{1}{3}(2u^3_3 - u^2_2 - u^1_1)\,\partial_{x^3} \otimes dx^0 \wedge dx^3.$$

Back