Classification of 3rd - order linear differential equations

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1. Notations

Let M be a 2 - dimensional manifold, x, y its local coordinates, $\mathcal{G}(M)$ a pseudogrup of local diffeomorphisms of M, $\tau : T \to M$ and $\tau^* : T^* \to M$ tangent and cotangent bundles, $\xi : M \times \mathbb{R} \to M$ a trivial line bundle, $C^\infty(\xi) \equiv C^\infty(M)$, $\mathrm{Diff}_k(M)$ the left module of linear differential operators of order $\leq k$ acting in $C^\infty(\xi)$, $A \in \mathrm{Diff}_3(M)$ a generic operator,

$$A = a_1\partial_x^3 + 3a_2\partial_x^2\partial_y + 3a_3\partial_x\partial_y^2 + a_4\partial_y^3 \\ + b_1\partial_x^2 + 2b_2\partial_x\partial_y + b_3\partial_y^2 + c_1\partial_x + c_2\partial_y + a_0.$$

The operator A defines the 3rd order generic linear PDE

$$\mathcal{E}_A = \{j_p^3 f \in J^3\xi \mid A(f) = 0\}.$$

Recall that

$$\mathcal{E}_A = \mathcal{E}_{fA}$$

for all $f \in C^\infty(M)$ that are nonzero at every point.

2. Symbols of operators

The class $\mathrm{H}_3 = A \bmod \mathrm{Diff}_2(M)$ is the *symbol of the operator* A. It is identified with the 3rd order polynomial in two variables p_1, p_2 in the cotangent bundle T^*

$$\mathrm{H}_3 \cong a_1\, p_1^3 + 3a_2\, p_1^2 p_2 + 3a_3\, p_1 p_2^2 + a_4\, p_2^3.$$

The polynomial $P(t) = a_1 t^3 + 3a_2 t^2 + 3a_3 t + a_4$ is associated with H_3.

The generality condition of A means that $P(t)$ has nonzero discriminant. Hence it have three distinct roots: either three real distinct roots, or one real root and two complex conjugated roots. Respectively, the symbol H_3 can be transformed by some $\varphi \in \mathcal{G}(M)$ either to form

$$\mathrm{H}_3 = (h_y\, p_1 - h_x\, p_2) p_1 p_2, \quad h \in C^\infty(M), \tag{1}$$

or to form

$$\mathrm{H}_3 = \big(cos(h)\, p_1 + sin(h)\, p_2\big)(p_1^2 + p_2^2), \quad h \in C^\infty(M). \tag{2}$$

Back

3. The Chern connection

Let H_3 be symbol of a generic operator A. Then there are a unique symmetric linear connection ∇_{T^*} in T^* and a unique differential 1-form $\theta \in C^\infty(\tau^*)$ such that

$$\nabla_{T^*}(\mathrm{H}_3) = \theta \otimes \mathrm{H}_3,$$

here ∇_{T^*} is the operator of covariant derivative of this connection. This connection is called the *Chern connection.*

Proposition. *Let $f \in C^\infty(M)$ be nonzero in every point. Then Chern connections for operators A and $f \cdot A$ are the same.*

For the case of three distinct real roots, nonzero components of the Chern connection are:

$$\Gamma^1_{11} = h_{xx}/h_x - h_{xy}/h_y, \quad \Gamma^2_{22} = h_{yy}/h_y - h_{xy}/h_x.$$

For the case of one real root:

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = -h_y/\big(3(sin^2(h) - cos^2(h)\big),$$
$$\Gamma^1_{12} = \Gamma^2_{11} = \Gamma^2_{22} = -h_x/\big(3(sin^2(h) - cos^2(h)\big).$$

4. The connection in the bundle $\otimes^k \tau^* \otimes \xi$

Let ∇_ξ be the trivial connection of the bundle ξ,

$$\nabla_\xi : C^\infty(\xi) \to C^\infty(\tau^* \otimes \xi), \quad \nabla_\xi(f) = dx \otimes \partial_x f + dy \otimes \partial_y f.$$

Lemma. *The Chern connection ∇_{T^*} and the trivial connection ∇_ξ generate a unique connection $\nabla^{(k)}$ in the bundle $\otimes^k \tau^* \otimes \xi$, $k \geq 1$, such that*

$$\nabla^{(k)}(\theta_1 \otimes \ldots \otimes \theta_k \otimes f) = \sum_{i=1}^{k} (\theta_1 \otimes \ldots \otimes \nabla_{T^*}\theta_i \otimes \ldots \otimes \theta_k) \otimes f + (\theta_1 \otimes \ldots \otimes \theta_k \otimes \nabla_\xi f),$$

$\forall\, \theta_1, \ldots, \theta_k \in C^\infty(\tau^*)$ and $\forall\, f \in C^\infty(\xi)$.

5. Differential operators D_k

Consider k-order linear differential operators D_k, $k = 0, 1, 2, \ldots$,

$$D_0 = \mathrm{id}_{C^\infty(\xi)}, \quad D_1 = \nabla_\xi, \quad \text{and}$$
$$D_k : C^\infty(\xi) \to C^\infty(S^k\tau^* \otimes \xi), \quad k = 2, 3, \ldots,$$

is defined by the formula

$$D_k = S^{(k)} \circ \nabla^{(k-1)} \circ \ldots \circ \nabla^{(1)} \circ \nabla_\xi.$$

where $S^{(k)} : \otimes^k \tau^* \otimes \xi \to S^k \tau^* \otimes \xi$ is symmetrization operator defined by the formula

$$S^{(k)}(\theta_1 \otimes \ldots \otimes \theta_k \otimes f) = (\theta_1 \cdot \ldots \cdot \theta_k \otimes f) \overset{df}{=} \frac{1}{k!} \sum \theta_{\pi(1)} \otimes \ldots \otimes \theta_{\pi(k)} \otimes f,$$

here the sum are taken on all $k!$ permutations $\pi = \binom{1, \ \ldots, \ k}{\pi(1), \ldots, \pi(k)}$ of numbers $1, 2, \ldots, k$.

6. The canonical decomposition of $J^k\xi$

$D_k = F_k\, j_k$, $k \geq 1$, where $F_k : J^k\xi \to S^kT^* \otimes J^0\xi$ are morphisms of vector bundles. They split well known exact sequences

$$0 \to S^kT^* \otimes J^0\xi \xrightarrow{i} J^k\xi \xrightarrow{\xi_{k,k-1}} J^{k-1}\xi \to 0,$$

where $i\big((dg_1)_p \cdot \ldots \cdot (dg_k)_p \otimes f(p)\big) = j^k_p(g_1 \ldots g_k f)$, $\forall\, f \in C^\infty(\xi)$ and $\forall\, g_1, \ldots, g_k \in C^\infty(M)$ with $g_1(p) = \ldots = g_k(p) = 0$.
These splittings mean that

$$J^k\xi \cong S^kT^* \oplus J^{k-1}\xi,$$
$$J^{k-1}\xi \cong S^{k-1}T^* \oplus J^{k-2}\xi,$$
$$\ldots$$
$$J^1\xi \cong T^* \oplus J^0\xi,$$
$$J^0\xi = M \times \mathbb{R}.$$

Theorem. *The connections* ∇_{T^*} *and* ∇_ξ *generate a canonical decomposition*

$$J^k\xi \cong \oplus_{m=1}^k S^mT^* \oplus J^0\xi \quad \text{such that} \quad j_k(f) = \oplus_{m=0}^k D_m(f).$$

7. Quantization of the symbol H_3

Let $A \in \mathrm{Diff}_3(M)$ be a generic operator,

$$A = 3(h_y\partial_x - h_x\partial_y)\partial_x\partial_y + b_1\partial_x^2 + 2b_2\partial_x\partial_y + b_3\partial_y^2 + c_1\partial_x + c_2\partial_y + a_0,$$

$$\mathrm{H}_3 = 3(h_y\, p_1 - h_x\, p_2)\, p_1 p_2 \in C^\infty(S^3\tau),$$

$$D_3(f) \in C^\infty(S^3\tau^*),\ \forall f \in C^\infty.$$

The canonical decomposition of $J^k\xi$ allows us to construct the operator $\widehat{\mathrm{H}_3} \in \mathrm{Diff}_3(M)$ by the formula

$$\widehat{\mathrm{H}_3}(f) = \left\langle\ \mathrm{H}_3,\ D_3(f)\ \right\rangle,\ \forall f \in C^\infty(M), \qquad (3)$$

where $\langle\cdot,\cdot\rangle : S^3\tau \times S^3\tau^* \to C^\infty(M)$ is a convolution of tensors,

$$\begin{aligned}\widehat{\mathrm{H}_3} &= 3(h_y\partial_x - h_x\partial_y)\partial_x\partial_y \\ &+ 3(h_{yy}h_x/h_y - h_{xx}h_y/h_x)\partial_x\partial_y \\ &+ (\ \ h_{xyy} - h_{xxy}h_y/h_x + h_{xy}h_{xx}h_y/h_x^2 - h_{xy}h_{yy}/h_y)\partial_x \\ &+ (-h_{xxy} + h_{xyy}h_x/h_y - h_{xy}h_{yy}h_x/h_y^2 + h_{xy}h_{xx}/h_x)\partial_y.\end{aligned}$$

It is said that $\widehat{\mathrm{H}_3}$ is obtained by *quantization* of the symbol H_3.

8. Quantization of the symbol H_2

The symbols of operators A and $\widehat{\mathrm{H}_3}$ are the same. It follows that the operator $A - \widehat{\mathrm{H}_3} \in \mathrm{Diff}_2(M)$. Its symbol is

$$\mathrm{H}_2 = a_{11}\, p_1^2 + \big(2a_{12} - 3(h_{yy}h_x/h_y - h_{xx}h_y/h_x)\big)\, p_1 p_2 + a_{22}\, p_2^2.$$

The quantization of this symbol gives the operator $\widehat{\mathrm{H}_2}$,

$$\widehat{\mathrm{H}_2}(f) = \big\langle\ \mathrm{H}_{A_2},\, D_2(f)\ \big\rangle,\ \forall f \in C^\infty,$$

$$\begin{aligned}\widehat{\mathrm{H}_2} =& a_{11}\partial_x^2 + \big(2a_{12} - 3(h_{yy}h_x/h_y - h_{xx}h_y/h_x)\big)\partial_x\partial_y + a_{22}\partial_y^2 \\ &+ a_{11}(h_{xy}/h_y - h_{xx}/h_x)\partial_x + a_{22}(h_{xy}/h_x - h_{yy}/h_y)\partial_y.\end{aligned}$$

9. Quantization of the symbol H_1

The symbols of operators $A - \widehat{\mathrm{H}_3}$ and $\widehat{\mathrm{H}_2}$ are the same. It follows that the operator $A - \widehat{\mathrm{H}_3} - \widehat{\mathrm{H}_2} \in \mathrm{Diff}_1(M)$. Its symbol is the following

$$\begin{aligned}\mathrm{H}_1 = &\Big(-h_{xyy} + h_{xxy}h_y/h_x - h_{xy}h_{xx}h_y/h_x^2 + h_{xy}h_{yy}/h_y \\ &- a_{11}(h_{xy}/h_y - h_{xx}/h_x) + a_1\Big)\, p_1 \\ +&\Big(\quad h_{xxy} - h_{xyy}h_x/h_y + h_{xy}h_{yy}h_x/h_y^2 - h_{xy}h_{xx}/h_x \\ &- a_{22}(h_{xy}/h_x - h_{yy}/h_y) + a_2\Big)\, p_2.\end{aligned}$$

The quantization of this symbol gives the operator $\widehat{\mathrm{H}_1}$:

$$\widehat{\mathrm{H}_1}(f) = \big\langle\ \mathrm{H}_1,\ D_1(f)\ \big\rangle,\ \forall f \in C^\infty(M),$$

$$\begin{aligned}\widehat{\mathrm{H}_1} = &\Big(-h_{xyy} + h_{xxy}h_y/h_x - h_{xy}h_{xx}h_y/h_x^2 + h_{xy}h_{yy}/h_y \\ &- a_{11}(h_{xy}/h_y - h_{xx}/h_x) + a_1\Big)\partial_x \\ &+ \Big(\quad h_{xxy} - h_{xyy}h_x/h_y + h_{xy}h_{yy}h_x/h_y^2 - h_{xy}h_{xx}/h_x \\ &- a_{22}(h_{xy}/h_x - h_{yy}/h_y) + a_2\Big)\partial_y.\end{aligned}$$

10. The quantization of the symbol H_0

$$A - \widehat{\mathrm{H}_3} - \widehat{\mathrm{H}_2} - \widehat{\mathrm{H}_1} = a_0, \quad \mathrm{H}_0 \stackrel{df}{=} a_0, \quad \text{and} \quad \widehat{\mathrm{H}_0} \stackrel{df}{=} a_0.$$

11. Natural decompositions of operators

Theorem. *Let* $A \in \mathrm{Diff}_3(M)$ *be a generic operator,* ∇_{T^*} *its Chern connection, and* ∇_ξ *the trivial connection in* ξ*. Then*

$$\mathrm{H} = \mathrm{H}_3 + \mathrm{H}_2 + \mathrm{H}_1 + \mathrm{H}_0$$

is a complite symbol of A*.*

$$A = \widehat{\mathrm{H}} \stackrel{df}{=} \widehat{\mathrm{H}_3} + \widehat{\mathrm{H}_2} + \widehat{\mathrm{H}_1} + \widehat{\mathrm{H}_0}$$

is a natural decomposition of A*.*
The naturality of this decompozition means that for any $\varphi \in \mathcal{G}(M)$

$$\varphi_*(\widehat{\mathrm{H}_k}) = \widehat{\varphi_*(\mathrm{H}_k)}, \quad k = 3, 2, 1, 0.$$

Free terms of quantized operators are zeros,

$$\widehat{\mathrm{H}_3}(1) = \widehat{\mathrm{H}_2}(1) = \widehat{\mathrm{H}_1}(1) = 0.$$

12. Example (3 distinct real roots)

$$
\begin{aligned}
A =& 3(h_y\partial_x - h_x\partial_y)\partial_x\partial_y \\
& + a_{11}\partial_x^2 + 2a_{12}\partial_x\partial_y + a_{22}\partial_y^2 + a_1\partial_x + a_2\partial_y + a_0, \\
\widehat{\mathrm{H}_3} =& 3(h_y\partial_x - h_x\partial_y)\partial_x\partial_y \\
& + 3(h_{yy}h_x/h_y - h_{xx}h_y/h_x)\partial_x\partial_y \\
& + (h_{xyy} - h_{xxy}h_y/h_x + h_{xy}h_{xx}h_y/h_x^2 - h_{xy}h_{yy}/h_y)\partial_x \\
& + (-h_{xxy} + h_{xyy}h_x/h_y - h_{xy}h_{yy}h_x/h_y^2 + h_{xy}h_{xx}/h_x)\partial_y, \\
\widehat{\mathrm{H}_2} =& a_{11}\partial_x^2 + \big(2a_{12} - 3(h_{yy}h_x/h_y - h_{xx}h_y/h_x)\big)\partial_x\partial_y + a_{22}\partial_y^2 \\
& + a_{11}(h_{xy}/h_y - h_{xx}/h_x)\partial_x + a_{22}(h_{xy}/h_x - h_{yy}/h_y)\partial_y, \\
\widehat{\mathrm{H}_1} =& \big(- h_{xyy} + h_{xxy}h_y/h_x - h_{xy}h_{xx}h_y/h_x^2 + h_{xy}h_{yy}/h_y \\
& - a_{11}(h_{xy}/h_y - h_{xx}/h_x) + a_1\big)\partial_x \\
& + \big(h_{xxy} - h_{xyy}h_x/h_y + h_{xy}h_{yy}h_x/h_y^2 - h_{xy}h_{xx}/h_x \\
& - a_{22}(h_{xy}/h_x - h_{yy}/h_y) + a_2\big)\partial_y, \\
\widehat{\mathrm{H}_0} =& a_0.
\end{aligned}
$$

Lemma.

1. *The symbol* σ_{Δ_2} *of the operator* Δ_2 *is contravariant pseudo-Riemannian metric. By* g_{Δ_2} *we denote its covariant metric. Then* $g_{\Delta_2}(\Delta_1, \Delta_1) \neq 0$.
2. *Scalar invariants* $I_1 = a^0$ *and* $I_2 = g_{\Delta_2}(\Delta_1, \Delta_1)$ *are functionally independent.*

Theorem

Let $\tilde{A} \in \mathrm{Diff}_3(M)$ *be another generic operator. Then the operators* A *and* $\tilde{A}$ *are locally equivalent with respect to* $\mathcal{G}(M)$ *if and only if their expressions in coordinates* I_1, I_2 *are the same.*

14. The classification of differential equations

Lemma. *Let $A \in \mathrm{Diff}_3(M)$ be a generic operator. Then the operator $g_{\Delta_2}(\Delta_1, \Delta_1)^{-1}A$ does not change under the operation of left multiplication $A \mapsto f \cdot A$.*

The operator $g_{\Delta_2}(\Delta_1, \Delta_1)^{-1}A$ is called the *normalized operator*.

Theorem

Differential equations, given by generic differential operators $A \in \mathrm{Diff}_3$ are locally equivalent with respect to $\mathcal{G}(M)$ if and only if their normalized operators $g_{\Delta_2}(\Delta_1, \Delta_1)^{-1}A$ are locally equivalent with respect to $\mathcal{G}(M)$.

15.

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