CLASSIFICATION OF 3RD - ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Abstract

In this talk, we consider generic 3rd - order scalar linear differential equations on a two dimensional manifold. We give a local classification of these equations up to diffeomorphisms of the manifold.

Let M be a 2 - dimensional manifold and $\xi : M \times \mathbb{R} \to M$ be a trivial line bundle. We identify its module of smooth sections $C^\infty(\xi)$ with a module of smooth functions $C^\infty(M)$. By $\mathrm{Diff}_k(M)$ we denote the left module of scalar linear differential operators of order $\leq k$ acting in $C^\infty(\xi)$.

Considering equations have the form $A(f) = 0$, where $A \in \mathrm{Diff}_3(M)$ is a generic operator. In local coordinates x^1, x^2 of M, it has the form

$$A = a^{ijk}\partial_{x^i}\partial_{x^j}\partial_{x^k} + a^{ij}\partial_{x^i}\partial_{x^j} + a^i\partial_{x^i} + a^0.$$

Let $\tau : T(M) \to M$ and $\tau^* : T^*(M) \to M$ be tangent and cotangent bundles. Then $\sigma_A = a^{ijk}\partial_i \odot \partial_j \odot \partial_k \in C^\infty(S^3\tau)$ is the symbol of A.

It follows from the generality condition of operator A that there are a unique symmetric linear connection ∇_{T^*} in the bundle τ^* (*Chern connection*) and a unique differential 1-form $\theta \in C^\infty(\tau^*)$ such that

$$\nabla_{T^*}(\sigma_A) = \theta \otimes \sigma_A.$$

Proposition 1. *Let f be everywhere nonzero smooth function in M. Then Chern connections for operators A and $f \cdot A$ are the same.*

Let ∇_ξ be a trivial connection in the bundle ξ. It is well known that the connections ∇_{T^*} and ∇_ξ generate a natural decomposition $J^k\xi = \oplus_{m=0}^k S^m T^*(M)$ such that $j_k(f) = \{D_m(f)\}_{0\leq m\leq k}$, $\forall f \in C^\infty(M)$, where operators D_m are defined by ∇_{T^*} and ∇_ξ.

Theorem 2. *Let $A \in \mathrm{Diff}_3(M)$ be a generic operator, ∇_{T^*} be its Chern connection and ∇_ξ be the trivial connection in the bundle ξ. Then there is a natural decomposition of the operator A*

$$A = \Delta_3 + \Delta_2 + \Delta_1 + \Delta_0,$$

such that

(1) $\Delta_k \in \mathrm{Diff}_k(M)$, $k = 3, 2, 1, 0$.
(2) $\Delta_3(1) = \Delta_2(1) = \Delta_1(1) = 0$, *and* $\Delta_0 = a^0$.

(3) *The symbol σ_{Δ_2} of the operator Δ_2 is contravariant pseudo-Riemannian metric. Then $g_{\Delta_2}(\Delta_1, \Delta_1) \neq 0$, where g_{Δ_2} is a covariant metric corresponding to σ_{Δ_2}.*
(4) *Scalar invariants $I^1 = a^0$ and $I^2 = g_{\Delta_2}(\Delta_1, \Delta_1)$ are functionally independent.*

Let $\mathcal{G}(M)$ be a pseudogroup of local diffeomorphisms of M.

Proposition 3. *Let $\tilde{A} \in \mathrm{Diff}_3(M)$ be another generic operator. Then the operators A and $\tilde{A}$ are locally equivalent with respect to $\mathcal{G}(M)$ if and only if their expressions in coordinates I^1, I^2 are the same.*

Theorem 4. *Differential equations, given by generic differential operators $A \in \mathrm{Diff}_3$ are locally equivalent with respect to $\mathcal{G}(M)$ if and only if their normalized operators $\dfrac{1}{I^2}A$ are locally equivalent with respect to $\mathcal{G}(M)$.*

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