On thick morphisms of (super)manifolds

Theodore Voronov

Manchester, UK
www.maths.manchester.ac.uk/~tv

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О толстыхъ морфизмахъ (супер)многообразій

Ѳ. Ѳ. Вороновъ

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- Thick morphisms make a *formal category* (i.e., the composition law is formal), which a formal thickening of the ordinary category of smooth supermanifolds and smooth maps.
- They provide a *nonlinear version of the classical functional-algebraic duality* between spaces and algebras, where ordinary algebra homomorphisms are replaced by certain "nonlinear homomorphisms".
- A quantum version of the theory exists, based on some formal Fourier integral operators.

PLAN OF THE TALK

Definition of a thick (or microformal) morphism

Let M_1, M_2 be supermanifolds with local coordinates x^a, y^i. Let p_a and q_i be their conjugate momenta and $\omega_1 = dp_a dx^a$, $\omega_2 = dq_i dy^i$ be the symplectic forms on T^*M_1, T^*M_2.

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Definition

A *microformal* or *thick morphism* $\Phi\colon\ M_1 \Rrightarrow M_2$ is a formal Lagrangian submanifold $\Phi \subset T^*M_2 \times T^*M_1$ w.r.t. $\omega_2 - \omega_1$ specified locally by a *generating function* of the form $S(x,q)$:

$$q_i dy^i - p_a dx^a = d(y^i q_i - S) \quad \text{on } \Phi\,,$$

where $S(x,q)$ is part of the structure and is a formal power series:

$$S(x,q) = S_0(x) + S^i(x)q_i + \frac{1}{2}\,S^{ij}(x)q_j q_i + \frac{1}{3!}\,S^{ijk}(x)q_k q_j q_i + \ldots$$

Example

An ordinary map $\varphi\colon\ M_1 \to M_2$ corresponds to $S(x,q) = \varphi^i(x)q_i$.

Pullback by a microformal morphism

Construction of pullback

Let $\Phi\colon\, M_1 \Longrightarrow M_2$ be a thick morphism with generating function S. The *pullback* Φ^* is a formal mapping $\Phi^*\colon\, \mathbf{C}^{\infty}(M_2) \to \mathbf{C}^{\infty}(M_1)$ of functional supermanifolds (of 'bosonic' functions) defined by

$$\boxed{\Phi^*[g](x) = g(y) + S(x,q) - y^i q_i\,,}$$

for $g \in \mathbf{C}^{\infty}(M_2)$, where q_i and y^i are determined from the equations

$$q_i = \frac{\partial g}{\partial y^i}(y)\,, \quad y^i = (-1)^{\tilde{\imath}}\frac{\partial S}{\partial q_i}(x,q)$$

(giving $y^i = (-1)^{\tilde{\imath}}\frac{\partial S}{\partial q_i}(x, \frac{\partial g}{\partial y}(y))$ solvable by iterations).

Heuristically, if $f = \Phi^*[g]$, then $\Lambda_f = \Lambda_g \circ \Phi$, where $\Lambda_f = \mathrm{gr}(df)$.

Description of pullbacks

Example

Let $S(x,q) = S^0(x) + \varphi^i(x)q_i$. Exercise: $\Phi^*[g] = S^0 + \varphi^* g$.
(NB: ordinary maps have generating functions $S = \varphi^i(x)q_i$.)

Description of pullbacks

Example

Let $S(x,q)=S^0(x)+\varphi^i(x)q_i$. Exercise: $\Phi^*[g]=S^0+\varphi^*g$. (NB: ordinary maps have generating functions $S=\varphi^i(x)q_i$.)

For a general $S(x,q)=S^0(x)+\varphi^i(x)q_i+\dots$, the equation $y^i=(-1)^{\tilde{\imath}}\frac{\partial S}{\partial q_i}(x,\frac{\partial g}{\partial y}(y))$ defines a formal perturbation $\varphi_g\colon\ M_1\to M_2$ of a map $\varphi=\varphi_0\colon\ M_1\to M_2$:

$$y^i=\varphi_g^i(x)=\varphi^i(x)+S^{ij}(x)\partial_j g(\varphi(x))+\dots\,,$$

and $\Phi^*[g](x)=\bigl(g(y)+S(x,q)-y^iq_i\bigr)\,|_{y=\varphi_g(x),q=\partial g/\partial y(\varphi_g(x))}$, which gives Φ^* as a **formal nonlinear differential operator** :

General form of $\Phi^*\colon\ \mathbf{C}^\infty(M_2)\to\mathbf{C}^\infty(M_1)$

$$\Phi^*[g](x)=S^0(x)+g(\varphi(x))+\frac{1}{2}\,S^{ij}(x)\partial_i g(\varphi(x))\partial_j g(\varphi(x))+\dots$$

Coordinate invariance

Transformation law for generating functions of thick morphisms

A generating function $S(x,q)$ as a geometric object on $M_1\times M_2$ transforms by

$$S'(x',q')=S(x,q)-y^iq_i+y^{i'}q_{i'}\,.$$

Here $S(x,q)$ is the expression for S in 'old' coordinates and $S'(x',q')$ is the expression for S in 'new' coordinates. At the r.h.s., the variables x^a and $y^{i'}$ are given by substitutions: $x^a=x^a(x')$ and $y^{i'}=y^{i'}(y)$, while q_i and y^i are determined from

$$q_i=\frac{\partial y^{i'}}{\partial y^i}(y)\,q_{i'}\,,\quad y^i=(-1)^{\tilde{\imath}}\frac{\partial S}{\partial q_i}(x,q)\,.$$

(The cocycle condition is satisfied.) The canonical relation $\Phi\subset T^*M_2\times(-T^*M_1)$ specified by S and the pullback operation Φ^* do not depend on a choice of coordinates.

Key fact: derivative of pullback

Theorem

Let $\Phi\colon M_1 \Longrightarrow M_2$ be a thick morphism. Consider the pullback

$$\Phi^*\colon \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)\,.$$

Then for every $g \in \mathbf{C}^\infty(M_2)$, the derivative $T\Phi^*[g]$ is given by

$$T\Phi^*[g] = \varphi_g^*\,,$$

where $\varphi_g^*\colon C^\infty(M_2) \to C^\infty(M_1)$ is the usual pullback with respect to the "perturbed" map $\varphi_g\colon M_1 \to M_2$, $\varphi_g = \varphi_0 + \varphi_1 + \varphi_2 + \ldots$ defined by $y^i = (-1)^{\tilde{\imath}}\frac{\partial S}{\partial q_i}(x, \frac{\partial g}{\partial y}(y))$.

Corollary

For every g, the derivative $T\Phi^*[g]$ of $\Phi^*\colon \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)$ is an algebra homomorphism $C^\infty(M_2) \to C^\infty(M_1)$.

Composition law

Consider thick morphisms $\Phi_{21}\colon M_1 \to M_2$ and $\Phi_{31}\colon M_2 \to M_3$ with generating functions $S_{21} = S_{21}(x, q)$ and $S_{32} = S_{32}(y, r)$.

Theorem

The composition $\Phi_{32} \circ \Phi_{21}$ is well-defined as a thick morphism $\Phi_{31}\colon M_1 \to M_3$ with the generating function $S_{31} = S_{31}(x, r)$, where

$$S_{31}(x, r) = S_{32}(y, r) + S_{21}(x, q) - y^i q_i$$

and y^i and q_i are expressed through (x^a, r_μ) from the system

$$q_i = \frac{\partial S_{32}}{\partial y^i}(y, r)\,, \quad y^i = (-1)^{\tilde{\imath}} \frac{\partial S_{21}}{\partial q_i}(x, q)\,,$$

which has a unique solution as a power series in r_μ and derivatives of S_{32}.

Resulting structure

Formal category

- Composition of thick morphisms is associative and $(\Phi_{32} \circ \Phi_{21})^* = \Phi_{21}^* \circ \Phi_{32}^*$.
- In the lowest order, the composition is as in the semidirect product $\mathcal{SM}\text{an} \rtimes \mathbf{C}^\infty$, whose arrows are pairs (φ_{21}, f_{21}) with the composition $(\varphi_{32}, f_{32}) \circ (\varphi_{21}, f_{21}) = (\varphi_{32} \circ \varphi_{21}, \varphi_{21}^* f_{32} + f_{21})$.
- Thick morphisms form a *formal category* $\mathcal{ET}\text{hick}$, a "formal thickening" of the category $\mathcal{SM}\text{an} \rtimes \mathbf{C}^\infty$.

Fermionic version

Fermionic version

- There is a fermionic version of the above based on anticotangent bundles ΠT^*M and odd generating functions $S(x, y^*)$.
- "Odd thick morphisms" $\Psi\colon\, M_1 \Rrightarrow M_2$ induce nonlinear pullbacks $\Psi^*\colon\, \boldsymbol{\Pi C}^{\infty}(M_2) \to \boldsymbol{\Pi C}^{\infty}(M_1)$ on **odd** functions ("fermionic fields").
- Composition of odd thick morphisms gives a formal category $\mathcal{OThick}$, which is a formal thickening of $\mathcal{SMan} \rtimes \boldsymbol{\Pi C}^{\infty}$.

Particular example: "thick diffeomorphisms"

Consider invertible thick morphisms $\Phi\colon\; M \longrightarrow M$.

Example

$\Phi = \mathrm{id}$ corresponds to $S(x,q) = x^a q_a$. An infinitesimal thick diffeomorphism Φ_ε is defined by $S_\varepsilon(x,q) = x^a q_a + \varepsilon H(x,q)$, where $H \in C^\infty(T^*M)$.

Theorem

- *The pullback by an infinitesimal thick diffeomorphism as above has the form of "Hamilton-Jacobi flow":*

$$\Phi_\varepsilon^*\colon\; f(x) \mapsto f(x) + \varepsilon H\Big(x, \frac{\partial f}{\partial x}\Big)\,.$$

- *The induced Lie bracket is exactly the canonical Poisson bracket.*

Particular example: "thick diffeomorphisms", cont'd

So the Lie algebras of the (formal) group of thick diffeomorphisms of M and the group of canonical transformations of T^*M coincide. We have nonlinear infinitesimal action of canonical transformations of T^*M on $\mathbf{C}^{\infty}(M)$.

Theorem

Let Φ_t be a 1-parameter group of thick diffeomorphisms of M with generator H; denote $f_t := \Phi_t^*(f)$ for $f \in \mathbf{C}^{\infty}(M)$. Then f_t satisfies

$$\frac{\partial f_t}{\partial t} = H\Big(x, \frac{\partial f_t}{\partial x}\Big)\,,$$

with the initial condition $f_0 = f$.

In view of the relation between the Hamilton-Jacobi equation and the Schrödinger equation, this is a "big hint" at the possibility of a **quantum version** — *which indeed exists.*

Motivation: Why we may need "nonlinear pullbacks"?

Answer: for L_∞-morphisms of homotopy Poisson brackets.
Recall, for a (super)manifold M:

P_∞- or S_∞-structure ("homotopy Poisson or Schouten"):

- A sequence of brackets on $C^\infty(M)$ satisfying Leibniz rule and making it an L_∞-algebra (in one of the two versions).

L_∞-algebras as Q-manifolds. The following are equivalent:

- An L_∞-algebra structure on a vector space L
- A *homological vector field* $Q \in \mathrm{Vect}((\Pi)L)$: $\tilde{Q} = 1$, $Q^2 = 0$.

L_∞-morphisms as Q-maps:

An L_∞-morphism $L \rightsquigarrow K$ $\Leftrightarrow$ a formal nonlinear map $(\Pi)L \to (\Pi)K$ intertwining vector fields Q_L and Q_K

$\Rightarrow$ For (super)manifolds, a "P_∞- or S_∞-morphism" should induce a NONLINEAR mapping of functions. Ordinary pullbacks won't do.

Example: Higher Koszul brackets

Classical fact: for a Poisson M, there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{A}^k(M) & \xrightarrow{d_P} & \mathfrak{A}^{k+1}(M) \\ {\scriptstyle (P^{ab})}\uparrow & & \uparrow{\scriptstyle (P^{ab})} \\ \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M)\,, \end{array}$$

and the vertical arrows map Koszul bracket to Schouten bracket.
Homotopy case: for a P_∞-manifold M, there are *higher Koszul brackets* (S_∞) and one can still construct a commutative diagram

$$\begin{array}{ccc} \mathfrak{A}(M) & \xrightarrow{d_P} & \mathfrak{A}(M) \\ \uparrow & & \uparrow \\ \Omega(M) & \xrightarrow{d} & \Omega(M)\,. \end{array}$$

However the vertical arrows here **cannot** map *many* higher Koszul brackets into *one* Schouten bracket. Need an L_∞-morphism.

Recollection: L_∞-algebras – 1

We consider $\mathbb{Z}_2$-graded version. (One can include a $\mathbb{Z}$-grading.)

There are two parallel notions: "symmetric" and "antisymmetric".

Definition (*L_∞-algebra: antisymmetric version*)

A vector space $L = L_0 \oplus L_1$ with a collection of multilinear operations

$$[-,\ldots,-]\colon \underbrace{L\times\ldots\times L}_{k\text{ times}} \to L \quad (\text{for } k=0,1,2,\ldots)$$

such that

- the parity of the kth bracket is $k \mod 2$;
- all brackets are antisymmetric (in $\mathbb{Z}_2$-graded sense);
- $\sum_{r+s=n}\sum_{\text{shuffles}}(-1)^{\beta}[[x_{\sigma(1)},\ldots,x_{\sigma(r)}],\ldots,x_{\sigma(r+s)}]=0$, for all $n = 0,1,2,3,\ldots$

(here $(-1)^{\beta} = (-1)^{rs}\operatorname{sgn}\sigma(-1)^{\alpha}$ and $(-1)^{\alpha}$ is the Koszul sign).

Recollection: L_∞-algebras – 2

A parallel notion is as follows.

Definition (*L_∞-algebra: symmetric version*)

A vector space $V = V_0 \oplus V_1$ with a collection of multilinear operations

$$\{-,\ldots,-\}\colon \underbrace{V\times\ldots\times V}_{k \text{ times}} \to V \quad (\text{for } k=0,1,2,\ldots)$$

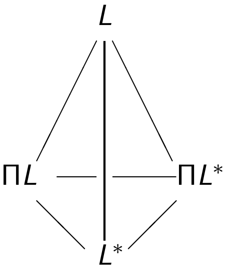
such that

- all brackets are odd;
- all brackets are symmetric (in $\mathbb{Z}_2$-graded sense);
- $\sum_{r+s=n}\sum_{\text{shuffles}}(-1)^{\alpha}\{\{v_{\sigma(1)},\ldots,v_{\sigma(r)}\},\ldots,v_{\sigma(r+s)}\}=0$, for all $n = 0, 1, 2, 3, \ldots$

(here $(-1)^{\alpha}$ is the Koszul sign).

NB: here signs come from parities only!

Recollection: L_∞-algebras – 3



Equivalent structures:

- Antisymmetric L_∞-algebra structure in L
- Symmetric L_∞-algebra structure in ΠL
- Homological vector field $Q \in \mathrm{Vect}(\Pi L)$, i.e., $\tilde{Q} = 1,\ Q^2 = 0$

(Also: P_∞- and S_∞-structures on L^* and ΠL^*, see later.)

NB: we identify $\mathbb{Z}_2$-graded vector spaces with supermanifolds.

Recollection: L_∞-algebras – 4.

Relation between brackets in L and ΠL

$\{\Pi x_1, \ldots, \Pi x_n\} = (-1)^{\varepsilon} \Pi [x_1, \ldots, x_n]$, where $\varepsilon = \sum \tilde{x}_k (n-k)$.

Relations with homological vector field Q

- $Q(\xi) = \sum \frac{1}{n!} \{\underbrace{\xi, \ldots, \xi}_{n}\}$, where $\xi \in V = \Pi L$
- Higher derived bracket formula:
  $\iota([x_1, \ldots, x_n]) = (-1)^{\varepsilon} [\ldots [Q, \iota(x_1)], \ldots, \iota(x_n)](0)$,
  for $x = x^i e_i \in L$, and $\iota(x) := (-1)^{\tilde{x}} x^i \partial/\partial\xi^i \in \mathrm{Vect}(\Pi L)$

L_∞-morphisms

An L_∞-morphism $L \rightsquigarrow K$ is given by a sequence $\Lambda^n L \to K$ or $S^n(\Pi L) \to \Pi K$ satisfying a certain sequence of identities ("higher homotopies"). **It is equivalent to a formal Q-map $\Pi L \to \Pi K$.**

"Homotopy Poisson/Schouten structures": P_∞ and S_∞

A P_∞- (S_∞-) structure on a supermanifold M is an antisymmetric (resp., symmetric) L_∞-structure on $C^\infty(M)$ such that the brackets are multiderivations of the associative product.

1. A P_∞-structure on M is specified by an **even** function $P \in C^\infty(\Pi T^*M)$ satisfying $[P, P] = 0$, by
$$\{f_1, \ldots, f_k\}_P := [\ldots[P, f_1], \ldots, f_k]|_M \,.$$
2. An S_∞-structure on M is specified by an **odd** function $H \in C^\infty(T^*M)$ satisfying $(H, H) = 0$, by
$$\{f_1, \ldots, f_k\}_H := (\ldots(H, f_1), \ldots, f_k)|_M \,.$$

Homological vector fields for P_∞ and S_∞ are "Hamilton–Jacobi":

1. $Q_P = \int_M Dx\, P(x, \frac{\partial\psi}{\partial x})\frac{\delta}{\delta\psi(x)} \in \mathrm{Vect}(\mathbf{\Pi C}^\infty(M))$
2. $Q_H = \int_M Dx\, H(x, \frac{\partial f}{\partial x})\frac{\delta}{\delta f(x)} \in \mathrm{Vect}(\mathbf{C}^\infty(M))$

Key theorem: nonlinear pullback as an L_∞-morphism

Let M_1 and M_2 be S_∞-manifolds, with $H_i \in C^\infty(T^*M_i)$, $i = 1, 2$.

Definition of an S_∞ ("homotopy Schouten") thick morphism

A thick morphism $\Phi\colon M_1 \Longrightarrow M_2$ is S_∞ if $\pi_1^* H_1 = \pi_2^* H_2$ on Φ.

Note: this is expressed by the Hamilton–Jacobi equation for $S(x, q)$

$$H_1\Bigl(x, \frac{\partial S}{\partial x}\Bigr) = H_2\Bigl(\frac{\partial S}{\partial q}, q\Bigr) . \qquad (1)$$

Theorem

If a thick morphism of S_∞-manifolds $\Phi\colon M_1 \Longrightarrow M_2$ is S_∞, then the pullback

$$\Phi^*\colon \mathbf{C}^\infty(M_2) \to \mathbf{C}^\infty(M_1)$$

is an L_∞-morphism of the homotopy Schouten brackets.
That is: if (1) holds, then Φ^* intertwines the homological vector fields $Q_{H_2} \in \mathrm{Vect}(\mathbf{C}^\infty(M_2))$ and $Q_{H_1} \in \mathrm{Vect}(\mathbf{C}^\infty(M_1))$.

Analog for P_∞-structures

Let M_1 and M_2 be P_∞-manifolds, with $P_i \in C^\infty(\Pi T^* M_i)$, $i = 1, 2$.

Definition of a P_∞ ("homotopy Poisson") odd thick morphism

An odd thick morphism $\Psi\colon\ M_1 \Rrightarrow M_2$ is P_∞ if $\pi_1^* P_1 = \pi_2^* P_2$ on Ψ.

This is expressed by the Hamilton–Jacobi equation for odd $S(x, y^*)$

$$P_1\left(x, \frac{\partial S}{\partial x}\right) = P_2\left(\frac{\partial S}{\partial y^*}, y^*\right). \tag{2}$$

Theorem

If an odd thick morphism of P_∞-manifolds $\Phi\colon\ M_1 \Rrightarrow M_2$ is P_∞, then the pullback

$$\Psi^*\colon\ \mathbf{\Pi C}^\infty(M_2) \to \mathbf{\Pi C}^\infty(M_1)$$

is an L_∞-morphism of the homotopy Poisson brackets.
That is: if (2) holds, then Ψ^* intertwines the homological vector fields $Q_{P_2} \in \mathrm{Vect}(\mathbf{\Pi C}^\infty(M_2))$ and $Q_{P_1} \in \mathrm{Vect}(\mathbf{\Pi C}^\infty(M_1))$.

Adjoint for a nonlinear transformation

Theorem (adjoint and pushforward)

1. For any fiberwise (not necessarily fiberwise-linear) map of vector bundles $\Phi\colon\, E_1 \to E_2$, there is a thick morphism (the adjoint)

$$\Phi^*\colon\, E_2^* \Rrightarrow E_1^*\,,$$

coinciding with the usual adjoint map if Φ is fiberwise-linear, and with the same properties. Construction:
$\Phi^* := \big(\boldsymbol{\kappa}\times\boldsymbol{\kappa}\big)(\Phi)^{op} \subset T^*E_1^* \times (-T^*E_2^*)$, *where*
$\boldsymbol{\kappa}\colon\, T^*E \to T^*E^*$ *is the Mackenzie–Xu diffeomorphism.*
2. The pushforward of functions on the duals (pullback by adjoint)

$$\Phi_* := (\Phi^*)^*\colon\, \mathbf{C}^{\infty}(E_1^*) \to \mathbf{C}^{\infty}(E_2^*)$$

maps the subspace of sections $\mathbf{C}^{\infty}(M,E_1) \subset \mathbf{C}^{\infty}(E_1^*)$ to $\mathbf{C}^{\infty}(M,E_2)$, and coincides on sections with $\boldsymbol{v} \mapsto \Phi\circ\boldsymbol{v}$.

"Fermionic analog": parity reversed adjoint

Theorem ("antiadjoint")

1. For any fiberwise map of vector bundles $\Phi\colon\, E_1 \to E_2$ *(not necessarily fiberwise-linear), there is an odd thick morphism (antiadjoint)*

$$\Phi^{*\Pi}\colon\, \Pi E_2^* \Rrightarrow \Pi E_1^*\,,$$

coinciding with the usual adjoint combined with parity reversion if Φ *is fiberwise-linear, and with the same properties. Construction:* $\Phi^{*\Pi} := (\chi\times\chi)(\Phi)^{op} \subset \Pi T^*(\Pi E_1^*) \times (-\Pi T^*(\Pi E_2^*))$*, where* $\chi\colon\, \Pi T^*E \to \Pi T^*(\Pi E^*)$ *is the odd analog of the Mackenzie–Xu diffeomorphism.*

2. The pushforward of functions on the antidual bundles

$$\Phi_*^{\Pi} := (\Phi^{*\Pi})^*\colon\, \mathbf{\Pi C}^{\infty}(\Pi E_1^*) \to \mathbf{\Pi C}^{\infty}(\Pi E_2^*)$$

maps the subspace of sections $\mathbf{C}^{\infty}(M, E_1) \subset \mathbf{\Pi C}^{\infty}(\Pi E_1^*)$ *to* $\mathbf{C}^{\infty}(M, E_2)$*, and on sections coincides with* $\boldsymbol{v} \mapsto \Phi\circ\boldsymbol{v}$*.*

Recollection: L_∞-algebroids

Definition

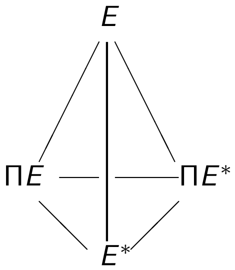
An L_∞-algebroid is a (super) vector bundle $E \to M$ with an antisymmetric L_∞-algebra structure on sections and a sequence of anchors $E \times_M \ldots \times_M E \to TM$ so that the Leibniz identities hold:

$$[u_1, \ldots, u_{n-1}, fu_n] = a(u_1, \ldots, u_{n-1})(f)\, u_n + (-1)^\alpha f\, [u_1, \ldots, u_n]\,,$$

where $(-1)^\alpha = (-1)^{(\tilde{u}_1 + \ldots + \tilde{u}_{n-1} + n)\tilde{f}}$.

- An L_∞-algebroid structure on $E \to M$ is equivalent to a (formal) homological vector field on the supermanifold ΠE.
- An L_∞-morphism of L_∞-algebroids $\Phi\colon\; E_1 \rightsquigarrow E_2$ is specified by a map (in general, nonlinear) $\Phi\colon\; \Pi E_1 \to \Pi E_2$ such that the corresponding homological vector fields are Φ-related.
- Example: all anchors assemble into an L_∞-morphism $\Pi E \to \Pi TM$.

Manifestations of an L_∞-algebroid structure



Equivalent manifestations

- L_∞-algebroid structure in vector bundle $E \to M$
- P_∞-structure on supermanifold E^*
- S_∞-structure on supermanifold ΠE^*
- Q-structure (homological vector field) on supermanifold ΠL

L_∞-morphisms of Lie-Poisson and Lie-Schouten brackets

Consider an L_∞-morphism $\Phi\colon\, E_1 \rightsquigarrow E_2$ of L_∞-algebroids over a base M. It is given by a Q-map $\Phi\colon\, \Pi E_1 \to \Pi E_2$. (We suppress indications on parity changes.)

Theorem

An L_∞-morphism $\Phi\colon\, E_1 \rightsquigarrow E_2$ over a base M induces morphisms of homotopy structures:

- *S_∞ thick morphism $\Phi^*\colon\, \Pi E_2^* \Rrightarrow \Pi E_2^*$*
- *P_∞ thick morphism $\Phi^*\colon\, E_2^* \Rrightarrow E_2^*$*

This gives L_∞-morphisms of the homotopy Lie-Schouten and homotopy Lie-Poisson brackets, respectively:

$$\Phi_*\colon\, \mathbf{C}^\infty(\Pi E_1^*) \to \mathbf{C}^\infty(\Pi E_2^*)$$

and

$$\Phi_*\colon\, \Pi\mathbf{C}^\infty(E_1^*) \to \Pi\mathbf{C}^\infty(E_2^*)\,.$$

Corollary: L_∞-morphisms induced by the anchor

Recall that the anchors $E \times_M \ldots \times_M E \to TM$ for an L_∞-algebroid $E \to M$ assemble into a single (nonlinear, in general) bundle map $a\colon \Pi E \to \Pi TM$ over M (which we also refer to as **anchor**).

Corollary

The anchor for an L_∞-algebroid $E \to M$ induces L_∞-morphisms

$$\mathbf{C}^\infty(\Pi E^*) \to \mathbf{C}^\infty(\Pi T^*M)$$

of the homotopy Schouten brackets, and

$$\mathbf{\Pi C}^\infty(E^*) \to \mathbf{\Pi C}^\infty(T^*M)\,.$$

of the homotopy Poisson brackets.

Application to higher Koszul brackets on a P_∞-manifold

Let M be a P_∞-manifold. Then T^*M becomes an L_∞-algebroid. In particular, there is the anchor $a\colon \Pi T^*M \to \Pi TM$ and its dual (a thick morphism) $a^*\colon \Pi T^*M \Rrightarrow \Pi TM$.

Corollary

For a homotopy Poisson manifold M, the non-linear pullback

$$a_*\colon \boldsymbol{\Omega}(M) = \mathbf{C}^\infty(\Pi TM) \to \mathbf{C}^\infty(\Pi T^*M) = \mathfrak{A}(M)\,,$$

is an L_∞-morphism between the higher Koszul brackets on forms induced by the homotopy Poisson structure and the canonical Schouten bracket on multivector fields.

Quantum pullbacks and quantum thick morphisms

Definition

A *quantum pullback* $\hat{\Phi}^*\colon\; OC^{\infty}_{\hbar}(M_2) \to OC^{\infty}_{\hbar}(M_1)$ is defined by

$$(\hat{\Phi}^*[w])(x) = \int_{T^*M_2} Dy\bar{D}q\; e^{\frac{i}{\hbar}(S_{\hbar}(x,q)-y^i q_i)}\, w(y)\,.$$

A *quantum thick* (or *microformal*) *morphism* $\hat{\Phi}\colon\; M_1 \twoheadrightarrow_{\hbar} M_2$ is the corresponding arrow in the dual category.

Here $S_{\hbar}(x,q)$ is a *quantum generating function*:

$$S_{\hbar}(x,q) = S^0_{\hbar}(x) + \varphi^i_{\hbar}(x)q_i + \frac{1}{2}\,S^{ij}_{\hbar}(x)q_jq_i + \frac{1}{3!}\,S^{ijk}_{\hbar}(x)q_kq_jq_i + \dots$$

$OC^{\infty}_{\hbar}(M)$ is the algebra of *oscillatory wave functions*, i.e. sums of formal expressions $w(x) = a_{\hbar}(x)e^{\frac{i}{\hbar}b_{\hbar}(x)}$, where
$a_{\hbar}, b_{\hbar} \in C^{\infty}(M)[[\hbar]]$.
($\bar{D}q := (2\pi\hbar)^{-n}(i\hbar)^m Dq$ in dimension $n|m$.)

Classical limit

Theorem

Let $\hat{\Phi}\colon M_1 \longrightarrow_{\hbar} M_2$ be a quantum thick morphism with a quantum generating function $S_{\hbar}$. Consider $S_0(x,q) := \lim_{\hbar\to 0} S_{\hbar}(x,q)$ as the (classical) generating function of a (classical) thick morphism $\Phi\colon M_1 \longrightarrow M_2$. Then for any oscillatory wave function of the form $w(y) = e^{\frac{i}{\hbar}g(y)}$ on M_2, the quantum pullback given by

$$\hat{\Phi}^*\left[e^{\frac{i}{\hbar}g}\right] = e^{\frac{i}{\hbar}f_{\hbar}(x)}\,,$$

where $f_{\hbar} = \Phi^*[g] + O(\hbar)$, and Φ^* is the pullback by the classical microformal morphism $\Phi\colon M_1 \longrightarrow M_2$ defined by $S_0(x,q)$.

We say that $\Phi = \lim_{\hbar\to 0} \hat{\Phi}$.

Explicit formula for quantum pullback

Suppose

$$S_\hbar(x,q) = S^0_\hbar(x) + \varphi^i_\hbar(x)q_i + S^+_\hbar(x,q)\,,$$

where $S^+_\hbar(x,q)$ is the sum of all terms of order $\geqslant 2$ in q_i.

Theorem

The action of $\hat{\Phi}^*$ defined by $S_\hbar(x,q)$ can be expressed as follows:

$$\bigl(\hat{\Phi}^* w\bigr)(x) = e^{\frac{i}{\hbar}S^0_\hbar(x)}\left(e^{\frac{i}{\hbar}S^+_\hbar\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}w(y)\right)\Bigg|_{y^i=\varphi^i_\hbar(x)}\,.$$

Hence the quantum pullback $\hat{\Phi}^*$ is a special type formal linear differential operator over a 'quantum-perturbed' map $\varphi_\hbar\colon\ M_1\to M_2$. Here $S^0_\hbar(x)$ gives the phase factor, $\varphi^i_\hbar(x)q_i$ gives the map, and $S^+_\hbar(x,q)$ is responsible for "quantum corrections".

Digression: brackets generated by an operator

Let A be a commutative algebra with 1 over $\mathbb{C}[[\hbar]]$. Let Δ be a linear operator on A. Consider two sequences of multilinear operations (of parity $\tilde{\Delta}$ and symmetric in the supersense):

Definition (a modification of Koszul's construction)

Quantum brackets generated by Δ :

$$\{a_1, \ldots, a_k\}_{\Delta,\hbar} := (-i\hbar)^{-k}[\ldots[\Delta, a_1], \ldots, a_k](1)\,;$$

classical brackets generated by Δ :

$$\{a_1, \ldots, a_k\}_{\Delta,0} := \lim_{\hbar\to 0} (-i\hbar)^{-k}[\ldots[\Delta, a_1], \ldots, a_k](1)$$

- Δ is a *formal $\hbar$-differential operator* if all quantum brackets are defined;
- Δ is an *$\hbar$-differential operator of order $\leqslant n$* if all quantum brackets vanish for $k > n$.

More on brackets and operators

Remark (Explicit formulas)

- For $k = 0$, $\{\varnothing\}_{\Delta,\hbar} = \Delta(1)$;
- for $k = 1$, $\{a\}_{\Delta,\hbar} = (-i\hbar)^{-1}\big(\Delta(a) - \Delta(1)a\big)$;
- for $k = 2$, $\{a, b\}_{\Delta,\hbar} =$
  $(-i\hbar)^{-2}\big(\Delta(ab) - \Delta(a)b - (-1)^{\tilde{a}\tilde{b}}\Delta(b)a + \Delta(1)ab\big)$;
- for general k, $\{a_1, \ldots, a_k\}_{\Delta,\hbar} = (-i\hbar)^{-k}$
  $$\sum_{s=0}^{k}(-1)^s \sum_{(k-s,s)\text{-shuffles }\tau} (-1)^{\alpha}\, \Delta\big(a_{\tau(1)} \ldots a_{\tau(k-s)}\big)\, a_{\tau(k-s+1)} \ldots a_{\tau(k)}\,,$$

  where $(-1)^{\alpha} = (-1)^{\alpha(\tau;\tilde{a}_1,\ldots,\tilde{a}_k)}$ is the Koszul sign.

Definition ($\hbar$-differential operators)

$\operatorname{ord}_\hbar \Delta \leqslant k$ iff for all $a \in A$, $[\Delta, a] = i\hbar B$ where $\operatorname{ord}_\hbar B \leqslant k - 1$.

$S_{\infty,\hbar}$-algebras

Let Δ be odd. *If $\Delta^2 = 0$, then the quantum brackets define an L_∞-algebra structure on A (in odd symmetric version).* Additionally, they satisfy the *modified Leibniz identity*

$$\{a_1,\ldots,a_{k-1},ab\}_{\Delta,\hbar} = \{a_1,\ldots,a_{k-1},a\}_{\Delta,\hbar}b \pm a\{a_1,\ldots,a_{k-1},b\}_{\Delta,\hbar} + \underbrace{(-i\hbar)\{a_1,\ldots,a_{k-1},a,b\}_{\Delta,\hbar}}_{\text{extra term}}\,.$$

We call such an algebraic structure an $S_{\infty,\hbar}$*-algebra*.
Note: the operator Δ and the whole $S_{\infty,\hbar}$-structure are fully defined by 0- and 1-brackets.

Lemma

The homological vector field on A corresponding to the quantum brackets generated by Δ has the "Batalin-Vilkovisky " form

$$Q = e^{-\frac{i}{\hbar}a}\Delta\left(e^{\frac{i}{\hbar}a}\right)\frac{\delta}{\delta a}\,.$$

BV-manifolds and BV quantum morphisms

Definition

(1) A *BV-manifold* is a supermanifold M equipped with an odd formal $\hbar$-differential operator Δ, $\Delta^2 = 0$. The operator Δ is the *BV-operator*.

(2) A *(quantum) BV-morphism* of BV-manifolds (M_1, Δ_1) and (M_2, Δ_2) is a quantum thick morphism $\hat{\Phi}\colon\ M_1 \to_{\hbar} M_2$ such that

$$\Delta_1 \circ \hat{\Phi}^* = \hat{\Phi}^* \circ \Delta_2\,.$$

Since Δ induces a sequence of quantum brackets, and is defined by the 0- and 1-brackets, a BV-structure and an $S_{\infty,\hbar}$-structure on M are equivalent.

Question

How to obtain an L_∞-morphism of quantum brackets generated by BV-operators? (Note: the operator $\hat{\Phi}^*$ is linear, so cannot be the answer.)

L_∞-morphism of quantum brackets induced by a quantum BV-morphism

Define $\hat{\Phi}^! \colon \ \mathbf{C}^\infty_\hbar(M_2) \to \mathbf{C}^\infty_\hbar(M_1)$ by

$$\hat{\Phi}^! := \frac{\hbar}{i} \ln \circ \hat{\Phi}^* \circ \exp \frac{i}{\hbar} \,,$$

or $\hat{\Phi}^!(g) = \frac{\hbar}{i} \ln \hat{\Phi}^*\left(e^{\frac{i}{\hbar}g}\right)$, for a $g \in \mathbf{C}^\infty_\hbar(M_2)$.

Theorem

If $\hat{\Phi} \colon \ M_1 \longrightarrow_\hbar M_2$ is a BV quantum morphism, then $\hat{\Phi}^!$ is an L_∞-morphism of the $S_{\infty,\hbar}$-algebras of functions.
Or, in greater detail: $\hat{\Phi}^!$ is a morphism of infinite-dimensional Q-manifolds $\mathbf{C}^\infty_\hbar(M_2) \to \mathbf{C}^\infty_\hbar(M_1)$, where

$$Q_\Delta = \int Dx \; e^{-\frac{i}{\hbar}f} \Delta\left(e^{\frac{i}{\hbar}f}\right) \frac{\delta}{\delta f(x)} \,.$$

From a quantum BV morphism to a classical S_∞ thick morphism

Let M be a BV-manifold with a BV-operator Δ. In the limit $\hbar \to 0$, Δ gives an S_∞-structure. Its "master Hamiltonian" is

$$H(x,p) = \lim_{\hbar\to 0} e^{-\frac{i}{\hbar}x^a p_a}\Delta\big(e^{\frac{i}{\hbar}x^a p_a}\big)\,.$$

Theorem ("analog of Egorov's theorem")

Let M_1 and M_2 be BV-manifolds and let $\hat{\Phi}\colon\, M_1 \xrightarrow{}_{\hbar} M_2$ be a BV quantum thick morphism. Then its classical limit $\Phi\colon\, M_1 \to M_2$ is an S_∞ thick morphism for the induced S_∞-structures.

Explicitly: the intertwining relation $\Delta_1 \circ \hat{\Phi}^* = \hat{\Phi}^* \circ \Delta_2$ implies the Hamilton-Jacobi equation for the classical thick morphism $\Phi = \lim_{\hbar\to 0} \hat{\Phi}$:

$$H_1\Big(x, \frac{\partial S}{\partial x}\Big) = H_2\Big(\frac{\partial S}{\partial q}, q\Big)\,.$$

"Quantum thick diffeomorphisms"

For an infinitesimal quantum diffeomorphism,

$$S_{\varepsilon}(x,q) = x^a q_a + \varepsilon H^{\hbar}(x,q)\,. \tag{3}$$

The function $H_{\hbar}(x,p)$ is the quantum analog of a classical Hamiltonian $H(x,p)$. (Not a usual function on T^*M!)

Lemma

The pullback by an infinitesimal quantum thick diffeomorphism with generator $H_{\hbar}(x,p)$ is given by

$$\hat{\Phi}^*_{\varepsilon}[w](x) = w(x) + \varepsilon\,\frac{i}{\hbar} H_{\hbar}\Bigl(\overset{1}{x}, \frac{\hbar}{i}\overset{2}{\frac{\partial}{\partial x}}\Bigr) w(x)\,.$$

Here in $H_{\hbar}\Bigl(\overset{1}{x}, \frac{\hbar}{i}\overset{2}{\frac{\partial}{\partial x}}\Bigr)$ the indices 1 and 2 atop of the non-commuting operators substituted for the arguments indicate the ordering where all x's stand to the left of all $\frac{\hbar}{i}\frac{\partial}{\partial x}$'s.

"Quantum thick diffeomorphisms", cont'd

We obtain that $H_\hbar(x, p)$ is the full symbol of a quantum Hamiltonian $\hat{H}$ based on xp-quantization. The transformation law of $H_\hbar(x, p)$ as a geometric object is different from that for classical Hamiltonians. In the zeroth order in $\hbar$, the function $H_\hbar(x, p)$ transforms as a genuine function on T^*M and then there are "quantum corrections" in the transformation law.

Theorem

Define $w = w_t(x)$ is $\hat{\Phi}_t^*[w_0]$ for a 1-parameter group of quantum diffeomorphisms $\hat{\Phi}_t$ with a generator $H^\hbar = H^\hbar(x, p)$. Then w satisfies the Schrödinger equation

$$\frac{\hbar}{i}\frac{\partial w}{\partial t} = \hat{H}w\,, \quad \textit{where } \hat{H} = H_\hbar\Bigl(\overset{1}{x}, \frac{\hbar}{i}\overset{2}{\frac{\partial}{\partial x}}\Bigr).$$

(Further: for the category of vector spaces and linear canonical relations, we get a variant of Berezin-Neretin spinor represent'n.)

“Nonlinear functional-algebraic duality”

Functional-algebraic duality

In various contexts there is a “duality” between spaces and algebras, maps of spaces and algebra homomorphisms. Example: Gelfand-Kolmogorov duality between compact Hausdorff spaces and the algebras of real-valued continuous functions on them. Or, for smooth manifolds and the algebras of C^{∞} functions. (Also: definition of supermanifolds, etc.etc.)

Definition

A *non-linear homomorphism* of algebras is a map $A_1 \to A_2$ such that its derivative at every point $a \in A_1$ is an algebra homomorphism. (Variant: a formal map $A_1 \to A_2$.)

We know that the pullbacks Φ^* for thick morphisms Φ are (formal) non-linear homomorphisms. Is the converse true?

"Nonlinear functional-algebraic duality", cont'd

Theorem

If a formal map $L\colon \mathbf{C}^{\infty}(M_2) \to \mathbf{C}^{\infty}(M_1)$ is a non-linear algebra homomorphism, then $L = \Phi^*$ (the pullback) for some thick morphism $\Phi\colon\; M_1 \Rrightarrow M_2$, which is uniquely defined.

Conjectured by the speaker and proved recently by Khudaverdian.

- If a thick morphism $\Phi\colon\; M_1 \Rrightarrow M_2$ is specified by a generating function S, then
$$\Phi^*(y^i c_i) = S(x, c)\,.$$
- So if a formal map $\alpha\colon\; \mathbf{C}^{\infty}(M_2) \to \mathbf{C}^{\infty}(M_1)$ is given, we can introduce $S(x, q) := \alpha(y^i q_i)$. Then the problem is to show that α and Φ^* (for Φ with the generating function S) coincide. Here it is where the condition of "nonlinear algebra homomorphism" for α is used.

Some open questions

- "Thick manifolds" and "thick bundles": if we have thick diffeomorphisms, what can be obtained by gluing?
- Action of thick morphisms on forms, cohomology, etc. ... (See [4])
- Closed formulas for Φ^*, perhaps via some diagrammatic technique.
- A characterization of quantum pullbacks among formal FIOs by some properties (similarly to the characterization of classical pullbacks as nonlinear algebra homomorphisms).
- ...

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FINIS

Thank you for attention!