Local and Nonlocal Geometry of PDEs and Integrability

Coverings and nonlocal symmetries of Lax-integrable PDEs

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Outline

- Introduction
- 3D Lax-integrable equations
- Example: the universal hierarchy equation
- 2D equations
- Conclusions

$$
\begin{tikzcd}
 & & \boxed{u_{yz} = u_{ts} + u_s u_{xz} - u_z u_{xs}} \arrow[dl, "s \leftrightarrow y,\ u_s = 0"'] \arrow[dr, "y \leftrightarrow z,\ s = x"] & & \\
 & \boxed{u_{ty} = u_z u_{xy} - u_y u_{xz}} \arrow[dl, "t = y"'] \arrow[d, "z = t"'] \arrow[dr, "z = x"] & & \boxed{u_{yz} = u_{tx} + u_x u_{xy} - u_y u_{xx}} \arrow[dl, "t \leftrightarrow z,\ u_z = 0"'] \arrow[dr, "z = y,\ y \to -y"] & \\
\boxed{u_{yy} = u_z u_{xy} - u_y u_{xz}} \arrow[drr] & \boxed{u_{ty} = u_t u_{xy} - u_y u_{tx}} \arrow[dr] & \boxed{u_{ty} = u_x u_{xy} - u_y u_{xx}} \arrow[d] & & \boxed{u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}} \arrow[dll] \\
 & & \boxed{32 \text{ two} - \text{dimensional equations}} & &
\end{tikzcd}
$$

3D equations - Lax pairs

- the *3D rdDym equation* (3D rdDym)

$$u_{ty} = u_x u_{xy} - u_y u_{xx}, \tag{1}$$

$$w_t = (u_x - \lambda) w_x, \quad w_y = \lambda^{-1} u_y w_x; \tag{2}$$

- the *3D Pavlov equation* (3D PE)

$$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}, \tag{3}$$

$$w_t = (\lambda^2 - \lambda u_x - u_y) w_x, \quad w_y = (\lambda - u_x) w_x; \tag{4}$$

- the *universal hierarchy equation* (UHE)

$$u_{yy} = u_t u_{xy} - u_y u_{tx}, \tag{5}$$

$$w_t = \lambda^{-2}(\lambda u_t - u_y) w_x, \quad w_y = \lambda^{-1} u_y w_x; \tag{6}$$

- the *modified Veronese web equation* (mVw)

$$u_{ty} = u_t u_{xy} - u_y u_{tx}, \tag{7}$$

$$w_t = (\lambda + 1) u_t w_x, \quad w_y = \lambda^{-1} u_y w_x; \tag{8}$$

3D equations - Expansion of w

Setting
$$w = \sum_{i=-\infty}^{\infty} \lambda^i w_i,$$
and substituting this expansion in (2), (4), (6) and (8), we obtain:

- 3D rdDym
$$\begin{aligned} w_{i,t} &= u_x w_{i,x} - w_{i-1,x}, \\ w_{i,y} &= u_y w_{i+1,x}. \end{aligned} \tag{9}$$
- 3D PE
$$\begin{aligned} w_{i,t} &= w_{i-2,x} - u_x w_{i-1,x} - u_y w_{i,x}, \\ w_{i,y} &= w_{i-1,x} - u_x w_{i,x}. \end{aligned} \tag{10}$$

- UHE

$$\begin{aligned} w_{i,t} &= u_t w_{i+1,x} - u_y w_{i+2,x}, \\ w_{i,y} &= u_y w_{i+1,x}. \end{aligned} \tag{11}$$

- mVw

$$\begin{aligned} w_{i,t} &= u_t w_{i,x} - w_{i-1,t}, \\ w_{i,y} &= u_y w_{i+1,x}. \end{aligned} \tag{12}$$

System (9), (10), (11), (12) are infinite in both directions, and the nonlocal quantities w_i are not defined properly. Therefore, we consider two reductions of (9), (10), (11), (12):

a) $w_i = 0$ for $i < 0 \Longrightarrow$ positive hierarchy of CL,

b) $w_i = 0$ for $i > 0 \Longrightarrow$ negative hierarchy of CL.

3D equations - Positive coverings τ^+

- 3D rdDym

$$\begin{aligned} q_{1,t} &= \frac{u_x}{u_y}, & q_{1,x} &= \frac{1}{u_y}, \\ q_{i,t} &= \frac{u_x}{u_y} q_{i-1,y} - q_{i-1,x}, & q_{i,x} &= \frac{q_{i-1,y}}{u_y}. \end{aligned} \tag{13}$$

- 3D PE

$$\begin{aligned} q_{0,t} + u_y q_{0,x} &= 0, & q_{0,y} + u_x q_{0,x} &= 0, \\ q_{1,t} + u_y q_{1,x} &= -u_x q_{0,x}, & q_{1,y} + u_x q_{1,x} &= q_{0,x}, \\ q_{i,t} + u_y q_{i,x} &= q_{i-2,x} - u_x q_{i-1,x}, & q_{i,y} + u_x q_{i,x} &= q_{i-1,x}. \end{aligned} \tag{14}$$

3D equations - Positive coverings τ^+

- UHE

$$
\begin{aligned}
q_{1,y} &= \frac{u_t}{u_y}, & q_{1,x} &= \frac{1}{u_y}, \\
q_{i,y} &= \frac{u_t}{u_y} q_{i-1,y} - q_{i-1,t}, & q_{i,x} &= \frac{q_{i-1,y}}{u_y}.
\end{aligned}
\tag{15}
$$

- mVw

$$
\begin{aligned}
q_{1,t} &= \frac{u_t}{u_y}, & q_{1,x} &= \frac{1}{u_y}, \\
q_{i,t} &= \frac{u_t}{u_y} q_{i-1,y} - q_{i-1,t}, & q_{i,x} &= \frac{q_{i-1,y}}{u_y}.
\end{aligned}
\tag{16}
$$

3D equations - Negative coverings τ^-

- 3D rdDym

$$\begin{aligned} r_{1,x} &= u_x^2 - u_t, & r_{1,y} &= u_x u_y, \\ r_{i,x} &= u_x r_{i-1,x} - r_{i-1,t}, & r_{i,y} &= u_y r_{i-1,x}. \end{aligned} \tag{17}$$

- 3D PE

$$\begin{aligned} r_{1,y} &= u_t + u_x u_y, & r_{1,x} &= u_y + u_x^2, \\ r_{i,y} &= r_{i-1,t} + u_y r_{i-1,x}, & r_{i,x} &= r_{i-1,y} + u_x r_{i-1,x}. \end{aligned} \tag{18}$$

- UHE

$$\begin{aligned} r_{1,y} &= u_x u_y, & r_{1,t} &= u_t u_x - u_y, \\ r_{i,y} &= u_y r_{i-1,x}, & r_{i,t} &= u_t r_{i-1,x} - r_{i-1,y}. \end{aligned} \tag{19}$$

- mVw

$$\begin{aligned} r_{1,y} &= u_x u_y, & r_{1,t} &= u_t u_x - u_t, \\ r_{i,y} &= u_y r_{i-1,x}, & r_{i,t} &= u_t r_{i-1,x} - r_{i-1,t}. \end{aligned} \tag{20}$$

Parenthesis I - Scaling symmetries and weights

A symmetry of the form

$$s = \delta u + \alpha x u_x + \beta y u_y + \gamma t u_t, \quad \alpha, \beta, \gamma, \delta \in \mathbb{Z},$$

is called a scaling symmetry of an equation $\mathcal{E}$.

If $\mathcal{E}$ admits such a symmetry, then we can assign weights to polynomial functions on $\mathcal{E}$ by

$$|x| = -\alpha, \quad |y| = -\beta, \quad |t| = -\gamma, \quad |u_{i,j,k}| = \delta - i\alpha - j\beta - k\gamma.$$

If $\mathbf{E}_\varphi$ is a symmetry and φ is a polynomial function, then we set

$$|\mathbf{E}_\varphi| = |\varphi| - |u|.$$

Then

$$|\,[\mathbf{E}_{\varphi_1}, \mathbf{E}_{\varphi_2}]\,| = |\mathbf{E}_{\varphi_1}| + |\mathbf{E}_{\varphi_2}|.$$

3D equations - Scaling symmetries and weights

- 3D rdDym

$$\upsilon_0 = -xu_x + 2u,$$

$$|y| = |t| = 0, \quad |x| = 1 \quad |u| = 2, \quad |q_i| = -i, \quad |r_i| = i + 2.$$

- 3D PE

$$\upsilon_0 = 3u - 2xu_x - yu_y,$$

$$|t| = 0, \quad |y| = 1, \quad |x| = 2, \quad |u| = 3, \quad |q_i| = -i, \quad |r_i| = i{+}3.$$

- UHE

$$\upsilon_0 = yu_y + u,$$

$$|x| = |t| = 0, \quad |y| = 1 \quad |u| = -1, \quad |q_i| = i{+}1, \quad |r_i| = -i{-}1.$$

- The mVw equation has no scaling symmetry, we assign zero weights to all the considered variables.

Example: UHE - Local symmetries

$$\upsilon_0 = yu_y + u,$$

$$\theta_0(X) = Xu_x - X'u, \quad \theta_1(X) = X$$

$$\varphi_0(T) = Tu_t + T'yu_y, \quad \varphi_{-1}(T) = Tu_y$$

	υ_0	$\theta_0(\bar{X})$	$\theta_1(\bar{X})$	$\varphi_0(\bar{T})$	$\varphi_{-1}(\bar{T})$
υ_0	0	0	$-\theta_1(\bar{X})$	0	$\varphi_{-1}(\bar{T})$
$\theta_0(X)$	...	$\theta_0([\bar{X}, X])$	$\theta_1([\bar{X}, X])$	0	0
$\theta_1(T)$	...	...	0	0	0
$\varphi_0(T)$	...	...	...	$\varphi_0([\bar{T}, T])$	$\varphi_{-1}([\bar{T}, T])$
$\varphi_{-1}(T)$	...	...	...	...	0

$$[T, \bar{T}] = T\bar{T}' - \bar{T}T'$$

We introduce the operators

$$
\begin{aligned}
\mathcal{P} &= -y\frac{\partial}{\partial t} + \sum_{k=0}^{\infty}(k+2)q_{k+1}\frac{\partial}{\partial q_k} \\
\mathcal{N} &= u\frac{\partial}{\partial x} + \sum_{k=0}^{\infty}(k+2)r_{k+1}\frac{\partial}{\partial r_k}
\end{aligned}
\tag{21}
$$

and define the polynomials $Q_n(T)$, $R_n(X)$, $n \in \mathbb{N}_0$, $T = T(t)$, $X = X(x)$, recursively as follows:

$$
\begin{aligned}
Q_0(T) &= T, & Q_n(T) &= \frac{1}{n}\mathcal{P}(Q_{n-1}(T)), \\
R_0(X) &= X, & R_n(X) &= \frac{1}{n}\mathcal{N}(R_{n-1}(X)).
\end{aligned}
\tag{22}
$$

Example: UHE - Lifts of the local symmetries

In both coverings τ^+ and τ^-, every local symmetry of the UHE can be lifted to the symmetry of $\tilde{\mathcal{E}}^+$ or $\tilde{\mathcal{E}}^-$, respectively.

For instance, the symmetry υ_0 can be lifted to the symmetry $\Upsilon_0^{\tau^+} = (\upsilon_0, \upsilon_0^{1,\tau^+}, \dots, \upsilon_0^{i,\tau^+}, \dots)$ of $\tilde{\mathcal{E}}^+$, etc.

The corresponding components of the lifts are given as follows:

$$
\begin{aligned}
\upsilon_0^{i,\tau^+} &= -(i+1)q_i + yq_{i,y}, & \upsilon_0^{i,\tau^-} &= (i+1)r_i + yu_y r_{i-1,x},\\
\theta_0^{i,\tau^+} &= \frac{X}{u_y} q_{i-1,y}, & \theta_0^{i,\tau^-} &= Xr_{i,x} - R_{i+1}(X),\\
\theta_1^{i,\tau^+} &= 0, & \theta_1^{i,\tau^-} &= R_i(X),\\
\varphi_0^{i,\tau^+} &= Tq_{i,t} + T' yq_{i,y} + Q_{i+1}(T), & \varphi_0^{i,\tau^-} &= Tr_{i,t} + T' yu_y r_{i-1,x},\\
\varphi_{-1}^{i,\tau^+} &= Tq_{i,y} - Q_i(T), & \varphi_{-1}^{i,\tau^-} &= Tr_{i,y}.
\end{aligned}
$$

Example: UHE - Invisible symmetries

We say that Φ^k_{inv}, $k = 1, 2, \ldots$, is an *invisible symmetry of depth* k if its first k components vanish, i.e.,

$$\Phi^k_{\mathrm{inv}} = (\underbrace{0, \ldots, 0}_{k-\text{times}}, \varphi^{k,1}_{\mathrm{inv}}, \varphi^{k,2}_{\mathrm{inv}}, \ldots, \varphi^{k,i}_{\mathrm{inv}}, \ldots).$$

The UHE has the invisible symmetries $\Phi^{k,\tau^+}_{\mathrm{inv}}(T)$ and $\Phi^{k,\tau^-}_{\mathrm{inv}}(X)$ in coverings τ^+ or τ^- respectively, where

$$\varphi^{k,1,\tau^+}_{\mathrm{inv}} = T(t), \qquad \varphi^{k,i,\tau^+}_{\mathrm{inv}} = Q_{i-1}(T),$$

$$i = 2, 3, \ldots$$

$$\varphi^{k,1,\tau^-}_{\mathrm{inv}} = X(x), \qquad \varphi^{k,i,\tau^-}_{\mathrm{inv}} = R_{i-1}(X).$$

Note that

$$|\mathbf{E}_{\Phi^{k,\tau^+}_{\mathrm{inv}}}| = -k - 1, \quad |\mathbf{E}_{\Phi^{k,\tau^-}_{\mathrm{inv}}}| = k + 1.$$

Example: UHE - Nonlocal symmetries in τ^+

There are two other series of nonlocal symmetries of the UHE in the positive covering τ^+.

1) The first series $\{\Psi_k^{\tau^+}\}_{k=0}^{\infty}$ is given as follows:

   a) $\Psi_0^{\tau^+} = (\psi_0^{\tau^+}, \psi_0^{1,\tau^+}, \psi_0^{2,\tau^+}, \psi_0^{3,\tau^+}, \dots)$, where

   $$\psi_0^{\tau^+} = 2q_1 u_y - y u_t, \qquad \psi_0^{j,\tau^+} = -(j+2)q_{j+1} - y q_{j,t} + 2q_1 q_{j,y}$$

   b) $\Psi_1^{\tau^+} = (\psi_1^{\tau^+}, \psi_1^{1,\tau^+}, \psi_1^{2,\tau^+}, \psi_1^{3,\tau^+}, \dots)$, where

   $$\begin{aligned} \psi_1^{\tau^+} &= -3q_2 u_y + 2q_1 u_t - y u_y q_{1,t}, \\ \psi_1^{j,\tau^+} &= (j+3)q_{j+2} + y q_{j+1,t} + 2q_1 q_{j,t} - (3q_2 + y q_{1,t}) q_{j,y} \end{aligned}$$

   c) $\Psi_k^{\tau^+} = \left\{\Psi_0^{\tau^+}, \Psi_{k-1}^{\tau^+}\right\}$, $k = 2, 3, \dots;\quad |\mathbf{E}_{\Psi_k^{\tau^+}}| = k+1.$

2) The second series $\{\Xi_m^{\tau^+}(T)\}_{m=1}^{\infty}$, $T = T(t)$, is given as follows:

   $$\Xi_m^{\tau^+}(T) = \left\{\Psi_m^{\tau^+}, \Phi_{-1}^{\tau^+}(T)\right\}; \quad |\mathbf{E}_{\Xi_m^{\tau^+}(T)}| = m.$$

Example: UHE - Nonlocal symmetries in τ^-

There are two other series of nonlocal symmetries of the UHE in the negative covering τ^-.

1) The first series $\{\Psi_k^{\tau^-}\}_{k=0}^{\infty}$ is given as follows:

   a) $\Psi_0^{\tau^-} = (\psi_0^{\tau^-}, \psi_0^{1,\tau^-}, \psi_0^{2,\tau^-}, \psi_0^{3,\tau^-}, \ldots)$, where

   $$\psi_0^{\tau^-} = 2r_1 - uu_x, \qquad \psi_0^{j,\tau^-} = (j+2)r_{j+1} - ur_{j,x}$$

   b) $\Psi_1^{\tau^-} = (\psi_1^{\tau^-}, \psi_1^{1,\tau^-}, \psi_1^{2,\tau^-}, \psi_1^{3,\tau^-}, \ldots)$, where

   $$\begin{aligned} \psi_1^{\tau^-} &= 3r_2 - 2r_1 u_x - ur_{1,x} + uu_x^2, \\ \psi_1^{j,\tau^-} &= (j+3)r_{j+2} - ur_{j+1,x} + (uu_x - 2r_1)r_{j,x} \end{aligned}$$

   c) $\Psi_k^{\tau^-} = \left\{\Psi_0^{\tau^-}, \Psi_{k-1}^{\tau^-}\right\}$, $k = 2, 3, \ldots;$ $\quad |\mathbf{E}_{\Psi_k^{\tau^-}}| = -k-1.$

2) The second series $\{\Xi_m^{\tau^-}(X)\}_{m=1}^{\infty}$, $X = X(x)$, is given as follows:

   $$\Xi_m^{\tau^-}(X) = \left\{\Psi_m^{\tau^-}, \Theta_1^{\tau^-}(X)\right\}; \quad |\mathbf{E}_{\Xi_m^{\tau^-}(T)}| = -m.$$

Parenthesis II - Lie algebras

Below we use the following Lie algebras:

- the Witt algebra $\mathfrak{W}$ of vector fields $\mathbf{e}_i = z^{i+1}\frac{\partial}{\partial z}$, $z \in \mathbb{Z}$;
- its subalgebras $\mathfrak{W}_k^-$ spanned by $\mathbf{e}_i$ with $i \le k \le 0$ and $\mathfrak{W}_k^+$ spanned by $\mathbf{e}_i$ with $i \ge k \ge 0$;
- the algebra $\mathfrak{V}[\rho]$ of vector fields $R(\rho)\frac{\partial}{\partial \rho}$ on $\mathbb{R}$ with a distinguished coordinate ρ;
- the loop algebra $\mathfrak{L}[\rho]$ spanned by the elements $z^i \otimes X$, $i \in \mathbb{Z}$, $X \in \mathfrak{V}[\rho]$, with the commutator
$$[z^i \otimes X, z^j \otimes Y] = z^{i+j} \otimes [X, Y];$$
- the algebra $\mathfrak{L}_k^+[\rho]$ spanned by the elements $p(z) \otimes X$, where $X \in \mathfrak{V}[\rho]$ and $p(z) \in \mathbb{R}[z]/(z^k)$ is a truncated polynomial. In a similar way, we define $\mathfrak{L}_k^-[\rho]$ with $p(z) \in \mathbb{R}[z^{-1}]/(z^{-k})$.

Consider the following subspaces in sym($\tilde{\mathcal{E}}^+$): W spanned by $\Upsilon_0^{\tau^+}$, $\Psi_i^{\tau^+}$, $i \geq 0$; $V[x]$ spanned by $\Theta_0^{\tau^+}(X)$, $\Theta_1^{\tau^+}(X)$; $V[t]$ spanned by $\Phi_0^{\tau^+}(T)$, $\Phi_{-1}^{\tau^+}(T)$, $\Phi_{\mathrm{inv}}^{i,\tau^+}(T)$, $\Xi_j^{\tau^+}(T)$, $i, j \geq 1$. Then we have the following:

Theorem: There exist bases $\mathbf{w}_i$ in W, $i \geq 0$, $\mathbf{v}_0(X)$, $\mathbf{v}_1(X)$ in $V[x]$, and $\mathbf{v}_i(T)$ in $V[t]$, $i \in \mathbb{Z}$, such that their commutators satisfy relations

	$\mathbf{w}_j$	$\mathbf{v}_j(\widetilde{X})$	$\mathbf{v}_j(\widetilde{T})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j}$	$j\mathbf{v}_{i+j}(\widetilde{X})$, $0 \leq i+j \leq 1$, 0, otherwise	$j\mathbf{v}_{i+j}(\widetilde{T})$
$\mathbf{v}_i(X)$	$\ldots$	$\mathbf{v}_{i+j}([X, \widetilde{X}])$, $0 \leq i+j \leq 1$, 0, otherwise	0
$\mathbf{v}_i(T)$	$\ldots$	$\ldots$	$\mathbf{v}_{i+j}([T, \widetilde{T}])$

Thus, sym($\tilde{\mathcal{E}}^+$) is isomorphic to $\mathfrak{W}_0^+ \ltimes (\mathfrak{L}_2^+[x] \oplus \mathfrak{L}[t])$ with the natural action of $\mathfrak{W}_0^+$ on $\mathfrak{L}_2^+[x]$ and $\mathfrak{L}[t]$.

Consider the following subspaces in sym($\tilde{\mathcal{E}}^-$): W spanned by $\Upsilon_0^{\tau^-}$, $\Psi_i^{\tau^-}$, $i \geq 0$; $V[x]$ spanned by $\Theta_0^{\tau^-}(X)$, $\Theta_1^{\tau^-}(X)$, $\Phi_{\mathrm{inv}}^{i,\tau^-}(X)$, $\Xi_j^{\tau^-}(X)$, $i, j \geq 1$; $V[x]$ spanned by $\Phi_0^{\tau^-}(\mathrm{T}_3)$,$\Phi_{-1}^{\tau^-}(T_4)$.
Then we have the following:

Theorem: There exist bases $\mathbf{w}_i$ in W, $i \leq 0$, $\mathbf{v}_i(X)$ in $V[x]$, $i \in \mathbb{Z}$, and $\mathbf{v}_{-1}(T)$, $\mathbf{v}_0(T)$ in $V[t]$, $i \in \mathbb{Z}$, such that their commutators satisfy relations

	$\mathbf{w}_j$	$\mathbf{v}_j(\widetilde{X})$	$\mathbf{v}_j(\widetilde{T})$
$\mathbf{w}_i$	$(j-i)\mathbf{w}_{i+j}$	$j\mathbf{v}_{i+j}(\widetilde{X})$	$\begin{cases} j\mathbf{v}_{i+j}(\widetilde{T}), & -1 \leq i+j \leq 0, \\ 0, & \text{otherwise} \end{cases}$
$\mathbf{v}_i(X)$	$\ldots$	$\mathbf{v}_{i+j}([X, \widetilde{X}])$	0
$\mathbf{v}_i(T)$	$\ldots$	$\ldots$	$\begin{cases} \mathbf{v}_{i+j}([T, \widetilde{T}]), & -1 \leq i+j \leq 0, \\ 0, & \text{otherwise} \end{cases}$

Thus, sym($\tilde{\mathcal{E}}^-$) is isomorphic to $\mathfrak{W}_0^- \ltimes (\mathfrak{L}_2^-[t] \oplus \mathfrak{L}[x])$ with the natural action of $\mathfrak{W}_0^-$ on $\mathfrak{L}_2^-[t]$ and $\mathfrak{L}[x]$.

3D equations - summary

	τ^+	τ^-
UHE	$\mathfrak{W}_0^+ \ltimes (\mathfrak{L}_2^+[x] \oplus \mathfrak{L}[t])$	$\mathfrak{W}_0^- \ltimes (\mathfrak{L}_2^-[t] \oplus \mathfrak{L}[x])$
3D rdDym	$\mathfrak{W}_0^- \ltimes (\mathfrak{L}_3^-[t] \oplus \mathfrak{L}[y])$	$\mathfrak{W}_0^+ \ltimes \mathfrak{L}[t] \oplus \mathfrak{V}[y]$
3D PE	$\mathfrak{W}_0^- \ltimes (\mathfrak{L}[q_0] \oplus \mathfrak{L}_4^-[t])$	$\mathfrak{W}_{-1}^+ \ltimes \mathfrak{L}[t]$
mVw	$\widetilde{\mathfrak{W}}_0^+ \ltimes (\mathfrak{L}[y] \oplus \mathfrak{L}_2^+[x]) \oplus \mathfrak{V}[t]$	$\widetilde{\mathfrak{W}}_0^- \ltimes \mathfrak{L}[x] \oplus \mathfrak{V}[y] \oplus \mathfrak{V}[t]$

Here $\widetilde{\mathfrak{W}}_0^+$ and $\widetilde{\mathfrak{W}}_0^-$ denote the subalgebras in $\mathfrak{W}_0^+$ and $\mathfrak{W}_0^-$ generated by the elements $\mathbf{e}_i - \mathbf{e}_0$, where $i \geq 1$ for $\widetilde{\mathfrak{W}}_0^+$ and $i \leq -1$ for $\widetilde{\mathfrak{W}}_0^-$.

H. Baran, I. S. Krasil'shchik, O. I. Morozov, and P. V., *Nonlocal symmetries of integrable linearly degenerate equations: A comparative study,* Theor. Math. Phys. 196 (2018), no. 2, 169–192.

2D equations

Using the local symmetries, the result of symmetry reductions of all four 3D equations comprises 32 two-dimensional equations of which

- sixteen can be solved explicitly,
- one reduces to the Riccati equation,
- five can be linearized by the Legendre transformation,
- and ten equations are 'nontrivial'.

H. Baran, I. S. Krasil'shchik, O. I. Morozov, and P. V., *Symmetry reductions and exact solutions of Lax integrable 3-dimensional systems,* J. Nonlinear Math. Phys. 21 (2014), no. 4, 643–671

2D equations

The latter include, besides other things, four 2D equations with finite-dimensional algebras of local symmetries, namely:

$$
\begin{aligned}
u_{yy} &= (u_y + y)u_{xx} - u_x u_{xy} - 2, \\
u_{yy} &= (u_y + 2x)u_{xx} + (y - u_x)u_{xy} - u_x, \\
u_{yy} &= (u_x + x)u_{xy} - (u_{xx} + 2)u_y, \\
u_{yy} &= u_y u_{xx} - (u_x + u)u_{xy} - u_x u_y.
\end{aligned}
\tag{23}
$$

H. Baran, I. S. Krasil'shchik, O. I. Morozov, and P. V., *Integrability properties of some equations obtained by symmetry reductions,* J. Nonlinear Math. Phys. 22 (2015), no. 2, 210–232

2D equations - Main result

P. Holba, I.S. Krasil'shchik, O.I. Morozov and P.V., 2D reductions of the equation $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$ and their nonlocal symmetries. J. Nonlinear Math. Phys. 24 (2017), suppl. 1, 36-47.

P. Holba, I.S. Krasil'shchik, O.I. Morozov and P.V., Reductions of the universal hierarchy and rdDym equations and their symmetry properties. Lobachevskii J. Math. 39 (2018), no. 5, 673-681.

- **Main result**

The Lie algebras of nonlocal symmetries of the 2D equations (23) are isomorphic either to the Witt algebra $\mathfrak{W} = \{z^{i+1}\partial/\partial z | i \in \mathbb{Z}\}$, or $\mathfrak{W} \oplus \mathfrak{s}_2$, where $\mathfrak{s}_2$ is the two-dimensional solvable Lie algebra, or $\mathfrak{W} \oplus \mathfrak{a}_1$, where $\mathfrak{a}_1$ is the one-dimensional Abelian Lie algebra.

Conclusions-part I

- We have considered four 3D Lax-integrable equations: (1) the universal hierarchy equation $u_{yy} = u_t u_{xy} - u_y u_{tx}$, (2) the rdDym equation $u_{ty} = u_x u_{xy} - u_y u_{xx}$, (3) the modified Veronese web equation $u_{ty} = u_t u_{xy} - u_y u_{tx}$, and (4) the Pavlov equation $u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$.
- For each equation, expanding the known Lax pairs in formal series in the spectral parameter, we construct two differential coverings and completely describe the nonlocal symmetry algebras associated with these coverings.
- The obtained Lie algebras of symmetries are (semi)direct sums of the Witt algebra, the algebra of vector fields on the line, and loop algebras.

Conclusions - part II

- Using the corresponding reductions of the known Lax pairs of the 3D equations, we describe nonlocal symmetries of the 2D reductions of the 3D equations and show that the Lie algebras of these symmetries are direct sums of the Witt algebra, the two-dimensional solvable Lie algebra and the one-dimensional Abelian Lie algebra.

- All computations were done using the Jets software, see

  H. Baran, M. Marvan, *Jets: A software for differential calculus on jet spaces and diffieties,*

  http://jets.math.slu.cz

THANK YOU FOR YOUR ATTENTION