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Workshop on Integrable Nonlinear Equations

Some integrability properties of a (3+1)-dimensional integrable generalization of the dKP equation

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(3+1)-dimensional generalization of the dKP equation

We have studied the following system of PDE (A. Sergyeyev, arXiv: 1401.2122v2):

$$\begin{aligned} q_z &= 2u_z + w_x + 2ww_z, \\ v_z &= 2q_x - 3u_x - 2w_y + 2wu_z - 2ww_x + 2uw_z, \\ u_t &= vu_z + qu_x - uv_z - wv_x + v_y, \\ w_t &= q_y - 2v_x + 4wu_x - wq_x + qw_x + vw_z - uq_z. \end{aligned} \tag{1}$$

If we put $w = 0$, $q = 3u/2$ and if we suppose that u and v are independent of z then system (1) boils down to

$$4v_x = 3u_y, \qquad 2u_t = 3uu_x + 2v_y. \tag{2}$$

Eliminating v from (2) yields an equation

$$(4u_t - 6uu_x)_x - 3u_{yy} = 0, \tag{3}$$

which, up to a rescaling of y and u, is nothing but the dispersionless Kadomtsev–Petviashvili (dKP) equation.

Lax pair and covering of (1)

The system (1) can be written as a set of compatibility conditions of the Lax pair (A. Sergyeyev, arXiv:1401.2122v2):

$$\psi_y = \psi_z f(p, \mathbf{u}), \qquad \psi_t = \psi_z g(p, \mathbf{u}), \tag{4}$$

where $\mathbf{u} = (q, u, v, w)$, $\mathbf{u}$ does not depend on p, $\psi = \psi(x, y, z, t)$, $p = \psi_x/\psi_z$ and

$$f(p, \mathbf{u}) = p^2 + wp + u, \qquad g(p, \mathbf{u}) = p^3 + 2wp^2 + qp + v. \tag{5}$$

We consider p as an *additional independent variable* and define vector fields

$$X_h = h_p \partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z. \tag{6}$$

associated to functions $h = h(p, \mathbf{u})$. Then equations

$$\chi_y = X_f(\chi), \qquad \chi_t = X_g(\chi), \tag{7}$$

where $\chi = \chi(x, y, z, t, p)$, define a covering over the system (1).

Expansion of the covering (7)

The covering (7) of the system (1) has the following explicit form:

$$
\begin{aligned}
\chi_y &= (p^2 w_z + p u_z - p w_x - u_x)\chi_p + (2p + w)\chi_x + (-p^2 + u)\chi_z, \\
\chi_t &= (2 w_z p^3 + (q_z - 2 w_x) p^2 + (v_z - q_x) p - v_x)\chi_p \\
&\quad + (3p^2 + 4pw + q)\chi_x + (-2p^3 - 2p^2 w + v)\chi_z.
\end{aligned}
\tag{8}
$$

Let us expand χ in formal series in p:

$$\chi = \sum_{i=-\infty}^{\infty} a_i p^{-i}, \tag{9}$$

where $a_i = a_i(x, y, z, t)$. Substituting (9) into (8) one can obtain the following recursive formula for a_i:

$$
\begin{aligned}
a_{i,x} = {} & (i-3)(v_x - 2u_x w)a_{i-3} \\
& + (i-2)(u_x - q_x + 2w_y - 2uw_z)a_{i-2} \\
& + (q - 2w^2)a_{i-2,x} + 2wa_{i-2,y} + va_{i-2,z} - a_{i-2,t} \\
& - (i-1)w_x a_{i-1} - 2wa_{i-1,x} + 2a_{i-1,y} - 2ua_{i-1,z}
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
a_{i,z} = {} & (i-3)u_x a_{i-3} + (i-2)(w_x - u_z)a_{i-2} \\
& + wa_{i-2,x} - a_{i-2,y} + ua_{i-2,z} - (i-1)w_z a_{i-1} + 2a_{i-1,x}
\end{aligned}
$$

We conjecture that all conservation laws associated with above covering are nontrivial for $i \geq 3$.

Let us suppose that $a_i = 0$ for $i < 0$. Then we obtain:

$$a_{0,x} = 0, \quad a_{0,z} = 0 \quad \Rightarrow \quad \boxed{a_0 = a_0(y,t)}$$

$$a_{1,x} = 2a_{0,y}, \quad a_{1,z} = 0 \quad \Rightarrow \quad \boxed{a_1 = 2xa_{0,y}}$$

$$a_{2,x} = -2(w + xw_x)a_{0,y} - a_{0,t} + 4xa_{0,yy},$$

$$a_{2,z} = (3 - 2xw_z)a_{0,y}$$

$$\Downarrow$$

$$\boxed{a_2 = (3z - 2xw)a_{0,y} - xa_{0,t} + 2x^2 a_{0,yy}}$$

$$a_{3,x} = (4xww_x - xq_x + xu_x - 3zw_x + q - 3u)a_{0,y} + (xw_x + w)a_{0,t} + (3z - 2x^2 w_x - 4xw)a_{0,yy} - 2xa_{0,ty} + 2x^2 a_{0,yyy}$$

$$a_{3,z} = (2xww_z - xu_z - xw_x - 3zw_z - w)a_{0,y} + (xw_z - 1)a_{0,t} + (3x - 2x^2 w_z)a_{0,yy}$$

Subtracting the trivial conservation law

$$b_{3,x} = D_x(H_3), \quad b_{3,z} = D_z(H_3),$$

where

$$H_3 = \frac{2}{3}x^3 a_{0,yyy} - x^2 a_{0,ty} + (3xz - 2x^2 w)a_{0,yy} + (xw - z)a_{0,t} + (2xw^2 - xq + xu - 3zw)a_{0,y},$$

we obtain an equivalent form of a_3:

$$\tilde{a}_{3,x} = a_{3,x} - b_{3,x} = (-w^2 + q - 2u)a_{0,y},$$

$$\tilde{a}_{3,z} = a_{3,z} - b_{3,z} = wa_{0,y}.$$

Case $a_0 = y$

Let us suppose that $a_0 = y$. Then

$$a_1 = 2x, \quad a_2 = -2xw + 3z,$$

$$a_{3,x} = -w^2 + q + 2u,$$
$$a_{3,z} = w,$$

$$a_{4,x} = v - 2uw + 2a_{3,y},$$
$$a_{4,z} = -w^2 + 2q - 3u,$$

$$a_{5,x} = 2w^4 - 4qw^2 + 6uw^2 + 2q^2 - 8qu + 7u^2 + vw - a_{3,t} + 2a_{4,y},$$
$$a_{5,z} = -2w^3 + 3qw - 7uw + 2v + 3a_{3,y},$$

$$a_{6,x} = 8uw^3 - 10quw + 16u^2w - 4vw^2 + 5qv - 9uv$$
$$+ (-4w^2 + 4q - 8u)a_{3,y} - a_{4,t} + 2a_{5,y},$$
$$a_{6,z} = 2w^4 - 6qw^2 + 4uw^2 + 5q^2 - 16qu + 12u^2 + 5vw$$
$$+ 4wa_{3,y} - 2a_{3,t} + 3a_{4,y}.$$

Symmetries

$$\phi_1(F_1) = \Big[F_1 q_z,\ \ F_1 u_z + F_{1,y},\ \ F_1 v_z + F_{1,t},\ \ F_1 w_z\Big],$$

$$\begin{aligned}\phi_2(F_2) = \Big[&F_2 q_x + F_{2,y}(2xq_z - 8w) + F_{2,t},\\ &F_2 u_x + F_{2,y}(2xu_z - 2w) + 2F_{2,yy}x,\\ &F_2 v_x + F_{2,y}(2xv_z - 2q) + 2F_{2,ty}x,\ \ F_2 w_x + F_{2,y}(2xw_z - 3)\Big],\end{aligned}$$

$$\begin{aligned}\phi_3(F_3) = \Big[&F_3 q_y + F_{3,t}(5w - xq_z) + F_{3,y}(3zq_z - 2q + 2xq_x)\\ &+ F_{3,yy}(2x^2 q_z - 16xw) + 2F_{3,ty}x,\\ &F_3 u_y + F_{3,t}(w - xu_z) + F_{3,y}(3zu_z - 2u + 2xu_x)\\ &+ F_{3,yy}(2x^2 u_z - 4xw + 3z) - F_{3,ty}x + 2F_{3,yyy}x^2,\\ &F_3 v_y + F_{3,t}(q + u - xv_z) + F_{3,y}(3zv_z - 3v + 2xv_x)\\ &+ F_{3,yy}(2x^2 v_z - 4xq) - F_{3,tt}x + 3F_{3,ty}z + 2F_{3,tyy}x^2,\\ &F_3 w_y + F_{3,t}(2 - xw_z) + F_{3,y}(3zw_z - w + 2xw_x)\\ &+F_{3,yy}(2x^2 w_z - 6x)\Big],\end{aligned}$$

$$\phi_4(F_4) = \left[\phi_4^1(F_4),\ \ \phi_4^2(F_4),\ \ \phi_4^3(F_4),\ \ \phi_4^4(F_4)\right],$$

where

$$\begin{aligned}
\phi_4^1(F_4) =& F_4 q_t + F_{4,t}(2q - xq_x - 2zq_z) \\
& + F_{4,y}(4a_3 q_z + 16w^3 - 18qw + 3zq_x + 2xq_y + 24uw - 3v) \\
& - F_{4,tt}x + F_{4,yy}(6xzq_z - 24zw - 4xq + 2x^2 q_x) \\
& + F_{4,ty}(18xw + 3z - 2x^2 q_z) + F_{4,yyy}\left(\frac{4}{3}x^3 q_z - 16x^2 w\right) \\
& + 2F_{4,tyy}x^2,
\end{aligned}$$

$$\begin{aligned}
\phi_4^2(F_4) =& F_4 u_t + F_{4,t}(2u - xu_x - 2zu_z) \\
& + F_{4,y}(4a_3 u_z + 4w^3 - 4qw + 3zu_x + 2xu_y + 2uw + v + 4a_{3,y}) \\
& + F_{4,yy}(2x^2 u_x + 6xzu_z - 4xu - 6zw + 4a_3) \\
& + F_{4,ty}(4xw - 2x^2 u_z - 2z) + F_{4,yyy}\left(\frac{4}{3}x^3 u_z - 4x^2 w + 6xz\right) \\
& - 2F_{4,tyy}x^2 + \frac{4}{3}F_{4,yyyy}x^3,
\end{aligned}$$

$$\begin{aligned}
\phi_4^3(F_4) =& F_4 v_t + F_{4,t}(3v - xv_x - 2zv_z) \\
&+ F_{4,y}(4a_3 v_z + 4qw^2 - 4q^2 + 6qu - 4vw + 3zv_x + 2xv_y + 4a_{3,t}) \\
&+ F_{4,yy}(6xzv_z + 2x^2 v_x - 6zq - 6xv) \\
&+ F_{4,ty}(4xq + 2xu + 4a_3 - 2x^2 v_z) \\
&- 2F_{4,tt} z + 6xzF_{4,tyy} - 2F_{4,tty} x^2 \\
&+ F_{4,yyy}\left(\frac{4}{3}x^3 v_z - 4x^2 q\right) + \frac{4}{3}F_{4,tyyy} x^3
\end{aligned}$$

$$\begin{aligned}
\phi_4^4(F_4) =& F_4 w_t + F_{4,t}(w - xw_x - 2zw_z) \\
&+ F_{4,y}(4a_3 w_z + 6w^2 + 3zw_x + 2xw_y - 7q + 9u) \\
&+ F_{4,yy}(2x^2 w_x + 6xzw_z - 2xw - 9z) + F_{4,ty}(-2x^2 w_z + 7x) \\
&+ F_{4,yyy}\left(\frac{4}{3}x^3 w_z - 6x^2\right)
\end{aligned}$$

Jacobi brackets

	$\phi_1(G_1)$	$\phi_2(G_2)$	$\phi_3(G_3)$
$\phi_1(F_1)$	0	0	$\phi_1(G_3F_{1,y}-3F_1G_{3,y})$
$\phi_2(F_2)$		$\phi_1(2G_2F_{2,y}-2F_2G_{2,y})$	$\phi_2(G_3F_{2,y}-2F_2G_{3,y})-\phi_1(F_2G_{3,t})$
$\phi_3(F_3)$			$\phi_3(G_3F_{3,y}-F_3G_{3,y})$

	$\phi_4(G_4)$
$\phi_1(F_1)$	$\phi_2(-3F_1G_{4,y})+\phi_1(G_4F_{1,t}+2F_1G_{4,t})$
$\phi_2(F_2)$	$\phi_3(-2F_2G_{4,y})+\phi_2(G_4F_{2,t}+F_2G_{4,t})$
$\phi_3(F_3)$	$\phi_4(-F_3G_{4,y})+\phi_3(G_4F_{3,t})$
$\phi_4(F_4)$	$\phi_4(G_4F_{4,t}-F_4G_{4,y})$

Different approach

Recall that the system (1) can be written as a set of compatibility conditions

$$D_t(\psi_y) = D_y(\psi_t) \tag{11}$$

of the Lax pair:

$$\psi_y = \psi_z f(p, \mathbf{u}), \qquad \psi_t = \psi_z g(p, \mathbf{u}),$$

where $\mathbf{u} = (q, u, v, w)$, $\psi = \psi(x, y, z, t)$, $p = \psi_x/\psi_z$ and

$$f(p, \mathbf{u}) = p^2 + wp + u, \qquad g(p, \mathbf{u}) = p^3 + 2wp^2 + qp + v.$$

Now let us consider a new independent variable s and suppose that

$$\psi_s = \psi_z h(s, p, \mathbf{u}), \tag{12}$$

where h is a polynomial in p:

$$h = \sum_{i=0}^{n} A_i(x, y, z, t, s) p^i \tag{13}$$

with $n \geq 3$.

Then from (11) and the additional compatibility conditions

$$D_s(\psi_y) = D_y(\psi_s), \quad D_s(\psi_t) = D_t(\psi_s). \tag{14}$$

one can obtain

$$\begin{aligned} A_n &= A_n(y,t,s), \\ A_{n-1} &= (n-1)wA_n + 2xA_{n,y}, \\ A_{n-2} &= (n-3)(n-2)\left(\frac{w^2}{2} - \frac{u}{n-2} + \frac{q}{n-3}\right) \\ &\quad + \left[2(n-2)xw + 3z\right] A_{n,y} - xA_{n,t} + 2x^2 A_{n,yy}, \end{aligned} \tag{15}$$

whereas A_i, $i = 0, \ldots, n-3$, are nonlocal.

Moreover, q_s, u_s, v_s, w_s form components of the nonlocal symmetry for system (1).

Examples

$\boxed{n=4}$, $\boxed{A_4=1}$

$$A_{1,x} = -2ww_y + q_y - 2u_y, \qquad A_{1,z} = w_y$$

$$\begin{aligned} A_{0,x} = {} & -8uw^2w_z + 4qww_x - 4q_xw^2 - 8u^2w_z - 16uu_zw - 4uww_x \\ & + 12u_xw^2 + 4vww_z + 4qu_x - 8q_xu + 4q_yw + 8uu_x + 6uw_y \\ & - 2u_yw + 4u_zv + 2vw_x - 10v_xw + 2A_{1,y} - 2q_t + 5v_y \end{aligned}$$

$$A_{0,z} = -4w^2w_x - 2qw_x + 6q_xw + 2uw_x - 14u_xw - 6ww_y - 3u_y + 4v_x$$

Nonlocal symmetry:

$$\begin{aligned} q_s = {} & A_{1,t} - 8A_{1,y}w + 10q_tw + (26w^3 - 10qw + 44uw + A_1 - 2v)q_x \\ & + (-19w^2 + q - 2u)q_y + (-80w^3 - 70uw + 6v)u_x + (8w^2 + 2q)u_y \\ & + (2w^4 - 4qw^2 + 84uw^2 + 2q^2 - 8qu + 10u^2 - 24vw + 2A_0)u_z \\ & + (-15w^4 - 2qw^2 - 20uw^2 + q^2 - 4qu + 5u^2 - 6vw + A_0)w_x \\ & + 2v_t + (50w^2 - 2q + 2u)v_x - 22v_yw + (2qw - 28uw + 3v)w_y \\ & + (2w^5 - 4qw^3 + 48uw^3 + 2q^2w - 8quw + 46u^2w - 20vw^2 \\ & + 2A_0w - 4uv)w_z \end{aligned}$$

$$
\begin{aligned}
u_s = {} & A_{0,y} - 2A_{1,y}w + 2q_t w + (6w^3 - 2qw + 4uw)q_x + (-6w^2 + 2q - 2u)q_y \\
& + (-18w^3 - 2uw + A_1 + 2v)u_x + (7w^2 - 2q + 9u)u_y \\
& + (w^4 - 2qw^2 + 20uw^2 + q^2 - 4qu + u^2 - 4vw + A_0)u_z \\
& + (10w^2 - 4u)v_x - 5v_y w + (-4w^4 - 2vw)w_x \\
& + (4w^3 - 4qw + 10uw)w_y + (8uw^3 + 2u^2 w - 4vw^2)w_z
\end{aligned}
$$

$$
\begin{aligned}
v_s = {} & A_{0,t} - 2A_{1,y}q + (-2w^2 + 4q - 2u)q_t + (4w^3 - 8qw + 10uw)q_y \\
& + (-2w^4 + 6qw^2 - 10uw^2 + 10qu - 6u^2 - 6vw + 2A0)q_x \\
& + (13w^4 - 22qw^2 + 40uw^2 - 7q^2 - 8qu + 9u^2 + 14vw - 3A_0)u_x \\
& + (2qw + 3v)u_y + (2w^5 - 4qw^3 - 8uw^3 + 2q^2 w + 24quw - 26u^2 w \\
& + 4vw^2 + 2A_0 w - 8qv + 4uv)u_z + (-14w^3 + 22qw - 28uw \\
& + A_1 - 2v)v_x + (5w^2 - 7q + 6u)v_y + (-2w^5 + 4qw^3 - 4uw^3 - 6q^2 w \\
& + 8quw - 4u^2 w + 6vw^2 - 2A_0 w - 2qv)w_x + (-2w^4 + 4qw^2 - 2q^2 \\
& - 6qu + 6u^2 + 6vw - 2A_0)w_y + (-6uw^4 + 12quw^2 - 20u^2 w^2 + 4vw^3 \\
& + 2q^2 u + 8qu^2 - 8qvw - 6u^3 + 4uvw + 2A_0 u)w_z
\end{aligned}
$$

$$
\begin{aligned}
w_s = & -3A_{1,y} + 4q_t + (10w^2 - 4q + 16u)q_x - 7q_y w + (-30w^2 - 24u)u_x \\
& + 4u_y w + (26uw - 8v)u_z + 18wv_x - 8v_y \\
& + (-6w^3 - 10uw + A_1 - 2v)w_x + (-w^2 + 2q - 11u)w_y \\
& + (w^4 - 2qw^2 + 20uw^2 + q^2 - 4qu + 17u^2 - 8vw + A0)w_z
\end{aligned}
$$

$\boxed{n = 5}$, $\boxed{A_5 = 1/2}$

$$
A_{2,x} = -2ww_y + q_y - 2u_y, \quad A_{2,z} = w_y
$$

$$
\begin{aligned}
A_{1,x} = & -6uw^2 w_z + 9u_x w^2 - 12uu_z w - 3q_x w^2 + 3qww_x - 3uww_x \\
& - 6u^2 w_z + 3vww_z + 3q_y w + 4uw_y - 2u_y w + 3qu_x + 6uu_x + 3vu_z \\
& - 15/2wv_x - 6q_x u + 3/2vw_x + 4v_y + 2A_{2,y} - 3/2q_t \\
A_{1,z} = & -3w^2 w_x - 5ww_y - 21/2wu_x + 9/2wq_x - 3/2qw_x + 3/2w_x u \\
& + 1/2q_y - 3u_y + 3v_x,
\end{aligned}
$$

$$
\begin{aligned}
A_{0,x} = {} & 3qv_x - 4q_yw^2 - 3vu_x - 9/2w^2v_x + 3vq_x + 8w^3w_y - 2vw_y + 4qq_y \\
& - 6u_zu^2 - 4q_yu - 9uv_x + 2w^3u_x - 8qu_y + 12u_yw^2 + v_yw + A_2u_x \\
& + 18uu_y - 3vww_x + 28uww_y - 8qww_y - 12wq_xu + 27wu_xu + 3w_zvu \\
& + 3u_zwv - 12w_zu^2w + 12w^2w_xu - 6u_zw^2u + 2A_{1,y} - A_{2,t} - 3/2v_t
\end{aligned}
$$

$$
\begin{aligned}
A_{0,z} = {} & -3q_t + 3w_zqu + 9w_xuw - 21u_zuw + 3u_zqw - 9uw^2w_z - 3qww_x \\
& + 6vww_z - 21/2wv_x + 15/2vu_z - 9/2qu_x + 45/2u_xu + 9w_yu \\
& - 15q_xu - 12w_zu^2 + 6qq_x + 3vw_x + 3w^3w_x + 33/2u_xw^2 + 6q_yw \\
& - 7u_yw - 3qw_y - u_zw^3 + u_zA_2 - 6q_xw^2 - 3w^2w_y + 3A_{2,y} + 13/2v_y
\end{aligned}
$$

Thank you for your attention