Geometry of homogeneous third-order Hamiltonian operators and applications to WDVV equations

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- What are third-order homogeneous Hamiltonian operators?
- How to find them? Applications to WDVV equations.

What are third-order homogeneous Hamiltonian operators?

First-order Dubrovin–Novikov (homogeneous) operators

Dubrovin–Novikov (homogeneous) operators were introduced in 1983 for the Hamiltonian formalism of hydrodynamic-type equations

$$u^i_t = v^i_j(\mathbf{u})u^j_x = A_1^{ij}\frac{\delta \mathcal{H}_1}{\delta u^j} \qquad \mathcal{H}_1 = \int h(\mathbf{u})dx$$

$\mathbf{u} = (u^i(t,x))$, $i, j = 1,\dots,n$ (n-components). The operators are of the form

$$A_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b^{ij}_k(\mathbf{u})u^k_x$$

Homogeneity: $\deg \partial_x = 1$.

Geometry of 1st-order Dubrovin–Novikov operators

Any change of coordinates of the type $\bar{u}^i = \bar{u}^i(u^j)$ will not change the 'nature' of the above operator. g^{ij} transforms as a contravariant 2-tensor; usually it is required that g^{ij} is non-degenerate; $\Gamma^j_{ik} = -g_{is}b^{sj}_k$ transforms as a linear connection.

Conditions:

- $A_1^* = -A_1$ is equivalent to: symmetry of g^{ij}, $\nabla[\Gamma]g = 0$;
- $[A_1, A_1] = 0$ is equivalent to: g_{ij} flat pseudo-Riemannian metric and $\Gamma^j_{ik} = \Gamma^j_{ki}$, or Γ is the Levi-Civita connection of g.

Third-order Dubrovin–Novikov operators

Dubrovin–Novikov operators were defined for higher orders too. In particular

$$\begin{aligned} A_3^{ij} =& g^{ij}(\mathbf{u})\partial_x^3 + b_k^{ij}(\mathbf{u})u_x^k\partial_x^2 \\ & + [c_k^{ij}(\mathbf{u})u_{xx}^k + c_{km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\ & + d_k^{ij}(\mathbf{u})u_{xxx}^k + d_{km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n, \end{aligned}$$

Examples of Hamiltonian equations of the form

$$u_t^i = A_3^{ij}\left(\frac{\delta \mathcal{H}_2}{\delta u^j}\right)$$

are in the 2-component case the Chaplygin gas equation (Mokhov DrSc thesis, '96) and the 3-component case WDVV equation (Ferapontov, Galvao, Mokhov, Nutku CMP '95).

Example: 2-component Chaplygin gas equation

(O. Mokhov, '96) The Monge–Ampère equation $u_{tt}u_{xx} - u_{xt}^2 = -1$ can be reduced to hydrodynamic form

$$a_t = b_x, \quad b_t = \left(\frac{b^2 - 1}{a}\right)_x,$$

via the change of variables $a = u_{xx}$, $b = u_{xt}$. It possesses the Hamiltonian formulation

$$\begin{pmatrix} a \\ b \end{pmatrix}_t = \partial_x \left(\begin{array}{cc} 0 & \partial_x \dfrac{1}{a} \\ \dfrac{1}{a}\partial_x & \dfrac{b}{a^2}\partial_x + \partial_x \dfrac{b}{a^2} \end{array} \right) \partial_x \begin{pmatrix} \delta H/\delta a \\ \delta H/\delta b \end{pmatrix},$$

and the nonlocal Hamiltonian,

$$H = -\int \left(\frac{1}{2}a(\partial_x^{-1}b)^2 + \partial_x^{-2}a\right) dx.$$

Example: 3-component WDVV equation

The simplest associativity (WDVV) equation:

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$$

can be presented by $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$ as

$$a_t = b_x, \quad b_t = c_x, \quad c_t = (b^2 - ac)_x.$$

From Ferapontov, Galvao, Mokhov, Nutku, CMP (1997), there are two local Dubrovin-Novikov Hamiltonian operators, first-order A_1 and third-order A_3,

$$A_3 = \begin{pmatrix} 0 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & -\partial_x^2 a \partial_x \\ \partial_x^3 & -\partial_x a \partial_x^2 & (\partial_x^2 b \partial_x + \partial_x b \partial_x^2 + \partial_x a \partial_x a \partial_x) \end{pmatrix}$$

Some known results

Some facts which are known on *non-degenerate* ($\det(g^{ij}) \neq 0$) third-order differential geometric Hamiltonian operators (Potemin '86, '97; Potemin–Balandin, '01; Doyle '95):

1. the coefficients $(-1/3)g_{is}b^{sj}_k$, $(-1/3)g_{is}c^{sj}_k$, $-g_{is}d^{sj}_k$ transform as Christoffel symbols;
2. $-g_{is}d^{sj}_k$ is symmetric and flat;
3. the operator can be brought to a constant form $\tilde{g}^{ij}\partial_x^3$ if and only if $(-1/3)g_{is}c^{sj}_k$ has zero torsion;
4. in flat coordinates of $-g_{is}d^{sj}_k$ the operator takes the form

$$A_3 = \partial_x \circ (g^{ij}\partial_x + c^{ij}_k u^k_x) \circ \partial_x,$$

where g_{ij} is quadratic and c_{ijk} is linear in the flat coordinates. Such flat coordinates are Casimirs of the operator.

Solving the linear equations

Let us set $c_{ijk} = g_{iq} g_{jp} c^{pq}_{k}$.
Proposition. Skew-adjointness of A_3 and $[A_3, A_3] = 0$ are equivalent, in Casimirs, to the fact that g_{ij} are second–order polynomials in field variables and

$$c_{nkm} = \frac{1}{3}(g_{nm,k} - g_{nk,m}),$$
$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0,$$
$$c_{mnk,l} = -g^{pq} c_{pml} c_{qnk}.$$

Corollary. The metric g_{mn} is the Monge form of a **quadratic line complex**.

What is a quadratic line complex?

A quadratic line complex is a submanifold defined by a homogeneous quadratic equation $X^TQX = f_{ij,hk}p^{ij}p^{hk} = 0$, in the manifold L of all lines in a projective space $\mathbb{P}^n(\mathbb{C})$;
$X = (p^{ij})$ are Plücker's coordinates $p^{ij} = u^iv^j - u^jv^i$, $P = (u^i)$, $V = (v^j)$ two distinct points.

Note that $L \subset \mathbb{P}^N(\mathbb{C})$ is a submanifold defined by quadratic equations (Plücker's embedding).

If $n = 3$ then quadratic line complexes are **classified** by the 11 Jordan forms of the matrix $Q\Omega^{-1}$, where Ω is the matrix of the Plücker's quadric (C. Segre – Weiler classification; see C.M. Jessop, A treatise on the line complex, Camb. Un. Pr. 1903).

Quadratic line complexes and Monge metrics

Replace the point V by the differentials $dP = (du^j)$. Then $p^{ij} = u^i du^j - u^j du^i$ (**Lie coordinates**). Use an affine chart: $u^{n+1} = 1$, $du^{n+1} = 0$. Then we have a Monge metric

$$g_{ij} = (du^i)^T Q_0 (du^i) + (du^i)^T Q_1 (u^i du^j - u^j du^i) + (u^i du^j - u^j du^i)^T Q_2 (u^i du^j - u^j du^i), \quad (2)$$

Q_0, Q_1, Q_2 constant matrices.

The surface $\det(g) = 0$ is the so-called **singular surface** of the complex. For $n = 2$ it is a conic, for $n = 3$ it is a Kummer quartic, for $n = 4$ it is a B. Segre sextic.

Quadratic line complexes and Monge metrics

Example: $n = 3$

$$g_{11} = -[R_{12}(u^2)^2 + R_{13}(u^3)^2 + 2B_{12}u^2u^3 + 2H_{12}u^2 + 2H_{13}u^3 + D_1],$$
$$g_{22} = -[R_{12}(u^1)^2 + R_{23}(u^3)^2 + 2B_{22}u^1u^3 + 2H_{21}u^1 + 2H_{23}u^3 + D_2],$$
$$g_{33} = -[R_{23}(u^2)^2 + R_{13}(u^1)^2 + 2B_{32}u^1u^2 + 2H_{31}u^1 + 2H_{32}u^2 + D_3],$$
$$g_{12} = R_{12}u^1u^2 + B_{12}u^1u^3 + B_{22}u^2u^3 - B_{32}(u^3)^2 + H_{12}u^1 + H_{21}u^2 + (E_2 - E_1)u^3 + F_{12},$$
$$g_{13} = R_{13}u^1u^3 + B_{12}u^1u^2 - B_{22}(u^2)^2 + B_{32}u^2u^3 + H_{13}u^1 + H_{31}u^3 + (E_1 - E_3)u^2 + F_{13},$$
$$g_{23} = R_{23}u^2u^3 - B_{12}(u^1)^2 + B_{22}u^1u^2 + B_{32}u^1u^3 + H_{23}u^2 + H_{32}u^3 + (E_3 - E_2)u^1 + F_{23},$$

Monge–Ampère example revisited

The operator:

$$A_3 = \partial_x \begin{pmatrix} 0 & \partial_x \frac{1}{a} \\ \frac{1}{a}\partial_x & \frac{b}{a^2}\partial_x + \partial_x \frac{b}{a^2} \end{pmatrix} \partial_x$$

is completely determined by its Monge metric:

$$g_{ij} = \begin{pmatrix} -2b & a \\ a & 0 \end{pmatrix}$$

In this case, the singular surface is $\det(g) = -a^2$, and is a line counted two times. Moreover, g is a flat pseudo-Riemannian metric. This is the simplest nontrivial homogeneous third-order operator.

WDVV example revisited

The operator:

$$A_3 = \begin{pmatrix} 0 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & -\partial_x^2 a \partial_x \\ \partial_x^3 & -\partial_x a \partial_x^2 & (\partial_x^2 b \partial_x + \partial_x b \partial_x^2 + \partial_x a \partial_x a \partial_x) \end{pmatrix}$$

The operator is completely determined by its metric:

$$g_{ij} = \begin{pmatrix} -2b & a & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In this case, the singular surface is $\det(g) = -1$, and is a quadruple plane at infinity. Moreover, g is a flat pseudo-Riemannian metric.

Two components case: affine classification

(The 1-component case was described by Gel'fand-Dorfman – point-equivalent to ∂_x^3).

Theorem: only two non-trivial metrics in 2-component case:

$$g_{ik}^{(1)} = \begin{pmatrix} 1-(b^2)^2 & 1+b^1b^2 \\ 1+b^1b^2 & 1-(b^1)^2 \end{pmatrix}, \quad g_{ik}^{(2)} = \begin{pmatrix} -2b^2 & b^1 \\ b^1 & 0 \end{pmatrix}$$

$g^{(1)}$ is non-flat, $g^{(2)}$ is flat and appears in the Chaplygin gas equation (O. Mokhov's Doctoral Thesis).

Theorem. In the 2-component cases the operators may be reduced to ∂_x^3 by the above reciprocal transformations.

Projective invariance

It is known that Monge metrics transform, under projective transformations $\tilde{u}^i = T^i(u^j) = (A^i_j u^j + A^i_0)/\Delta$, with $\Delta = c_i u^i + c_0$, as

$$\tilde{g}_{ij} = \frac{g_{ij}\big|_{u^i = T^i(\tilde{u}^j)}}{\Delta^4}.$$

Theorem. Reciprocal transformations of the type

$$d\tilde{x} = \Delta dx, \quad \tilde{u}^i = T^i(u^j) = (A^i_j u^j + A^i_0)/\Delta$$

preserve the canonical form of third-order homogeneous operators and effect a projective transformation on the Monge metric g_{ij}. **The projective group is maximal.**

Projective normal forms for $n = 3$

$$g^{(1)} = \begin{pmatrix} (u^2)^2 + c & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 + c(u^3)^2 & -cu^2u^3 - u^1 \\ 2u^2 & -cu^2u^3 - u^1 & c(u^2)^2 + 1 \end{pmatrix},$$

$$g^{(2)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 & -u^1 \\ 2u^2 & -u^1 & 1 \end{pmatrix},$$

$$g^{(3)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 & 0 \\ -u^1u^2 & (u^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g^{(4)} = \begin{pmatrix} -2u^2 & u^1 & 0 \\ u^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{(5)} = \begin{pmatrix} -2u^2 & u^1 & 1 \\ u^1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad g^{(6)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

How to find them? Application to:

- **hydrodynamic-type systems in conservative form;**
- **WDVV equations.**

Suitable coordinate systems

It is clear that Casimirs of A_3 are good:

$$A_3 = \partial_x \circ (g^{ij}\partial_x + c^{ij}_k u^k_x) \circ \partial_x,$$

Casimirs are *conservation law densities*, so it is natural to look for operators A_3 for hydrodynamic-type systems in conservative form:

$$a^i_t = (V^i(\mathbf{a}))_x$$

In potential coordinates $a^i = b^i_x$

$$b^i_t = V^i(\mathbf{b}_x), \quad A_3 = -g^{ij}(\mathbf{b}_x)\partial_x - c^{ij}_k(\mathbf{b}_x)b^k_{xx},$$

A necessary condition

For a system of PDEs $F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \ldots) = 0$ we have that

$$u^i_t = A_3^{ij}\left(\frac{\delta H}{\delta u^j}\right) \quad \text{with} \quad A_3^* = -A_3 \quad \text{and} \quad [A_3, A_3] = 0$$

$$\Rightarrow \quad \ell_F \circ A_3 = A_3^* \circ \ell_F^*$$

The right-hand side as a **necessary condition** to Hamiltonianity (Kersten, Krasil'shchik, Verbovetsky, JGP '04).

How to implement the necessary condition

The operator equation to be fulfilled by Hamiltonian operators can be reformulated as follows. Extend the equation by

$$\begin{cases} F = u^i_t - f(t, x, u^i, u^i_x, u^i_{xx}, \ldots) = 0 \\ (\ell_F)^*(\mathbf{p}) = 0, \end{cases}$$

cotangent covering or adjoint system. Then the condition is

$$\tilde{\ell}_F(A_3(\mathbf{p})) = 0$$

Necessary condition in suitable coordinates

Theorem. The Hamiltonianity of a hydrodynamic-type system in conservative form with respect to A_3:

$$u^i_t = A_3^{ij}\left(\frac{\delta H}{\delta u^j}\right) \quad \text{with} \quad A_3^* = -A_3 \quad \text{and} \quad [A_3, A_3] = 0$$

is *equivalent* to the following conditions on the Monge metric g:

$$g_{im}\frac{\partial V^m}{\partial a^j} = g_{jm}\frac{\partial V^m}{\partial a^i}, \quad c_{mkj}\frac{\partial V^m}{\partial a^i} + c_{mik}\frac{\partial V^m}{\partial a^j} + c_{mji}\frac{\partial V^m}{\partial a^k} = 0.$$

A general example

In N component case we have

$$g_{ij} = \begin{pmatrix} 2a^2 & -a^1 & 0 & & 1 \\ -a^1 & 0 & & 1 & \\ 0 & & 1 & & \\ & 1 & & & 0 \\ 1 & & & 0 & 0 \end{pmatrix}$$

and the Hamiltonian is

$$H = -\frac{1}{2}a^1(D^{-1}a^2)^2 + \frac{1}{2}\sum_{m=2}^{N}(D^{-1}a^m)(D^{-1}a^{N+2-m}).$$

implies the hydrodynamic type systems

$$a^1_t = a^2_x, \quad a^2_t = a^3_x, \ldots, \quad a^{N-1}_t = a^N_x, \quad a^N_t = [a^1a^3 - (a^2)^2]_x.$$

Main example: WDVV in $N = 3$

The associativity equation (η_{ij} an $N \times N$ constant nondegenerate matrix):

$$\eta^{\mu\lambda} \frac{\partial^3 F}{\partial t^\lambda \partial t^\alpha \partial t^\beta} \frac{\partial^3 F}{\partial t^\nu \partial t^\mu \partial t^\gamma} = \eta^{\mu\lambda} \frac{\partial^3 F}{\partial t^\nu \partial t^\alpha \partial t^\mu} \frac{\partial^3 F}{\partial t^\lambda \partial t^\beta \partial t^\gamma}$$

If $N = 3$ we have a single equation. Let us introduce coordinates

$$a = f_{xxx}, \quad b = f_{xxt}, \quad c = f_{xtt}.$$

Then the compatibility conditions for the WDVV equation become

$$a_t = b_x, \quad b_t = c_x, c_t = (\varphi(a, b, c; \eta))_x$$

The WDVV Monge metric

By using the compatibility conditions we have:

Theorem. The previous hydrodynamic-type system for generic values of η has a third-order Hamiltonian operator for which Casimirs are the letters a, b, c. The Monge metric of the third order operator is, up to a reciprocal transformation of projective type, the metric

$$g^{(3)} = \begin{pmatrix} b^2+1 & -ab & 0 \\ -ab & a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Particular cases

The first example (Ferapontov et al., CMP 1995)

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{type } g^{(5)}\colon \quad \begin{pmatrix} -2b & a & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The second example (Kalayci, Nutku PLA 1997)

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{type } g^{(4)}\colon \quad \begin{pmatrix} -2(b+c) & a-b-2c & a+b+1 \\ a-b-2c & 2a-2c+1 & a+b+1 \\ a+b+1 & a+b+1 & 0 \end{pmatrix}$$

Particular cases

O. Mokhov, O. Pavlenko, 2012, considered the following choice of η:

$$\eta = \begin{pmatrix} 0 & \alpha & 1 \\ \alpha & \beta & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and found first-order homogeneous Hamiltonian operators. Here is the third order operator of type $g^{(4)}$:

$$\begin{pmatrix} 2(\alpha c - b) & a - 2\alpha^2 c + \alpha b & -a\alpha + \alpha^2 b + \beta^2 \\ a - 2\alpha^2 c + \alpha b & -2a\alpha + 2\alpha^3 c + \beta^2 & \alpha(a\alpha - \alpha^2 b - \beta^2) \\ -a\alpha + \alpha^2 b + \beta^2 & \alpha(a\alpha - \alpha^2 b - \beta^2) & 0 \end{pmatrix}$$

Particular cases

Here the metric is of type $g^{(3)}$ (the most generic)

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$f_{ttt} = \frac{f_{xxt}^2 - f_{xxx} f_{xtt} + f_{xtt}^2 - 1}{f_{xxt}} \qquad \begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = \left(\frac{-ac + b^2 + c^2 - 1}{b} \right)_x \end{cases}$$

$$g_{ij} = \begin{pmatrix} b^2 + 1 & -ab + bc & -b^2 \\ -ab + bc & a^2 - 2ac + c^2 + 1 & ab - bc \\ -b^2 & ab - bc & b^2 \end{pmatrix}$$

Particular cases

Here the metric is of type $g^{(4)}$

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$f_{ttt} = \frac{f_{xxt} f_{xtt} + 1}{f_{xxx}} \qquad \begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = \left(\frac{bc+1}{a}\right)_x \end{cases}$$

$$g_{ij} = \begin{pmatrix} c^2 & -1 & -ac \\ -1 & 0 & 0 \\ -ac & 0 & a^2 \end{pmatrix}$$

An example in the case $N = 4$

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

WDVV equations are a system:

$$\begin{aligned}
&-2f_{xyz} - f_{xyy}f_{xxy} + f_{yyy}f_{xxx} = 0,\\
&-f_{xzz} - f_{xyy}f_{xxz} + f_{yyz}f_{xxx} = 0,\\
&-2f_{xyz}f_{xxz} + f_{xzz}f_{xxy} + f_{yzz}f_{xxx} = 0,\\
&f_{zzz} - (f_{xyz})^2 + f_{xzz}f_{xyy} - f_{yyz}f_{xxz} + f_{yzz}f_{xxy} = 0,\\
&f_{yyy}f_{xzz} - 2f_{yyz}f_{xyz} + f_{yzz}f_{xyy} = 0.
\end{aligned}$$

6-components WDVV system

We introduce new field variables a^k:

$$a^1 = f_{xxx}, a^2 = f_{xxy}, a^3 = f_{xxz}, a^4 = f_{xyy}, a^5 = f_{xyz}, a^6 = f_{xzz}.$$

The compatibility conditions for this system can be written as a pair of hydrodynamic type systems in conservative form:

$$a^i_y = (v^i(\mathbf{a}))_x, \quad a^i_z = (w^i(\mathbf{a}))_x,$$

where

$$v^1 = a^2, \quad w^1 = a^3, \quad v^2 = a^4, \quad v^3 = w^2 = a^5, \quad w^3 = a^6,$$
$$v^4 = f_{yyy} = \frac{2a^5 + a^2a^4}{a^1}, \quad v^5 = w^4 = f_{yyz} = \frac{a^3a^4 + a^6}{a^1},$$
$$v^6 = w^5 = f_{yzz} = \frac{2a^3a^5 - a^2a^6}{a^1},$$
$$w^6 = f_{zzz} = (a^5)^2 - a^4a^6 + \frac{(a^3)^2a^4 + a^3a^6 - 2a^2a^3a^5 + (a^2)^2a^6}{a^1}.$$

Monge metric for 6-components WDVV

$$
g_{ik}(\mathbf{a}) = \begin{pmatrix} (a^4)^2 & -2a^5 & 2a^4 & -(a^1a^4+a^3) & a^2 & 1 \\ -2a^5 & -2a^3 & a^2 & 0 & a^1 & 0 \\ 2a^4 & a^2 & 2 & -a^1 & 0 & 0 \\ -(a^1a^4+a^3) & 0 & -a^1 & (a^1)^2 & 0 & 0 \\ a^2 & a^1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

More information on the 6-component WDVV like: nonlocal Hamiltonians, momentum, factorization of the operator can be found in: MV Pavlov, RF Vitolo, `arxiv::1409.7647`

Factorization of the IIIrd order operator

Theorem: The Hamiltonian operator A_3 can be rewritten in the simplified form

$$A_2^{ij} = \varphi^{\beta\gamma}\partial_x\psi^i_\beta\partial_x\psi^j_\gamma\partial_x,$$

where

$$\psi^i_\gamma = \frac{1}{a^1}\begin{pmatrix} a^1 & 0 & 0 & a^4 & a^5 & a^3a^4 - a^2a^5 \\ 0 & 0 & 0 & 0 & -1 & a^2 \\ 0 & 0 & 0 & -1 & 0 & -a^3 \\ 0 & a^1 & 0 & 0 & a^3 & a^1a^5 - a^2a^3 \\ 0 & 0 & 0 & 0 & 0 & -a^1 \\ 0 & 0 & a^1 & 0 & -a^2 & (a^2)^2 - a^1a^4 \end{pmatrix},$$

$$\varphi^{\beta\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Perspectives on IIIrd order HO

1. Classification in higher number of components.
2. The nonlocal version of this talk! Analogue of Ferapontov's curvature condition for first-order operators.
3. Classification of compatible pairs of operators: first-order and third-order. Preliminary results in $n = 2$.
4. Conjecture: pairs of a first-order and a third-order homogeneous HO define a Frobenius manifold.

Perspectives on conservative systems

1. Compatibility conditions for nonlocal operators.
2. Classification of conservative hydrodynamic-type systems that admit third-order operators.
3. Classification of other systems in Casimirs admitting third-order operators.
4. Classification of integrable systems admitting compatible pairs of one third-order operator and another operator (of the first, second, third order ...) as Hamiltonian operators.

Perspectives on WDVV

1. Conjecture: all WDVV are the same bi-Hamiltonian system up to a coordinate change.
2. Conjecture: correspondence between Frobenius manifolds and pairs of first-order, third order homogeneous HO; another relation between Frobenius manifolds and WDVV?
3. The identity operator $g^{(6)}$ does not appear in the WDVV systems? Other metrics $g^{(1)}$, $g^{(2)}$?
4. Why a quadratic line complex is attached to each WDVV system? I can't believe that it is there by chance. Relation with Gromov–Witten invariants?

Symbolic computations

Within the REDUCE CAS (now free software) we use the packages CDIFF and CDE, freely available at `http://gdeq.org`.

CDIFF was developed by the Twente group (Gragert, Kersten, Post, Roelofs); it generates total derivatives on a supermanifold.

CDE (by R. Vitolo) can compute (in the forthcoming version 2.0): Fréchet derivatives, formal adjoints, symmetries and conservation laws, Hamiltonian operators, their brackets, their Lie derivatives.

Cooperation with AC Norman (Trinity College, Cambridge) to improvements and documentation of REDUCE's kernel.

The end!

THANK YOU!