Geometry of homogeneous third-order Hamiltonian operators and applications to hydrodynamic-type systems of PDEs

RF Vitolo (joint work with EV Ferapontov, MV Pavlov)

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- What are third-order homogeneous Hamiltonian operators?
- How to find them? Applications to hydrodynamic-type systems.

What are third-order homogeneous Hamiltonian operators?

First-order Dubrovin–Novikov (homogeneous) operators

Dubrovin–Novikov (homogeneous) operators were introduced in 1983 for the Hamiltonian formalism of hydrodynamic-type equations

$$u^i_t = v^i_j(\mathbf{u})u^j_x = A_1^{ij}\frac{\delta\mathcal{H}_1}{\delta u^j} \quad \mathcal{H}_1 = \int h(\mathbf{u})dx$$

$\mathbf{u} = (u^i(t,x))$, $i,j = 1,\ldots,n$ (n-components). The operators are of the form

$$A_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b^{ij}_k(\mathbf{u})u^k_x$$

Homogeneity: $\deg\partial_x = 1$.

Geometry of 1st-order Dubrovin–Novikov operators

Any change of coordinates of the type $\bar{u}^i = \bar{u}^i(u^j)$ will not change the 'nature' of the above operator. g^{ij} transforms as a contravariant 2-tensor; usually it is required that g^{ij} is non-degenerate; $\Gamma^j_{ik} = -g_{is}b^{sj}_k$ transforms as a linear connection.

Conditions:

- $A_1^* = -A_1$ is equivalent to: symmetry of g^{ij}, $\nabla[\Gamma]g = 0$;
- $[A_1, A_1] = 0$ is equivalent to: g_{ij} flat pseudo-Riemannian metric and $\Gamma^j_{ik} = \Gamma^j_{ki}$, or Γ is the Levi-Civita connection of g.

Third-order Dubrovin–Novikov operators

Dubrovin–Novikov operators were defined for higher orders too. In particular

$$\begin{aligned} A_3^{ij} =& g^{ij}(\mathbf{u})\partial_x^3 + b_k^{ij}(\mathbf{u})u_x^k\partial_x^2 \\ &+ [c_k^{ij}(\mathbf{u})u_{xx}^k + c_{km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\ &+ d_k^{ij}(\mathbf{u})u_{xxx}^k + d_{km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n, \end{aligned}$$

Examples of Hamiltonian equations of the form

$$u_t^i = A_3^{ij}\left(\frac{\delta \mathcal{H}_2}{\delta u^j}\right)$$

are in the 2-component case the Chaplygin gas equation (Mokhov DrSc thesis, '96) and the 3-component case WDVV equation (Ferapontov, Galvao, Mokhov, Nutku CMP '95).

Example: 2-component Chaplygin gas equation

(O. Mokhov, '96) The Monge–Ampère equation $u_{tt}u_{xx} - u_{xt}^2 = -1$ can be reduced to hydrodynamic form

$$a_t = b_x, \quad b_t = \left(\frac{b^2 - 1}{a}\right)_x,$$

via the change of variables $a = u_{xx}$, $b = u_{xt}$. It possesses the Hamiltonian formulation

$$\begin{pmatrix} a \\ b \end{pmatrix}_t = \partial_x \begin{pmatrix} 0 & \partial_x \dfrac{1}{a} \\ \dfrac{1}{a}\partial_x & \dfrac{b}{a^2}\partial_x + \partial_x \dfrac{b}{a^2} \end{pmatrix} \partial_x \begin{pmatrix} \delta H/\delta a \\ \delta H/\delta b \end{pmatrix},$$

and the nonlocal Hamiltonian,

$$H = -\int \left(\frac{1}{2}a(\partial_x^{-1}b)^2 + \partial_x^{-2}a\right) dx.$$

Example: 3-component WDVV equation

The simplest associativity (WDVV) equation:

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$$

can be presented by $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$ as

$$a_t = b_x, \quad b_t = c_x, \quad c_t = (b^2 - ac)_x.$$

From FERAPONTOV, GALVAO, MOKHOV, NUTKU, CMP (1997), there are two local Dubrovin-Novikov Hamiltonian operators, first-order A_1 and third-order A_3,

$$A_3 = \begin{pmatrix} 0 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & -\partial_x^2 a \partial_x \\ \partial_x^3 & -\partial_x a \partial_x^2 & (\partial_x^2 b \partial_x + \partial_x b \partial_x^2 + \partial_x a \partial_x a \partial_x) \end{pmatrix}$$

Some known results

Non-degenerate ($\det(g^{ij}) \neq 0$) third-order homogeneous Hamiltonian operators have the canonical form (Potemin '86, '97; Potemin–Balandin, '01; Doyle '95):

$$A_3 = \partial_x \circ (g^{ij}\partial_x + c^{ij}_k u^k_x) \circ \partial_x,$$

where (Ferapontov, Pavlov, V., JGP 2014)

$$c_{nkm} = \frac{1}{3}(g_{nm,k} - g_{nk,m}),$$
$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0,$$
$$c_{mnk,l} = -g^{pq}c_{pml}c_{qnk}.$$

g_{ij} is the Monge form of a **quadratic line complex**.

Monge metrics

Example: $n = 3$

$$g_{11} = -[R_{12}(u^2)^2 + R_{13}(u^3)^2 + 2B_{12}u^2u^3 + 2H_{12}u^2 + 2H_{13}u^3 + D_1],$$
$$g_{22} = -[R_{12}(u^1)^2 + R_{23}(u^3)^2 + 2B_{22}u^1u^3 + 2H_{21}u^1 + 2H_{23}u^3 + D_2],$$
$$g_{33} = -[R_{23}(u^2)^2 + R_{13}(u^1)^2 + 2B_{32}u^1u^2 + 2H_{31}u^1 + 2H_{32}u^2 + D_3],$$
$$g_{12} = R_{12}u^1u^2 + B_{12}u^1u^3 + B_{22}u^2u^3 - B_{32}(u^3)^2 + H_{12}u^1 + H_{21}u^2 + (E_2 - E_1)u^3 + F_{12},$$
$$g_{13} = R_{13}u^1u^3 + B_{12}u^1u^2 - B_{22}(u^2)^2 + B_{32}u^2u^3 + H_{13}u^1 + H_{31}u^3 + (E_1 - E_3)u^2 + F_{13},$$
$$g_{23} = R_{23}u^2u^3 - B_{12}(u^1)^2 + B_{22}u^1u^2 + B_{32}u^1u^3 + H_{23}u^2 + H_{32}u^3 + (E_3 - E_2)u^1 + F_{23},$$

Monge–Ampère example revisited

The operator:

$$A_3 = \partial_x \begin{pmatrix} 0 & \partial_x \dfrac{1}{a} \\ \dfrac{1}{a}\partial_x & \dfrac{b}{a^2}\partial_x + \partial_x \dfrac{b}{a^2} \end{pmatrix} \partial_x$$

is completely determined by its Monge metric:

$$g_{ij} = \begin{pmatrix} -2b & a \\ a & 0 \end{pmatrix}$$

In this case, the singular surface is $\det(g) = -a^2$, and is a line counted two times. Moreover, g is a flat pseudo-Riemannian metric. This is the simplest nontrivial homogeneous third-order operator.

WDVV example revisited

The operator:

$$A_3 = \begin{pmatrix} 0 & 0 & \partial_x^3 \\ 0 & \partial_x^3 & -\partial_x^2 a \partial_x \\ \partial_x^3 & -\partial_x a \partial_x^2 & (\partial_x^2 b \partial_x + \partial_x b \partial_x^2 + \partial_x a \partial_x a \partial_x) \end{pmatrix}$$

The operator is completely determined by its metric:

$$g_{ij} = \begin{pmatrix} -2b & a & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In this case, the singular surface is $\det(g) = -1$, and is a quadruple plane at infinity. Moreover, g is a flat pseudo-Riemannian metric.

Affine classification for $n = 2$

(The 1-component case was described by Gel'fand-Dorfman – point-equivalent to ∂_x^3).

Theorem (Ferapontov, Pavlov, V. JGP 2014): only two non-trivial metrics in 2-component case:

$$g_{ik}^{(1)} = \begin{pmatrix} 1-(b^2)^2 & 1+b^1b^2 \\ 1+b^1b^2 & 1-(b^1)^2 \end{pmatrix}, \quad g_{ik}^{(2)} = \begin{pmatrix} -2b^2 & b^1 \\ b^1 & 0 \end{pmatrix}$$

$g^{(1)}$ is non-flat, $g^{(2)}$ is flat and appears in the Chaplygin gas equation (O. Mokhov's Doctoral Thesis).

Theorem. In the 2-component cases the operators may be reduced to ∂_x^3 by a reciprocal transformation.

Projective classification for $n = 3$

Ferapontov, Pavlov, V., JGP 2014

$$g^{(1)} = \begin{pmatrix} (u^2)^2 + c & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 + c(u^3)^2 & -cu^2u^3 - u^1 \\ 2u^2 & -cu^2u^3 - u^1 & c(u^2)^2 + 1 \end{pmatrix},$$

$$g^{(2)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 & -u^1 \\ 2u^2 & -u^1 & 1 \end{pmatrix},$$

$$g^{(3)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 & 0 \\ -u^1u^2 & (u^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g^{(4)} = \begin{pmatrix} -2u^2 & u^1 & 0 \\ u^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{(5)} = \begin{pmatrix} -2u^2 & u^1 & 1 \\ u^1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad g^{(6)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Projective classification for $n = 4$

Ferapontov, Pavlov, V., arXiv 2015
Any Monge metric of a third-order homogeneous Hamiltonian operator admits the following decomposition:

$$g_{ij} = \varphi_{\alpha\beta}\psi^\alpha_i\psi^\beta_j$$

where $\psi^\alpha_i du^i$ are *linear line complexes*, $\varphi_{\alpha\beta}$ is a non-degenerate bilinear form and

$$\varphi_{\alpha\beta}\psi^\alpha_{[i}\psi^\beta_{j,k]} = 0.$$

The above condition can always be fulfilled for any Monge metric as above (*generalized Clebsch normal form*). From the projective classification of metabelian Lie algebrae (Galitski-Timashev 1999) we have a classification of 4-frames of linear line complexes $\psi^\alpha_i du^i$ and $\varphi_{\alpha\beta}$ with 38 classes. $n \geqslant 5$ wild!

How to find them? Application to:

- **hydrodynamic-type systems in conservative form;**
- **WDVV equations.**

Suitable coordinate systems

It is clear that canonical coordinates of A_3 are good:

$$A_3 = \partial_x \circ (g^{ij}\partial_x + c^{ij}_k u^k_x) \circ \partial_x,$$

Casimirs are *conservation law densities*, so it is natural to look for operators A_3 for hydrodynamic-type systems in conservative form:

$$a^i_t = (V^i(\mathbf{a}))_x$$

A necessary condition

For a system of PDEs $F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \ldots) = 0$ we have that

$$u^i_t = A_3^{ij}\left(\frac{\delta H}{\delta u^j}\right) \quad \text{with} \quad A_3^* = -A_3 \quad \text{and} \quad [A_3, A_3] = 0$$

$$\Rightarrow \quad \ell_F \circ A_3 = A_3^* \circ \ell_F^*$$

The right-hand side as a **necessary condition** to Hamiltonianity (Kersten, Krasil'shchik, Verbovetsky, JGP '04).

Necessary condition in suitable coordinates

Theorem. The Hamiltonianity of a hydrodynamic-type system in conservative form with respect to A_3:

$$u^i_t = A_3^{ij}\left(\frac{\delta H}{\delta u^j}\right) \quad \text{with} \quad A_3^* = -A_3 \quad \text{and} \quad [A_3, A_3] = 0$$

is *equivalent* to the following conditions on the Monge metric g:

$$g_{im}\frac{\partial V^m}{\partial a^j} = g_{jm}\frac{\partial V^m}{\partial a^i}, \quad c_{mkj}\frac{\partial V^m}{\partial a^i} + c_{mik}\frac{\partial V^m}{\partial a^j} + c_{mji}\frac{\partial V^m}{\partial a^k} = 0,$$

$$\frac{\partial^2 V^k}{\partial u^i \partial u^j} = g^{ks}c_{smj}\frac{\partial V^m}{\partial u^i} + g^{ks}c_{smi}\frac{\partial V^m}{\partial u^j}$$

The generic case: systems compatible with $g^{(1)}$

$$
\begin{aligned}
u^1_t &= (\alpha u^2 + \beta u^3)_x,\\
u^2_t &= \left(\frac{((u^2)^2 - c)(\alpha u^2 + \beta u^3) + \gamma(1 - c(u^2)^2) + \delta(u^1 - cu^2u^3)}{u^1u^2 - u^3}\right)_x,\\
u^3_t &= \left(\frac{\alpha u^3((u^2)^2 - c) + \beta u^3(u^2u^3 - cu^1) + \gamma(u^1 - cu^2u^3) + \delta((u^1)^2 - c(u^3)^2)}{u^1u^2 - u^3}\right)_x,
\end{aligned}
$$

where the system is **completely exceptional** and **non-diagonalizable** if and only if $\alpha\delta - \beta\gamma = 0$. The nonlocal Hamiltonian

$$
\begin{aligned}
H = \int \Big(&\frac{1}{2}\alpha(2cxu^1\partial_x{}^{-1}u^2 + u^3(\partial_x{}^{-1}u^2)^2 + cx^2u^3) + \beta u^3(1 - c^2)\partial_x{}^{-1}u^2\partial_x{}^{-1}u^3\\
&+ \delta(xu^1\partial_x{}^{-1}u^1 + cu^3\partial_x{}^{-1}u^1\partial_x{}^{-1}u^2 + cu^1\partial_x{}^{-1}u^2\partial_x{}^{-1}u^3 + cxu^3\partial_x{}^{-1}u^3)\\
&+ \frac{1}{2}\gamma(cu^1(\partial_x{}^{-1}u^2)^2 + x^2u^1 + 2cxu^3\partial_x{}^{-1}u^2\Big)dx.
\end{aligned}
$$

Integrability is not known.

First singular case: systems compatible with $g^{(2)}$

$$u^1_t = (\alpha u^2 + \beta u^3)_x,$$
$$u^2_t = \left(\frac{((u^2)^2 - 1)(\alpha u^2 + \beta u^3) - (\gamma + \delta u^1)}{u^1 u^2 - u^3}\right)_x,$$
$$u^3_t = \left(\frac{(u^2 u^3 - u^1)(\alpha u^2 + \beta u^3) - u^1(\gamma + \delta u^1)}{u^1 u^2 - u^3}\right)_x,$$

where the system is **completely exceptional** and **non-diagonalizable** if and only if $\alpha\delta - \beta\gamma = 0$. The nonlocal Hamiltonian:

$$H = \int \left(\frac{1}{2}\alpha u^3 ({\partial_x}^{-1} u^2)^2 + \beta u^3 {\partial_x}^{-1} u^2 {\partial_x}^{-1} u^3 - \frac{1}{2}\gamma x^2 u^1 - \delta x u^1 {\partial_x}^{-1} u^1\right) dx.$$

Integrability is not known.

Second singular case: systems compatible with $g^{(3)}$

$$
\begin{aligned}
u^1_t &= (u^2 + u^3)_x, \\
u^2_t &= \left(\frac{u^2(u^2 + u^3) - 1}{u^1} \right)_x, \\
u^3_t &= u^1_x,
\end{aligned}
$$

which is **completely exceptional** and **non-diagonalizable** with the he nonlocal Hamiltonian,

$$
H = \int \left(-\partial_x{}^{-1} u^1 \partial_x{}^{-1} u^3 + x u^1 \partial_x{}^{-1} u^2 \right) dx.
$$

Setting $u^1 = f_{xxt}$, $u^2 = f_{xtt} - f_{xxx}$, $u^3 = f_{xxx}$ we obtain the WDVV-type equation $f^2_{xxt} - f_{xxx} f_{xtt} + f^2_{xtt} - f_{xxt} f_{ttt} - 1 = 0$ (Dubrovin 1994; Agafonov 1998). **Admits a Lax pair.**

Further singular cases

- $g^{(4)}$: WDVV-type equation $f_{xxx} = f_{ttt}f_{xxt} - f_{xtt}^2$ (Kalayci and Nutku, JPA 1998). It is **bi-Hamiltonian**.
- $g^{(5)}$: WDVV-type equation $f_{ttt} = f_{xxt}^2 - f_{xxx}f_{xtt}$ (Ferapontov, Galvao, Mokhov, Nutku, CMP 1995). It is **bi-Hamiltonian** and up to a reciprocal transformation is the **3-wave equation** (Zakharov, Manakov, ∼1970).

A general example

In N component case we have

$$g_{ij} = \begin{pmatrix} 2a^2 & -a^1 & 0 & & 1 \\ -a^1 & 0 & & 1 & \\ 0 & & 1 & & \\ & 1 & & & 0 \\ 1 & & & 0 & 0 \end{pmatrix}$$

and the Hamiltonian is

$$H = -\frac{1}{2}a^1(D^{-1}a^2)^2 + \frac{1}{2}\sum_{m=2}^{N}(D^{-1}a^m)(D^{-1}a^{N+2-m}).$$

implies the hydrodynamic type systems

$$a_t^1 = a_x^2, \quad a_t^2 = a_x^3, \ldots, \quad a_t^{N-1} = a_x^N, \quad a_t^N = [a^1 a^3 - (a^2)^2]_x.$$

WDVV in $N = 3$

The associativity equation (η_{ij} an $N \times N$ constant nondegenerate matrix):

$$\eta^{\mu\lambda}\frac{\partial^3 F}{\partial t^\lambda \partial t^\alpha \partial t^\beta}\frac{\partial^3 F}{\partial t^\nu \partial t^\mu \partial t^\gamma} = \eta^{\mu\lambda}\frac{\partial^3 F}{\partial t^\nu \partial t^\alpha \partial t^\mu}\frac{\partial^3 F}{\partial t^\lambda \partial t^\beta \partial t^\gamma}$$

If $N = 3$ we have a single equation. Let us introduce coordinates

$$a = f_{xxx}, \quad b = f_{xxt}, \quad c = f_{xtt}.$$

Then the compatibility conditions for the WDVV equation become

$$a_t = b_x, \quad b_t = c_x, c_t = (\varphi(a, b, c; \eta))_x$$

The WDVV Monge metric

By using the compatibility conditions we have:

Theorem. The previous hydrodynamic-type system for generic values of η has a third-order Hamiltonian operator for which Casimirs are the letters a, b, c. The Monge metric of the third order operator is, up to a reciprocal transformation of projective type, the metric

$$g^{(3)} = \begin{pmatrix} b^2+1 & -ab & 0 \\ -ab & a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Perspectives on IIIrd order HO

1. The nonlocal version of this talk! Analogue of Ferapontov's curvature condition for first-order operators.
2. Conjecture: pairs of a first-order and a third-order homogeneous HO define a Frobenius manifold.
3. Compatibility conditions for nonlocal operators.
4. Classification of higher-order systems of conservation laws admitting third-order operators.

Perspectives on WDVV

1. Conjecture: WDVV in all dimensions have a bi-Hamiltonian formulation by a pair of a (nonlocal?) first-order and a local third-order homogeneous Hamiltonian operator. See MV Pavlov, RV, Lett. Math. Phys 2015 `arxiv::1409.7647` for the 6-component case.
2. Conjecture: all WDVV are the same bi-Hamiltonian system up to a coordinate change.
3. Conjecture: correspondence between Frobenius manifolds and pairs of first-order, third order homogeneous HO; another relation between Frobenius manifolds and WDVV?
4. Why a quadratic line complex is attached to each WDVV system? I can't believe that it is there by chance. Relation with Gromov–Witten invariants?

Symbolic computations

Within the REDUCE CAS (now free software) we use the packages CDIFF and CDE, freely available at `http://gdeq.org`.

CDIFF was developed by the Twente group (Gragert, Kersten, Post, Roelofs); it generates total derivatives on a supermanifold.

CDE (by R. Vitolo) can compute in the **new version 2.0**: Fréchet derivatives, formal adjoints, symmetries and conservation laws, Hamiltonian operators, their brackets, their Lie derivatives.

Cooperation with AC Norman (Trinity College, Cambridge) to improvements and documentation of REDUCE's kernel.

The end!

THANK YOU!