CDIFF: a suite of REDUCE packages for computations in the geometry of DE

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Contents

History

CDIFF has been developed at the University of Twente by

P. Gragert, P. Kersten, G. Post, M. Roelofs

in the end of '80 - beginning of '90.

REDUCE 3.8 is now free software, hosted at

`http://reduce-algebra.sourceforge.net`

and CDIFF is included in the official distribution.
The name CDIFF was given to the package only recently.

Structure

CDIFF is based upon two packages, a set of tools and an interface:

- `supervf.red`: build vectorfields on supermanifolds; used for building total derivatives on DEs and their tangent and cotangent coverings.
- `integrator.red`: integrates overdetermined systems of DEs.
- `tools21.red`: various tools.
- `prova.red`: automatic generation of jet coordinates and prolonged equations, plus equation solver (**experimental**).

Documentation

In the CDIFF folder of the official distribution there is a documentation file (by R.V.) which describes

- installation of REDUCE;
- work with CDIFF;
- examples of computations with CDIFF: conservation laws, higher or generalized symmetries, Hamiltonian operators.

Example programs are included as well.

Example: Hamiltonian operators for KdV

KdV: $u_t = u_{xxx} + uu_x$.
After [KKV], we have to solve $\bar{\ell}_{KdV}(\varphi) = 0$, or

$$\bar{D}_t(\varphi) - u * \bar{D}_x(\varphi) - u_x * \varphi - \bar{D}_{xxx}(\varphi) = 0$$

over the system

$$\begin{cases} u_t = u_{xxx} + uu_x \\ p_t = p_{xxx} + up_x \end{cases}$$

$\bar{D}_t$, $\bar{D}_x$ are total derivatives on the ℓ^*-covering, hence

$$\bar{D}_t = \frac{\partial}{\partial t} + u^i_{\sigma t}\frac{\partial}{\partial u^i_\sigma} + p^j_{\mu t}\frac{\partial}{\partial p^j_\mu}$$

Choice of coordinates

Note that in $\bar{D}_t = \partial/\partial t + u^i_{\sigma t}\partial/\partial u^i_\sigma + p^j_{\mu t}\partial/\partial p^j_\mu$ the sum is extended to all indexes (i, σ) and (j, μ) of **internal (or parametric) coordinates** of the ℓ^*-covering. We recall that these are coordinates on the equation manifold $F(x^\lambda, u^i_\sigma) = 0$.

When the equation is expressed as $u^k_\tau = G(x^\lambda, u^j_\mu)$ we have a splitting of coordinates into **external (or principal)** and **internal (or parametric)**.

We will need to prolong our equations; a way to avoid the occurrence of new conditions is that the equations be in **passive orthonomic form** [M].

Define the total derivatives

Begin a CDIFF program with the statement

```
super_vectorfield(ddx,{x,t,u,u1,u2,u3,u4,u5,u6,u7,
u8,u9,u10,u11,u12,u13,u14,u15,u16,u17},
{ext 1,ext 2,ext 3,ext 4,ext 5,ext 6,ext 7,ext 8,
ext 9,ext 10,ext11,ext 12,ext 13,ext 14,ext 15,ext 16});
```

and analogously for each independent variable.

```
ddx(0,1):=1$
ddx(0,2):=0$
ddx(0,3):=u1$
ddx(0,4):=u2$
```

where $u1 = u_x$, $u2 = u_{xx}$, $\ldots$

Note that in the definition of $\bar{D}_t$ we have

```
super_vectorfield(ddt,{x,t,u,u1,u2,u3,u4,u5,u6,u7,
u8,u9,u10,u11,u12,u13,u14,u15,u16,u17},
{ext 1,ext 2,ext 3,ext 4,ext 5,ext 6,ext 7,ext 8,
ext 9,ext 10,ext11,ext 12,ext 13,ext 14,ext 15,ext 16});
```

where

```
ddt(0,1):=0$
ddt(0,2):=1$
ddt(0,3):=ut$
ddt(0,4):=ut1$
```

where $ut = u_t$, $ut1 = u_{tx}$, ...so that coefficients are principal.

For scalar evolutionary equations with two independent variables internal variables are of the type $(t, x, u, u_x, u_{xx}, \ldots)$. So we shall restrict total derivatives using

```
ut:=u*u1+u3;
ut1:=ddx ut;
ut2:=ddx ut1;
ut3:=ddx ut2;
```

Odd (ℓ^*-covering) variables

$\bar{D}_x$ and $\bar{D}_t$ contain odd variables, introduced as follows:

```
ddx(1,1):=0$
ddx(1,2):=0$
ddx(1,3):=ext 4$
ddx(1,4):=ext 5$
```

the first index '1' says that we are dealing with odd variables, `ext` indicates anticommuting variables. Here, `ext 3` is p_0, `ext 4` is p_x, `ext 5` is p_{xx}, …so `ddx(1,3):=ext 4` indicates $p_x \partial/\partial p$, etc.. We replace p_t by `ext 6 + u*ext 4`:

```
ddt(1,1):=0$
ddt(1,2):=0$
ddt(1,3):=ext 6 + u*ext 4$
ddt(1,4):=ddx(ddt(1,3))$
ddt(1,5):=ddx(ddt(1,4))$
```

Coverings of passive orthonomic DEs

In case of an evolutionary equation, the ℓ^*-covering is evolutionary.

Computational question: is the ℓ^*-covering of an equation in passive orthonomic form still in passive orthonomic form?

Gradings

In DEs with scale symmetries variables are graded:

```
graadlijst:={{u},{u1},{u2},{u3},{u4},{u5},
{u6},{u7},{u8},{u9},{u10},{u11},{u12},{u13},{u14}};
```

This is the list of all monomials of degree 0, 1, 2, . . . which can be constructed from the above list of elementary variables with their grading.

```
grd0:={1};
grd1:= mkvarlist1(1,1)$
grd2:= mkvarlist1(2,2)$
grd3:= mkvarlist1(3,3)$
grd4:= mkvarlist1(4,4)$
```

Ansatz for Hamiltonian operators

We want to solve the equation

```
ddt(phi)-u*ddx(phi)-u1*phi-ddx(ddx(ddx(phi)))=0
```

and we make the following ansatz:

```
phi:=
(for each el in grd0 sum (c(ctel:=ctel+1)*el))*ext 3+
(for each el in grd1 sum (c(ctel:=ctel+1)*el))*ext 3+
(for each el in grd2 sum (c(ctel:=ctel+1)*el))*ext 3+
(for each el in grd3 sum (c(ctel:=ctel+1)*el))*ext 3+

(for each el in grd0 sum (c(ctel:=ctel+1)*el))*ext 4+
(for each el in grd1 sum (c(ctel:=ctel+1)*el))*ext 4+
(for each el in grd2 sum (c(ctel:=ctel+1)*el))*ext 4+

(for each el in grd0 sum (c(ctel:=ctel+1)*el))*ext 5+
(for each el in grd1 sum (c(ctel:=ctel+1)*el))*ext 5+

(for each el in grd0 sum (c(ctel:=ctel+1)*el))*ext 6+
```

Define the equation

After having set

```
equ 1:=ddt(phi)-u*ddx(phi)-u1*phi-ddx(ddx(ddx(phi)));
vars:={x,t,u,u1,u2,u3,u4,u5,u6,u7,u8,u9,u10,u11,u12};
tel:=1;
```

we initialize the equations:

```
initialize_equations(equ,tel,{},{c,ctel,0},{f,0,0});
```

and we plug the ansatz for `phi` and due to the **polinomiality** and the **graded structure** we obtain a long list of equations on the coefficients of all graded monomials which multiply each single variable p_σ.
We solve the equations using `es`, the equation solver from `integrator.red` package.

```
for i:=2:tel do es i;
```

Solution of the equation

The results are the two well-known Hamiltonian operators for the KdV:

```
phi := c(4)*ext(4) + 3*c(3)*ext(6) +
2*c(3)*ext(4)*u + c(3)*ext(3)*u1$
```

Of course, the results correspond to the operators

$$\texttt{ext(4)} \to$$
$$D_x,$$
$$\texttt{3*c(3)*ext(6) + 2*c(3)*ext(4)*u + c(3)*ext(3)*u1} \to$$
$$3D_{xxx} + 2uD_x + u_x$$

Note that each operator is multiplied by one arbitrary real constant, `c(4)` and `c(3)`.

Other solved problems

- Higher symmetries;
- Local conservation laws;
- Non-local Hamiltonian operators (KdV);
- Hamiltonian operators for systems of DEs (Boussinesq equation).

P. Kersten did more computations for Monge-Ampère DEs (2D-associativity eq., or WDVV eq.), but these are not included as examples of CDIFF distribution.

Automatic interface (experimental)

The interface `prova.red` has been recently developed by R.V..
It needs initial data like: independent variables, dependent variables, the differential equation, principal and parametric coordinates.
Coordinates are generated automatically in the form

```
u_xnynzn
```

e.g. we have `u_x2y0z1` $\to u_{xxz}$.

Structure of the interface

We first generate all coordinates and total derivatives, then compute prolongations of the equation, then restrict total derivatives to the equation manifold.
Then, we compute gradings of all internal coordinates.
We proceed by writing the equation and an ansatz for it.
We have been able to reproduce previous ‘handcrafted’ computations.
Range of applicability: potentially for each DE which is written in passive orthonomic form we can compute higher symmetries, conservation laws, local and nonlocal hamiltonian operators.

References

- P. Kersten, I. Krasil′shchik, and A. Verbovetsky, *Hamiltonian operators and ℓ^*-coverings*, J. Geom. Phys. **50** (2004), 273–302
- P. Kersten, I. Krasil′shchik, A. Verbovetsky, R. Vitolo, *Hamiltonian structures for general PDEs*, in Differential Equations—Geometry, Symmetries and Integrability: The Abel Symposium 2008 (B. Kruglikov, V. Lychagin, and E. Straume, eds.), Abel Symposia 5, Springer, 2009, pp. 187–198
- S. Igonin, P. Kersten, J. Krasil′shchik, A. Verbovetsky, R. Vitolo, *Variational brackets in geometry of PDEs*, 2010, to be finished really soon...
- R. Vitolo, *CDIFF: a REDUCE package for computations in the geometry of DEs*, the REDUCE distribution, available at `http://reduce-algebra.sourceforge.net`.