Weakly nonlocal Poisson brackets: three computational approaches

R. Vitolo

Joint work with M. Casati, P. Lorenzoni

Independent University of Moscow
J. Krasil'shchik – A. Verbovetsky's seminar
April 2019

Contents

- Weakly nonlocal Poisson brackets
- Algorithm for Jacobi identity:
  - with distributions,
  - with differential operators,
  - with Poisson Vertex algebras.
- Perspectives in the anticommutative case.

Hamiltonian PDEs

An evolutionary system of PDEs

$$F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \ldots) = 0$$

admits a Hamiltonian formulation if there exist A, $\mathcal{H} = \int h\, dx$ such that

$$u^i_t = A^{ij}\left(\frac{\delta \mathcal{H}}{\delta u^j}\right)$$

where $A = (A^{ij})$ is a Hamiltonian operator, i.e. a matrix of differential operators $A^{ij} = A^{ij\sigma}\partial_\sigma$, where $\partial_\sigma = \partial_x \circ \cdots \circ \partial_x$ (total x-derivatives σ times), such that

$$\{F, G\}_A = \int \frac{\delta F}{\delta u^i} A^{ij\sigma} \partial_\sigma \frac{\delta G}{\delta u^j}\, dx$$

is a Poisson bracket (skew-symmetric and Jacobi).

First-order homogeneous operators

First-order homogeneous operators were introduced in 1983 by Dubrovin and Novikov for the Hamiltonian formalism of hydrodynamic-type equations

$$u^i_t = v^i_j(\mathbf{u})u^j_x = A_1^{ij}\frac{\delta\mathcal{H}_1}{\delta u^j}, \quad \mathcal{H}_1 = \int h(\mathbf{u})dx$$

$\mathbf{u} = (u^i(t,x))$, $i, j = 1,\ldots,n$ (n-components). The operators are of the form

$$\boxed{A_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b^{ij}_k(\mathbf{u})u^k_x}$$

Homogeneity: $\deg \partial_x = 1$.

Geometry of homogeneous operators

Any change of coordinates of the type $\bar{u}^i = \bar{u}^i(u^j)$ will not change the 'nature' of the above operator. g^{ij} transforms as a contravariant 2-tensor; usually it is required that $\det(g^{ij}) \neq 0$; $\Gamma^j_{ik} = -g_{is} b^{sj}_k$ transforms as a linear connection.

Conditions on A_1 to be Hamiltonian:

- Skew-symmetry of $\{,\}_{A_1}$ is equivalent to:
  symmetry of g^{ij}, $\nabla[\Gamma]g = 0$;
- Jacobi identity of $\{,\}_{A_1}$ is equivalent to:
  g_{ij} flat pseudo-Riemannian metric and $\Gamma^j_{ik} = \Gamma^j_{ki}$, or Γ is the Levi-Civita connection of g.

It follows that the canonical form for such operators is $A_1 = \eta^{ij}\partial_x$, where η^{ij} are constants.

Ferapontov–Mokhov nonlocal operators

First-order nonlocal homogeneous operators were introduced in 1983 by Ferapontov and Mokhov for the Hamiltonian formalism of hydrodynamic-type equations. Here

$$\boxed{P^{ij} = g^{ij}(\mathbf{u})\partial_x + b^{ij}_k(\mathbf{u})u^k_x + c^\alpha w^i_{\alpha k}u^k_x\partial_x^{-1}w^j_{\alpha h}u^h_x}$$

The condition on P to be Hamiltonian:

- Skew-symmetry of $\{,\}_P$ is equivalent to:
  symmetry of g^{ij}, $\nabla[\Gamma]g = 0$;
- Jacobi identity of $\{,\}_P$ is equivalent to:
  - $\Gamma^j_{ik} = \Gamma^j_{ki}$, or Γ is the Levi-Civita connection of g.
  - g_{ij} pseudo-Riemannian metric such that

    $$R^{ij}_{kl} = w^i_k w^j_l - w^i_l w^j_k$$

  - w^i_j is a symmetric endomorphism: $g^{is}w^j_s = g^{js}w^i_s$;
  - $\nabla_k w^i_j = \nabla_j w^i_k$.

Weakly nonlocal Hamiltonian operators

Maltsev and Novikov (2001) further enlarged the class of nonlocal operators to the weakly nonlocal operators:

$$P^{ij} = B^{ij\sigma}\partial_\sigma + e^\alpha w^i_\alpha \partial_x^{-1} w^j_\alpha, \tag{1}$$

where $B^{ij\sigma}$ and w^i_α, w^j_β can depend on (u^k, u^k_σ), and e^α are constants. The operator ∂_x^{-1} is defined to be

$$\partial_x^{-1} = \frac{1}{2}\int_{-\infty}^{x} dx - \frac{1}{2}\int_{x}^{+\infty} dx \tag{2}$$

Jacobi property for nonlocal brackets

Problem: how to compute the Jacobi property for a bracket defined by a weakly nonlocal operator P?

$$\{\{F,G\}_P,H\}_P+\{\{H,F\}_P,G\}_P+\{\{G,H\}_P,F\}_P=0.$$

Standard solution: the above expression is a differential operator in three arguments, denoted by $[P,P]$ (Schouten bracket), and it must vanish *up to total divergencies*.

- If P is local, there are standard formulae.
- If P is nonlocal, the standard formulae do not work!

Linearization and adjoint of operators

$\ell_{P,\psi}(\varphi)$: the linearization of the (coefficients of the) operator P. We have the coordinate expressions:

$$\ell_{P,\psi}(\varphi)^i = \frac{\partial B^{ij\sigma}}{\partial u^k_\tau}\partial_\sigma\psi^1_j\partial_\tau\varphi^k + e^\alpha\frac{\partial w^i_\alpha}{\partial u^k_\tau}\partial_\tau\varphi^k\partial_x^{-1}(w^j_\alpha\psi_j) + e^\alpha w^i_\alpha\partial_x^{-1}\left(\frac{\partial w^j_\alpha}{\partial u^k_\tau}\partial_\tau\varphi^k\psi_j\right), \quad (3)$$

If $A(\psi)^i = A^{ij\sigma}\partial_\sigma\psi_j$ is a differential operator, then its adjoint is

$$A^*(\psi)^j = (-1)^\sigma\partial_\sigma(A^{ij\sigma}\psi_i).$$

The adjoint is defined by

$$\psi^1_i A^{ij\sigma}\partial_\sigma\psi^2_j = (-1)^\sigma\partial_\sigma(A^{ij\sigma}\psi^1_i)\psi^2_j + \text{tot. div.},$$

The Schouten bracket

There are many formulae for Schouten bracket:

$$
\begin{aligned}
[P,Q](\psi^1,\psi^2,\psi^3) = {}& \langle \ell_{P,\psi^1}(Q(\psi^2)),\psi^3\rangle - \langle \ell_{P,\psi^2}(Q(\psi^1)),\psi^3\rangle \\
&+ \langle \ell_{Q,\psi^1}(P(\psi^2)),\psi^3\rangle - \langle \ell_{Q,\psi^2}(P(\psi^1)),\psi^3\rangle \\
&- \langle P(\ell^*_{Q,\psi^2}(\psi^1)),\psi^3\rangle - \langle Q(\ell^*_{P,\psi^2}(\psi^1)),\psi^3\rangle
\end{aligned} \tag{a}
$$

$$
\begin{aligned}
[P,Q](\psi^1,\psi^2,\psi^3) = {}& \langle P(\mathcal{E}(\langle Q(\psi^1),\psi^2\rangle)),\psi^3\rangle + \mathrm{cyclic}(\psi^1,\psi^2,\psi^3) \\
&+ \langle Q(\mathcal{E}(\langle P(\psi^1),\psi^2\rangle)),\psi^3\rangle + \mathrm{cyclic}(\psi^1,\psi^2,\psi^3)
\end{aligned} \tag{b}
$$

$$
\begin{aligned}
[P,Q](\psi^1,\psi^2,\psi^3) = {}& \langle \ell_{P,\psi^1}(Q(\psi^2)),\psi^3\rangle + \mathrm{cyclic}(\psi^1,\psi^2,\psi^3) \\
&+ \langle \ell_{Q,\psi^1}(P(\psi^2)),\psi^3\rangle + \mathrm{cyclic}(\psi^1,\psi^2,\psi^3)
\end{aligned} \tag{c}
$$

Equivalence is up to total divergencies.

Which is the best formula?

- a: Used by F. Magri (1978). Good for local operators. Divergence-free.
- b: Used by E.V. Ferapontov and O.I. Mokhov (1990) for nonlocal operators. Good 'enough'.
- c: Used by B. Dubrovin and Y. Zhang. The best for nonlocal operators as it can be made algorithmic.

The algorithm

We start from the formula (c) for a weakly nonlocal operator $P^{ij\sigma}\partial_\sigma = B^{ij\sigma}\partial_\sigma + e^\alpha w^i_\alpha \partial_x^{-1} w^j_\alpha$:

$$
\begin{aligned}
\frac{1}{2}[P,P](\psi^1,\psi^2,\psi^3) = {} & \frac{\partial B^{ij\sigma}}{\partial u^k_\tau}\partial_\sigma\psi^1_j\partial_\tau(P^{kl\sigma}\partial_\sigma\psi^2_l)\psi^3_i \\
& + e^\alpha \frac{\partial w^i_\alpha}{\partial u^k_\tau}\partial_\tau(P^{kl\sigma}\partial_\sigma\psi^2_l)\partial_x^{-1}(w^j_\alpha\psi^1_j)\psi^3_i \\
& + e^\alpha w^i_\alpha\partial_x^{-1}\left(\frac{\partial w^j_\alpha}{\partial u^k_\tau}\partial_\tau(P^{kl\sigma}\partial_\sigma\psi^2_l)\psi^1_j\right)\psi^3_i + \mathrm{cyclic}(\psi^1,\psi^2,\psi^3),
\end{aligned}
$$

where the equality is up to total divergencies.

The algorithm – (1)

Let us introduce the notation

$$\tilde{\psi}^a_\alpha = \partial_x^{-1}(w^i_\alpha \psi^a_i), \quad a = 1, 2, 3. \tag{4}$$

1. Integrate by parts nonlocal terms with two integrands:

$$\begin{aligned} e^\alpha w^i_\alpha \partial_x^{-1} \left(\frac{\partial w^j_\alpha}{\partial u^k_\tau} \partial_\tau (B^{kp\sigma} \partial_\sigma \psi^b_p + e^\alpha w^k_\alpha \tilde{\psi}^b_\alpha)^k \psi^c_j \right) \psi^a_i = \\ - e^\alpha \tilde{\psi}^a_\alpha \left(\frac{\partial w^j_\alpha}{\partial u^k_\tau} \partial_\tau (B^{kp\sigma} \partial_\sigma \psi^b_p + e^\alpha w^k_\alpha \tilde{\psi}^b_\alpha)^k \psi^c_j \right) \end{aligned} \tag{5}$$

The algorithm – (2)

2. After (1) the generic summands of (3) are of three types:

$$C^{\alpha\beta k}\tilde{\psi}^a_\alpha\tilde{\psi}^b_\beta\psi^c_k, \tag{6}$$

$$C^{\alpha kj\sigma}\tilde{\psi}^a_\alpha\partial_\sigma(\psi^b_j)\psi^c_k, \tag{7}$$

$$C^{kj\sigma i\tau}\partial_\tau(\psi^a_i)\partial_\sigma(\psi^b_j)\psi^c_k, \tag{8}$$

where C's are functions of (u^i, u^i_σ).
Then, bring all nonlocal terms of type (7) to one of the forms

$$C^{pi}\tilde{\psi}^1\partial_\sigma\psi^2_p\psi^3_i, \quad C^{pi}\tilde{\psi}^2\partial_\sigma\psi^3_p\psi^1_i, \quad C^{pi}\tilde{\psi}^3\partial_\sigma\psi^1_p\psi^2_i. \tag{9}$$

by integration by parts, *e.g.*

$$D^{pi}\tilde{\psi}^1\psi^2_p\partial_\sigma\psi^3_i = (-1)^k\partial_\sigma(D^{pi}\tilde{\psi}^1\psi^2_p)\psi^3_i = C^{pi\tau}\tilde{\psi}^1\partial_\tau\psi^2_p\psi^3_i + l.t.$$

The algorithm – (3)

3. The third step of the algorithm amounts at bringing the local part into a divergence-free form. This is achieved by integrating by parts local terms with respect to ψ^3 so that the result is of order 0 in ψ^3, for example:

$$
\begin{aligned}
D^{i\sigma j\tau k\mu}\partial_\sigma\psi_i^1\partial_\tau\psi_j^2\partial_\mu\psi_k^3 &= (-1)^\mu\partial_\mu(D^{i\sigma j\tau k\mu}\partial_\sigma\psi_i^1\partial_\tau\psi_j^2)\psi_k^3 \\
&= C^{i\sigma j\tau k}\partial_\sigma\psi_i^1\partial_\tau\psi_j^2\psi_k^3
\end{aligned}
$$

Distributions

Weakly nonlocal Poisson brackets have the form

$$\{u^i(x), u^j(y)\}_P = B_k^{ij}(u^h, u_\sigma^h)\delta^{(k)}(x-y) \\ + e^\alpha w_\alpha^i(u^k, u_\sigma^k)\nu(x-y)w_\alpha^j(u^k, u_\sigma^k)$$

where $\nu(x-y) = \frac{1}{2}\operatorname{sgn}(x-y)$. The Jacobi identity

$$\{\{u^i(x), u^j(y)\}_P, u^k(z)\}_P + \{\{u^k(z), u^i(x)\}_P, u^j(y)\}_P \\ + \{\{u^j(y), u^k(z)\}_P, u^i(x)\}_P = 0$$

can be written as

$$J_{xyz}^{ijk} = \frac{\partial P_{x,y}^{ij}}{\partial u_\sigma^l(x)}\partial_x^\sigma P_{x,z}^{lk} + \frac{\partial P_{x,y}^{ij}}{\partial u_\sigma^l(y)}\partial_y^\sigma P_{y,z}^{lk} + \frac{\partial P_{z,x}^{ki}}{\partial u_\sigma^l(z)}\partial_z^\sigma P_{z,y}^{lj} + \\ \frac{\partial P_{z,x}^{ki}}{\partial u_\sigma^l(x)}\partial_x^\sigma P_{x,y}^{lj} + \frac{\partial P_{y,z}^{jk}}{\partial u_\sigma^l(y)}\partial_y^\sigma P_{y,x}^{li} + \frac{\partial P_{y,z}^{jk}}{\partial u_\sigma^l(z)}\partial_z^\sigma P_{z,x}^{li} = 0, \quad (10)$$

where $P_{x,y}^{ij} = \{u^i(x), u^j(y)\}_P$.

The algorithm for distributions

The first step is to reduce the nonlocal terms from 6 to 3 forms using the identities

$$\begin{aligned}
\nu(z-y)\delta(z-x) &= \nu(x-y)\delta(x-z),\\
\nu(y-x)\delta(y-z) &= \nu(z-x)\delta(z-y),\\
\nu(x-z)\delta(x-y) &= \nu(y-z)\delta(y-x),
\end{aligned}$$

and their differential consequences. After this the nonlocal part becomes

$$\begin{aligned}
a_1(x,y,z)\nu(x-y)\nu(x-z) &+ \mathrm{cyclic}(x,y,z)\\
&+ \sum_{n\geq 0} b_n(x,y)\nu(x-y)\delta^{(n)}(x-z) + \mathrm{cyclic}(x,y,z).
\end{aligned}$$

The local part is transformed in order not to act by derivatives on test functions of x.

Poisson vertex algebras

(De Sole, Kac, 2013) The main object is the λ-bracket on elements of a differential algebra:

$$\{f_\lambda g\} = \sum_{s \leqslant S} C_s(f,g)\lambda^s.$$

The λ-bracket is defined by its action on generators of the algebra by the master formula:

$$\{f_\lambda g\} = \frac{\partial g}{\partial u^j_\sigma}\,(\lambda+\partial)^\sigma\,\{u^i_{\lambda+\partial}u^j\}\,(-\lambda-\partial)^\tau\,\frac{\partial f}{\partial u^i_\tau}$$

The correspondence in terms of differential operators:

$$\{u^i_{\lambda+\partial}u^j\} = P^{ij}$$

In the expansion, λ^s corresponds to the s-th x-derivative acting on an argument and ∂ corresponds to linearizing the coefficients of the operator.

The divergence-free basis

Theorem The Jacobi identity can be written in terms

1. $A^{ijk}\lambda^p\mu^q \longleftrightarrow A^{ijk}\partial^p(\psi_i^1)\partial^q(\psi_j^2)\psi_k^3$;
2. $B^k(\lambda+\mu+\partial)^{-1}A^{ij}\lambda^p \longleftrightarrow A^{ij}\partial^p(\psi_i^1)\psi_j^2\tilde{\psi}^3$;
3. $[(\lambda+\partial)^{-1}B^i]A^{jk}\mu^p \longleftrightarrow A^{jk}\tilde{\psi}^1\partial^p(\psi_j^2)\psi_k^3$;
4. $[(\mu+\partial)^{-1}B^j]A^{ki}\lambda^p \longleftrightarrow A^{ki}\partial^p(\psi_i^1)\tilde{\psi}^2\psi_k^3$;
5. $B^k(\lambda+\mu+\partial)^{-1}A^j(\lambda+\partial)^{-1}C^i \longleftrightarrow A^j\tilde{\psi}^1\psi_j^2\tilde{\psi}^3$;
6. $B^k(\lambda+\mu+\partial)^{-1}A^i(\mu+\partial)^{-1}C^j \longleftrightarrow A^i\psi_i^1\tilde{\psi}^2\tilde{\psi}^3$;
7. $[(\lambda+\partial)^{-1}B^i]A^k[(\mu+\partial)^{-1}C^j] \longleftrightarrow A^k\tilde{\psi}^1\tilde{\psi}^2\psi_k^3$.

A threevector which is expressed using the above basis is divergence-free.

Poisson Vertex Algebrae and the algorithm

Remark. The proof of the above theorem uses Laurent series expansion of the type

$$\begin{aligned}\partial_x^{-1}(w\psi) =& w\partial_x^{-1}\psi - \partial_x^{-1}(\partial_x w \cdot \partial_x^{-1}\psi) \\ =& w\partial_x^{-1}\psi - \partial_x w \partial_x^{-2}\psi + \partial_x^2 w \partial_x^{-3}\psi + \cdots\end{aligned}$$

Theorem. The threevector coming from the algorithm (with the distribution or the differential operator formalism) vanishes if and only if the threevector in the previous basis vanishes.

Work in progress: anticommuting variables

There is a canonical isomorphism between variational multivectors and superfunctions on a supermanifold:

$$A^{j(\sigma_1 i_1)\cdots(\sigma_h i_h)} D_{\sigma_1}\psi^1_{i_1}\cdots D_{\sigma_h}\psi^h_{i_h} \quad \longrightarrow$$
$$\longrightarrow \quad A^{j(\sigma_1 i_1)\cdots(\sigma_h i_h)} p_{i_1\sigma_1}\cdots p_{i_h\sigma_h}$$

and the Schouten bracket is expressed via the formula

$$[F,H] = \left[\frac{\delta H}{\delta u^i}\frac{\delta F}{\delta p_i} - (-1)^{(F+1)(H+1)}\frac{\delta F}{\delta u^i}\frac{\delta H}{\delta p_i}\right]$$

The above formula can be generalized to nonlocal operator, while the algorithm does not work with superfunctions.

Work in progress: more anticommuting variables

J. Krasil'shchik and A. Verbovetsky: a unified framework for Hamiltonian, symplectic and recursion operators as Bäcklund tr.

$$\begin{tikzcd}
 & \mathcal{W} \arrow[dl] \arrow[dr] & \\
\mathcal{T}^*\mathcal{E} \arrow[rr, "P"] \arrow[dr] & & \mathcal{T}\mathcal{E} \arrow[dl] \\
 & \mathcal{E} &
\end{tikzcd}$$

- $\mathcal{E}$: The equation $F^i = u^i_t - f^i(u^j, u^j_\sigma)$
- $\mathcal{T}\mathcal{E}$: The tangent covering $\ell_F(q) = 0$, space of symmetries
- $\mathcal{T}^*\mathcal{E}$: The cotangent covering $\ell^*_F(p)$, space of conserved quantities
- $\mathcal{W}$: The space of nonlocal variables, e.g. $r_x = w^i_k u^k_x p_i$.

Symbolic computations - partly in progress

Distributions: working Maple software written by P. Lorenzoni (for private use).
Differential operators: Reduce software in development using the package CDE (by RV). CDE (by RV) can compute symmetries and conservation laws, local and nonlocal Hamiltonian operators, Schouten brackets of local multivectors, Fréchet derivatives (or linearization of a system of PDEs), formal adjoints, Lie derivatives of Hamiltonian operators. A book, in cooperation with JS Krasil'shchik and AM Verbovetsky: *The symbolic computation of integrability structures for partial differential equations*, is published in the series Texts and Monographs in Symbolic Computation, Springer, 2017.
Poisson Vertex Algebrae: ?

Thank you!

Contacts: raffaele.vitolo@unisalento.it