Projective-geometric aspects of WDVV equations

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- WDVV equations and their bi-Hamiltonian structure.
- Projective-geometric interpretation.
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Witten–Dijkgraaf–Verlinde–Verlinde equations

The problem: in $\mathbb{R}^N$ find a function $F = F(t^1, \ldots, t^N)$ such that

1. $\dfrac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}$ constant symmetric nondegenerate matrix

2. $c^\gamma_{\alpha\beta} = \eta^{\gamma\epsilon} \dfrac{\partial^3 F}{\partial t^\epsilon \partial t^\alpha \partial t^\beta}$ structure constants of an associative algebra

3. $F(c^{d_1} t^1, \ldots, c^{d_N} t^N) = c^{d_F} F(t^1, \ldots, t^N)$ quasihomogeneity $(d_1 = 1)$

If $e_1, \ldots, e_N$ is the basis of $\mathbb{R}^N$ then the algebra operation is

$$e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(\mathbf{t}) e_\gamma \quad \text{with unity} \quad e_1$$

WDVV equations of associativity

$$\eta^{\mu\lambda} \frac{\partial^3 F}{\partial t^\lambda \partial t^\alpha \partial t^\beta} \frac{\partial^3 F}{\partial t^\nu \partial t^\mu \partial t^\gamma} = \eta^{\mu\lambda} \frac{\partial^3 F}{\partial t^\nu \partial t^\alpha \partial t^\mu} \frac{\partial^3 F}{\partial t^\lambda \partial t^\beta \partial t^\gamma} \quad \text{(WDVV)}$$

Why study WDVV?

1. Solutions yield Gromov–Witten invariants
2. Solutions correspond to integrable hierarchies
3. Applications to Quantum Field Theory (?)

WDVV equations in detail

$d_F \neq 3$: By linear transformations preserving e_1:

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta,N+1} = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

$$F = \frac{1}{2}(t^1)^2 t^N + \frac{1}{2} t^1 \sum_{\alpha=2}^{N-1} t^\alpha t^{N-\alpha+1} + f(t^2, \ldots, t^N);$$

$d_F = 3$: By linear transformations preserving e_1:

$$\eta_{\alpha\beta} = \delta_{\alpha\beta}$$

$$F = \frac{1}{6}(t^1)^3 + \frac{1}{2} t^1 \sum_{\alpha=2}^{N} (t^\alpha)^2 + f(t^2, \ldots, t^N);$$

Main example: WDVV in the case $N = 3$

If $N = 3$ we have a single equation on $f = f(t^2, t^3) = f(x, t)$.
Two cases:

- $\eta_{\alpha\beta} = \delta_{\alpha+\beta,N+1}$, then

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$$

- $\eta_{\alpha\beta} = \delta_{\alpha\beta}$, then

$$f_{ttt} = \frac{f_{xxt}^2 - f_{xxx} f_{xtt} + f_{xtt}^2 - 1}{f_{xxt}}$$

WDVV equations as hydrodynamic systems

Construction by O. Mokhov (1995). Let us introduce coordinates

$$a = f_{xxx}, \quad b = f_{xxt}, \quad c = f_{xtt}.$$

Then the compatibility conditions in the two cases are

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = (b^2 - ac)_x \end{cases} \quad \text{and} \quad \begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = \left(\frac{-ac+b^2+c^2-1}{b}\right)_x \end{cases}$$

The system on the left is bi-Hamiltonian (Ferapontov, Galvao, Mokhov, Nutku CMP'98) by a third-order and a first-order Hamiltonian operator of Dubrovin–Novikov type. Further study by Kersten, Krasil'shchik, Verbovetsky, V. TMP'12. **What about the system on the right?**

Digression: third-order Hamiltonian operators

Homogeneous, or Dubrovin–Novikov Hamiltonian operator ($\mathbf{u} = (u^1, \ldots, u^n)$):

$$\begin{aligned} A^{ij} = & g^{ij}(\mathbf{u})\partial_x^3 + b_k^{ij}(\mathbf{u})u_x^k\partial_x^2 \\ & + [c_k^{ij}(\mathbf{u})u_{xx}^k + c_{km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\ & + d_k^{ij}(\mathbf{u})u_{xxx}^k + d_{km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n. \end{aligned}$$

They admit the **canonical form** (Potemin '89,'92; Doyle '92)

$$A^{ij} = \partial_x \left(g^{ij}\partial_x + c_k^{ij}u_x^k \right) \partial_x$$

They correspond to **quadratic line complexes**, and are projectively invariant (Ferapontov, Pavlov, V., JGP '14, IMRN '16).

Digression: Plücker's line geometry

Two points $U, V \in \mathbb{P}^n(\mathbb{C})$, $U = [u^0, \ldots, u^n]$, $V = [v^0, \ldots, v^n]$ define a line with coordinates $p_{ij} = \det \begin{vmatrix} u^i & u^j \\ v^i & v^j \end{vmatrix}$ inside a projective space: $\mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C})$ (Plücker's embedding).

A **quadratic line complex** is a variety

$$X^T Q X = f_{ij,hk} p^{ij} p^{hk} = 0$$

Take $V = U + dU$: then $p^{ij} = u^i du^j - u^j du^i$ and we have a **Monge metric**

$$g_{ab} du^a du^b = f_{ij,hk}(u^i du^j - u^j du^i)(u^h du^k - u^k du^h)$$

We set $u^0 = 1$, $du^0 = 0$.

Digression: hydrodynamic systems of cons. laws

Theorem: (Fereapontov, Pavlov, V. LMP '17) Let A be a third-order homogeneous Hamiltonian operator. Then a hydrodynamic system $u^i_t = V^i_j u^j_x = (V^i)_{,j} u^j_x$ is compatible with A if and only if

$$\begin{cases} g_{im} V^m_j = g_{jm} V^m_i \\ c_{mkj} V^m_i + c_{mik} V^m_j + c_{mji} V^m_k = 0, \\ V^k_{i,j} = g^{ks} c_{smj} V^m_i + g^{ks} c_{smi} V^m_j \end{cases} \tag{1}$$

Each system is identified with a *congruence of lines* (Agafonov, Ferapontov (1996-2001)) in $\mathbb{P}^{n+1}$ with coordinates $[y^1, \ldots, y^{n+2}]$

$$y^i = u^i y^{n+1} + V^i y^{n+2}$$

The congruence is *linear*: there are n linear relations between u^i, V^i, $u^i V^j - u^j V^i$, hence the system is *linearly degenerate*, and *non diagonalizable*.

Digression: classification of operators and systems

(Results by Ferapontov, Pavlov, V.): The Hamiltonian operator

$$A^{ij} = \partial_x \left(g^{ij} \partial_x + c^{ij}_k u^k_x \right) \partial_x$$

is **completely determined** by its leading term g^{ij}. A multiparameter family of hydrodynamic systems is determined by a single operator A. Both objects are invariant up to reciprocal transformations of projective type:

$$d\tilde{x} = (a_i u^i + a)dx + (a_i V^i + b)dt$$
$$d\tilde{t} = (b_i u^i + c)dx + (b_i V^i + d)dt$$

Classification: for $n = 3$ only two operators (and systems) exist, WDVV and a linear system. For $n = 4$ a classification is still possible; not possible for $n \geqslant 5$.

Results for N=3

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = (b^2 - ac)_x \end{cases} \implies g_{ij} = \begin{pmatrix} -2b & a & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{known}$$

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = \left(\frac{-ac+b^2+c^2-1}{b} \right)_x \end{cases} \implies$$

$$g_{ij} = \begin{pmatrix} b^2+1 & -ab+bc & -b^2 \\ -ab+bc & a^2-2ac+c^2+1 & ab-bc \\ -b^2 & ab-bc & b^2 \end{pmatrix} \quad \textbf{new!}$$

The above system is indeed bi-Hamiltonian by the above operator and a first-order non-local operator (P. Lorenzoni, V. 2018).

WDVV, $N=3$, generic $\eta_{\alpha\beta}$

$$\begin{aligned}
&f_{2xt}^2(\eta_{11}\eta_{33}-\eta_{13}^2)+f_{2xt}f_{3t}(-\eta_{11}\eta_{22}+\eta_{12}^2)+f_{2xt}f_{x2t}(-\eta_{11}\eta_{23}+\eta_{12}\eta_{13})\\
&+f_{2xt}(-3\eta_{12}\eta_{23}\eta_{33}+\eta_{13}\eta_{22}\eta_{33}+2\eta_{13}\eta_{23}^2)+f_{3t}f_{3x}(\eta_{11}\eta_{23}-\eta_{12}\eta_{13})\\
&+f_{3t}\eta_{22}(-\eta_{12}\eta_{23}+\eta_{13}\eta_{22})+f_{3x}f_{x2t}(-\eta_{11}\eta_{33}+\eta_{13}^2)\\
&+f_{3x}\eta_{33}(\eta_{12}\eta_{33}-\eta_{13}\eta_{23})+f_{x2t}^2(\eta_{11}\eta_{22}-\eta_{12}^2)\\
&+f_{x2t}(\eta_{12}\eta_{22}\eta_{33}+2\eta_{12}\eta_{23}^2-3\eta_{13}\eta_{22}\eta_{23})\\
&-\eta_{22}^2\eta_{33}^2+2\eta_{22}\eta_{23}^2\eta_{33}-\eta_{23}^4=0 \qquad (2)
\end{aligned}$$

Theorem: The hydrodynamic system corresponding to the above WDVV equation yields a linear line congruence that is determined by a homogeneous third-order Hamiltonian operator. *Proof by computer.*

WDVV in the case $N = 4$

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

WDVV equations are an overdetermined nonlinear system:

$$\begin{aligned}
&-2f_{xyz} - f_{xyy}f_{xxy} + f_{yyy}f_{xxx} = 0,\\
&-f_{xzz} - f_{xyy}f_{xxz} + f_{yyz}f_{xxx} = 0,\\
&-2f_{xyz}f_{xxz} + f_{xzz}f_{xxy} + f_{yzz}f_{xxx} = 0,\\
&f_{zzz} - (f_{xyz})^2 + f_{xzz}f_{xyy} - f_{yyz}f_{xxz} + f_{yzz}f_{xxy} = 0,\\
&f_{yyy}f_{xzz} - 2f_{yyz}f_{xyz} + f_{yzz}f_{xyy} = 0.
\end{aligned}$$

6-components WDVV system

We introduce new field variables u^k:

$$u^1 = f_{xxx}, u^2 = f_{xxy}, u^3 = f_{xxz}, u^4 = f_{xyy}, u^5 = f_{xyz}, u^6 = f_{xzz}.$$

The compatibility conditions for this system can be written as a pair of *commuting* hydrodynamic type systems in conservative form:

$$\begin{cases} u^1_y = u^2_x, \\ u^2_y = u^4_x, \\ u^3_y = u^5_x, \\ u^4_y = \left(\frac{2u^5+u^2u^4}{u^1}\right)_x \\ u^5_y = \left(\frac{u^3u^4+u^6}{u^1}\right)_x \\ u^6_y = \left(\frac{2u^3u^5-u^2u^6}{u^1}\right)_x \end{cases} \qquad \begin{cases} u^1_z = u^3_x, \\ u^2_z = u^5_x, \\ u^3_z = u^6_x, \\ u^4_z = \left(\frac{u^3u^4+u^6}{u^1}\right)_x \\ u^5_z = \left(\frac{2u^3u^5-u^2u^6}{u^1}\right)_x \\ u^6_z = \Big((u^5)^2 - u^4u^6 + \\ \qquad \frac{(u^3)^2u^4+u^3u^6-2u^2u^3u^5+(u^2)^2u^6}{u^1}\Big)_x \end{cases}$$

WDVV as a bi-Hamiltonian system

Theorem: (Pavlov, V. LMP 2015) The leading term of a third-order Hamiltonian operator for *both* the previous hydrodynamic-type systems:

$$g_{ik}(\mathbf{u}) = \begin{pmatrix} (u^4)^2 & -2u^5 & 2u^4 & -(u^1u^4+u^3) & u^2 & 1 \\ -2u^5 & -2u^3 & u^2 & 0 & u^1 & 0 \\ 2u^4 & u^2 & 2 & -u^1 & 0 & 0 \\ -(u^1u^4+u^3) & 0 & -u^1 & (u^1)^2 & 0 & 0 \\ u^2 & u^1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Both systems are bi-Hamiltonian with the above homogeneous third-order Hamiltonian operator and a (compatible) local first-order Hamiltonian operator.

Advanced computations

- In the case $N = 4$ also the choice $\eta_{\alpha\beta} = \delta_{\alpha\beta}$ yields a hydrodynamic system with a homogeneous third-order operator. It is, however, not so easy to prove that there exists a first-order operator.
- For generic N we obtain $N - 1$ hydrodynamic systems. Work is in progress in the case $N = 5$ to find a homogeneous third-order Hamiltonian operator.

Very advanced computations

The equation of associativity for F-manifolds, or *Oriented Associativity Equation* (Hertling–Manin '98):

$$\frac{\partial^2 c^i}{\partial a^j \partial a^m}\frac{\partial^2 c^m}{\partial a^k \partial a^n} = \frac{\partial^2 c^i}{\partial a^k \partial a^m}\frac{\partial^2 c^m}{\partial a^j \partial a^n} \tag{3}$$

also shows similar features: in the simplest case $N = 3$ it can be transformed into a hydrodynamic system which admits a third-order *non-local* Hamiltonian operator (Casati, Ferapontov, Pavlov, V. '18):

$$A^{ij} = \partial_x \big(g^{ij}\partial_x + c^{ij}_k u^k_x + w^i_k u^k_x \partial_x^{-1} w^j_h u^h_x \big) \partial_x,$$

with **extremely complicated** coordinate expression.

Open questions

Integrable PDEs	**Projective Geometry**
Third-order Hamiltonian operator	Quadratic Line Complex
Hydrodynamic system	Linear Line Congruence
First-order Hamiltonian operator	???

Open questions: What is the Projective Geometric meaning of a first-order operator that is compatible with the third-order operator and with the hydrodynamic system? And what is the Projective Geometric meaning of the compatibility between operators and systems?

THANK YOU!