Projective-geometric aspects
of WDVV equations

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Local and Nonlocal Geometry of PDEs
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Contents

- WDVV equations and their bi-Hamiltonian structure.
- Projective-geometric interpretation.
- Open questions.
Witten–Dijkgraaf–Verlinde–Verlinde equations

The problem: in $\mathbb{R}^N$ find a function $F = F(t^1, \ldots, t^N)$ such that

1. $\frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha \beta}$ constant symmetric nondegenerate matrix

2. $c_{\alpha \beta}^\gamma = \eta^{\gamma \epsilon} \frac{\partial^3 F}{\partial t^\epsilon \partial t^\alpha \partial t^\beta}$ structure constants of an associative algebra

3. $F(c^{d_1} t^1, \ldots, c^{d_N} t^N) = c^{d_F} F(t^1, \ldots, t^N)$ quasihomogeneity $(d_1 = 1)$

If $e_1, \ldots, e_N$ is the basis of $\mathbb{R}^N$ then the algebra operation is

$$e_\alpha \cdot e_\beta = c_{\alpha \beta}^\gamma(t)e_\gamma$$

with unity $e_1$
WDVV equations of associativity

\[ \eta^{\mu\lambda} \frac{\partial^3 F}{\partial t^\lambda \partial t^\alpha \partial t^\beta} \frac{\partial^3 F}{\partial t^\nu \partial t^\mu \partial t^\gamma} = \eta^{\mu\lambda} \frac{\partial^3 F}{\partial t^\nu \partial t^\alpha \partial t^\mu} \frac{\partial^3 F}{\partial t^\lambda \partial t^\beta \partial t^\gamma} \quad \text{(WDVV)} \]

Why study WDVV?

1. Solutions yield Gromov–Witten invariants
2. Solutions correspond to integrable hierarchies
3. Applications to Quantum Field Theory (?)
WDVV equations in detail

d_F \neq 3: \text{ By linear transformations preserving } e_1:

\eta_{\alpha\beta} = \delta_{\alpha+\beta,N+1} = 
\begin{pmatrix}
0 & 1 \\
\vdots \\
1 & 0
\end{pmatrix}

F = \frac{1}{2}(t^1)^2 t^N + \frac{1}{2} t^1 \sum_{\alpha=2}^{N-1} t^\alpha t^{N-\alpha+1} + f(t^2, \ldots, t^N);

d_F = 3: \text{ By linear transformations preserving } e_1:

\eta_{\alpha\beta} = \delta_{\alpha\beta}

F = \frac{1}{6}(t^1)^3 + \frac{1}{2} t^1 \sum_{\alpha=2}^{N} (t^\alpha)^2 + f(t^2, \ldots, t^N);
Main example: WDVV in the case $N = 3$

If $N = 3$ we have a single equation on $f = f(t^2, t^3) = f(x, t)$.

Two cases:

- $\eta_{\alpha\beta} = \delta_{\alpha+\beta,N+1}$, then

$$f_{ttt} = f_{xxt} - f_{xxx} f_{xtt}$$

- $\eta_{\alpha\beta} = \delta_{\alpha\beta}$, then

$$f_{ttt} = \frac{f_{xxt}^2 - f_{xxx} f_{xtt} + f_{xtt}^2 - 1}{f_{xxt}}$$
WDVV equations as hydrodynamic systems


\[ a = f_{xxx}, \quad b = f_{xxt}, \quad c = f_{xtt}. \]

Then the compatibility conditions in the two cases are

\[
\begin{aligned}
& a_t = b_x, \\
& b_t = c_x, \\
& c_t = (b^2 - ac)_x
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
& a_t = b_x, \\
& b_t = c_x, \\
& c_t = \left( \frac{-ac + b^2 + c^2 - 1}{b} \right)_x
\end{aligned}
\]

The system on the left is bi-Hamiltonian (Ferapontov, Galvao, Mokhov, Nutku CMP’98) by a third-order and a first-order Hamiltonian operator of Dubrovin–Novikov type. Further study by Kersten, Krasil’shchik, Verbovetsky, V. TMP’12. **What about the system on the right?**
Digression: third-order Hamiltonian operators

Homogeneous, or Dubrovin–Novikov Hamiltonian operator $(u = (u^1, \ldots, u^n))$:

$$A^{ij} = g^{ij}(u) \partial_x^3 + b^{ij}_k(u) u^k_x \partial_x^2$$

$$+ [c^{ij}_k(u) u^{k}_{xx} + c^{ij}_{km}(u) u^k_x u^m_x] \partial_x$$

$$+ d^{ij}_k(u) u^{k}_{xxx} + d^{ij}_{km}(u) u^k_x u^m_{xx} + d^{ij}_{kmn}(u) u^k_x u^m_x u^n_x.$$ 

They admit the **canonical form** (Potemin ’89,’92; Doyle ’92)

$$A^{ij} = \partial_x \left(g^{ij} \partial_x + c^{ij}_k u^k_x\right) \partial_x$$

They correspond to **quadratic line complexes**, and are projectively invariant (Ferapontov, Pavlov, V., JGP ’14, IMRN ’16).
Digression: Plücker’s line geometry

Two points $U, V \in \mathbb{P}^n(\mathbb{C})$, $U = [u^0, \ldots, u^n]$, $V = [v^0, \ldots, v^n]$ define a line with coordinates $p_{ij} = \det \begin{vmatrix} u^i & u^j \\ v^i & v^j \end{vmatrix}$ inside a projective space: $\mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{P}^N(\mathbb{C})$ (Plücker’s embedding).

A quadratic line complex is a variety

$$X^TQX = f_{ij,hk}p^{ij}p^{hk} = 0$$

Take $V = U + dU$: then $p^{ij} = u^i du^j - u^j du^i$ and we have a Monge metric

$$g_{ab}du^a du^b = f_{ij,hk}(u^i du^j - u^j du^i)(u^h du^k - u^k du^h)$$

We set $u^0 = 1$, $du^0 = 0$. 
Digression: hydrodynamic systems of cons. laws

**Theorem:** (Fereapontov, Pavlov, V. LMP ’17) Let $A$ be a third-order homogeneous Hamiltonian operator. Then a hydrodynamic system $u^i_t = V^i_j u^j_x = (V^i)_j u^j_x$ is compatible with $A$ if and only if

$$
\begin{align*}
&g_{im} V^m_j = g_{jm} V^m_i \\
&c_{mkj} V^m_i + c_{mk} V^m_j + c_{mji} V^m_k = 0, \\
&V^k_{i,j} = g^{ks} c_{ksm} V^m_i + g^{ks} c_{sni} V^m_j
\end{align*}
$$

Each system is identified with a congruence of lines (Agafonov, Ferapontov (1996-2001)) in $\mathbb{P}^{n+1}$ with coordinates $[y^1, \ldots, y^{n+2}]$

$$
y^i = u^i y^{n+1} + V^i y^{n+2}
$$

The congruence is *linear*: there are $n$ linear relations between $u^i, V^i, u^i V^j - u^j V^i$, hence the system is *linearly degenerate*, and *non diagonalizable*. 
(Results by Ferapontov, Pavlov, V.): The Hamiltonian operator

\[ A^{i,j} = \partial_x \left( g^{i,j} \partial_x + c^{ij}_k u^k_x \right) \partial_x \]

is completely determined by its leading term \( g^{i,j} \). A multiparameter family of hydrodynamic systems is determined by a single operator \( A \). Both objects are invariant up to reciprocal transformations of projective type:

\[
\begin{align*}
    d\tilde{x} &= (a_i u^i + a)dx + (a_i V^i + b)dt \\
    d\tilde{t} &= (b_i u^i + c)dx + (b_i V^i + d)dt
\end{align*}
\]

**Classification:** for \( n = 3 \) only two operators (and systems) exist, WDVV and a linear system. For \( n = 4 \) a classification is still possible; not possible for \( n \geq 5 \).
Results for \( N=3 \)

\[
\begin{align*}
\begin{cases}
a_t = b_x, \\
b_t = c_x, \\
c_t = (b^2 - ac)_x
\end{cases} & \implies g_{ij} = \begin{pmatrix} -2b & a & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{known}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
a_t = b_x, \\
b_t = c_x, \\
c_t = \left(\frac{-ac + b^2 + c^2 - 1}{b}\right)_x
\end{cases} & \implies \\
g_{ij} = \begin{pmatrix} b^2 + 1 & -ab + bc & -b^2 \\ -ab + bc & a^2 - 2ac + c^2 + 1 & ab - bc \\ -b^2 & ab - bc & b^2 \end{pmatrix} \quad \text{new!}
\end{align*}
\]

The above system is indeed bi-Hamiltonian by the above operator and a first-order non-local operator (P. Lorenzoni, V. 2018).
WDVV, $N = 3$, generic $\eta_{\alpha\beta}$

$$f_{2xt}^2(\eta_{11}\eta_{33} - \eta_{13}^2) + f_{2xt}f_{3t}(-\eta_{11}\eta_{22} + \eta_{12}^2) + f_{2xt}f_{x2t}(-\eta_{11}\eta_{23} + \eta_{12}\eta_{13})$$
$$+ f_{2xt}(-3\eta_{12}\eta_{23}\eta_{33} + \eta_{13}\eta_{22}\eta_{33} + 2\eta_{13}\eta_{23}^2) + f_{3t}f_{3x}(\eta_{11}\eta_{23} - \eta_{12}\eta_{13})$$
$$+ f_{3t}\eta_{22}(-\eta_{12}\eta_{23} + \eta_{13}\eta_{22}) + f_{3x}f_{x2t}(-\eta_{11}\eta_{33} + \eta_{13}^2)$$
$$+ f_{3x}\eta_{33}(\eta_{12}\eta_{33} - \eta_{13}\eta_{23}) + f_{x2t}^2(\eta_{11}\eta_{22} - \eta_{12}^2)$$
$$+ f_{x2t}(\eta_{12}\eta_{22}\eta_{33} + 2\eta_{12}\eta_{23}^2 - 3\eta_{13}\eta_{22}\eta_{23})$$
$$- \eta_{22}\eta_{33}^2 + 2\eta_{22}\eta_{23}\eta_{33} - \eta_{23}^4 = 0 \quad (2)$$

**Theorem:** The hydrodynamic system corresponding to the above WDVV equation yields a linear line congruence that is determined by a homogeneous third-order Hamiltonian operator. *Proof by computer.*
WDVV in the case $N = 4$

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

WDVV equations are an overdetermined nonlinear system:

\begin{align*}
-2f_{xyz} - f_{xyy}f_{xxy} + f_{yyy}f_{xxx} &= 0, \\
-f_{xzz} - f_{xyy}f_{xxz} + f_{yyz}f_{xxx} &= 0, \\
-2f_{xyz}f_{xxz} + f_{xzz}f_{xxy} + f_{yzz}f_{xxx} &= 0, \\
f_{zzz} - (f_{xyz})^2 + f_{xzz}f_{xyy} - f_{yyz}f_{xxz} + f_{yzz}f_{xyy} &= 0, \\
f_{yyy}f_{xzz} - 2f_{yyz}f_{xyz} + f_{yzz}f_{xyy} &= 0.
\end{align*}
6-components WDVV system

We introduce new field variables $u^k$:

$$u^1 = f_{xxx}, u^2 = f_{xyy}, u^3 = f_{xz}, u^4 = f_{xy}, u^5 = f_{yz}, u^6 = f_{zz}.$$ 

The compatibility conditions for this system can be written as a pair of *commuting* hydrodynamic type systems in conservative form:

\[
\begin{align*}
\begin{cases}
  u^1_y = u^2_x, \\
  u^2_y = u^4_x, \\
  u^3_y = u^5_x, \\
  u^4_y = \left(\frac{2u^5 + u^2 u^4}{u^1}\right)_x \\
  u^5_y = \left(\frac{u^3 u^4 + u^6}{u^1}\right)_x \\
  u^6_y = \left(\frac{2u^3 u^5 - u^2 u^6}{u^1}\right)_x
\end{cases} \\
\begin{cases}
  u^1_z = u^3_x, \\
  u^2_z = u^5_x, \\
  u^3_z = u^6_x, \\
  u^4_z = \left(\frac{u^3 u^4 + u^6}{u^1}\right)_x \\
  u^5_z = \left(\frac{2u^3 u^5 - u^2 u^6}{u^1}\right)_x \\
  u^6_z = \left(\frac{(u^5)^2 - u^4 u^6 + (u^3)^2 u^4 + u^3 u^6 - 2u^2 u^3 u^5 + (u^2)^2 u^6}{u^1}\right)_x
\end{cases}
\end{align*}
\]
**Theorem:** (Pavlov, V. LMP 2015) The leading term of a third-order Hamiltonian operator for both the previous hydrodynamic-type systems:

\[
g_{ik}(u) = \begin{pmatrix}
(u^4)^2 & -2u^5 & 2u^4 & -(u^1u^4 + u^3) & u^2 & 1 \\
-2u^5 & -2u^3 & u^2 & 0 & u^1 & 0 \\
2u^4 & u^2 & 2 & -u^1 & 0 & 0 \\
-(u^1u^4 + u^3) & 0 & -u^1 & (u^1)^2 & 0 & 0 \\
u^2 & u^1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Both systems are bi-Hamiltonian with the above homogeneous third-order Hamiltonian operator and a (compatible) local first-order Hamiltonian operator.
In the case $N = 4$ also the choice $\eta_{\alpha\beta} = \delta_{\alpha\beta}$ yields a hydrodynamic system with a homogeneous third-order operator. It is, however, not so easy to prove that there exists a first-order operator.

For generic $N$ we obtain $N - 1$ hydrodynamic systems. Work is in progress in the case $N = 5$ to find a homogeneous third-order Hamiltonian operator.
The equation of associativity for $F$-manifolds, or Oriented Associativity Equation (Hertling–Manin ’98):

$$\frac{\partial^2 c^i}{\partial a^j \partial a^m} \frac{\partial^2 c^m}{\partial a^k \partial a^n} = \frac{\partial^2 c^i}{\partial a^k \partial a^m} \frac{\partial^2 c^m}{\partial a^j \partial a^n}$$

(3)

also shows similar features: in the simplest case $N = 3$ it can be transformed into a hydrodynamic system which admits a third-order non-local Hamiltonian operator (Casati, Ferapontov, Pavlov, V. ’18):

$$A^{ij} = \partial_x \left( g^{ij} \partial_x + c_k^{ij} u^k_x + w^i_k u^k_x \partial_x^{-1} w^j_h u^h_x \right) \partial_x,$$

with extremely complicated coordinate expression.
### Open questions

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**Open questions:** What is the Projective Geometric meaning of a first-order operator that is compatible with the third-order operator and with the hydrodynamic system? And what is the Projective Geometric meaning of the compatibility between operators and systems?
THANK YOU!