Projective geometry of homogeneous second order Hamiltonian operators

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Hamiltonian PDEs

An evolutionary system of PDEs

$$F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \dots) = 0$$

admits a Hamiltonian formulation if there exist A, $\mathcal{H} = \int h\, dx$ such that

$$u^i_t = A^{ij}\left(\frac{\delta \mathcal{H}}{\delta u^j}\right)$$

where $A = (A^{ij})$ is a Hamiltonian operator, i.e. a matrix of differential operators $A^{ij} = A^{ij\sigma}\partial_\sigma$, where $\partial_\sigma = \partial_x \circ \cdots \circ \partial_x$ (total x-derivatives σ times), such that

$$\{F, G\}_A = \int \frac{\delta F}{\delta u^i} A^{ij\sigma} \partial_\sigma \frac{\delta G}{\delta u^j}\, dx$$

is a Poisson bracket (skew-symmetric and Jacobi).

First-order homogeneous operators

First-order homogeneous operators were introduced in 1983 by Dubrovin and Novikov for the Hamiltonian formalism of quasilinear first-order equations

$$u^i_t = v^i_j(\mathbf{u})u^j_x = A_1^{ij}\frac{\delta\mathcal{H}_1}{\delta u^j} \quad \mathcal{H}_1 = \int h(\mathbf{u})dx$$

$\mathbf{u} = (u^i(t,x))$, $i,j = 1,\ldots,n$ (n-components). The operators are of the form

$$\boxed{A_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b^{ij}_k(\mathbf{u})u^k_x}$$

Homogeneity: $\deg\partial_x = 1$.

Canonical form: $A_1^{ij} = \eta^{ij}\partial_x$.

Higher-order homogeneous operators

Higher order homogeneous operators were introduced in 1984 by Dubrovin and Novikov. In particular, the cases under consideration here are the second-order and third-order homogeneous operators:

$$
\begin{aligned}
A_2^{ij} =& g_2^{ij}(\mathbf{u})\partial_x^2 + b_{2\,k}^{ij}(\mathbf{u})u_x^k\partial_x \\
&+ c_{2\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{2\,km}^{ij}(\mathbf{u})u_x^k u_x^m,
\end{aligned}
$$

$$
\begin{aligned}
A_3^{ij} =& g_3^{ij}(\mathbf{u})\partial_x^3 + b_{3\,k}^{ij}(\mathbf{u})u_x^k\partial_x^2 \\
&+ [c_{3\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{3\,km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\
&+ d_{3\,k}^{ij}(\mathbf{u})u_{xxx}^k + d_{3\,km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{3\,kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n.
\end{aligned}
$$

Homogeneous operators and integrable systems

Two main mechanisms generate many bi-Hamiltonian systems from higher-order homogeneous operators:

- Compatible triples (*regular mechanism*):
  - with third-order operators: KdV, Camassa–Holm, dispersive water waves (Antonowicz–Fordy 1989), coupled Harry–Dym, etc..
  - with second-order operators: AKNS, 2-component Camassa-Holm, Kaup–Broer (Kuperschmidt 1984), etc..
- Compatible pairs (*singular mechanism*):
  - with third-order operators: Monge–Ampère, WDVV, Oriented Associativity (or F-manifolds) equation (as quasilinear systems of the first order);
  - with second-order operators: new systems here!

Bi-Hamiltonian structures from compatible triples

A classification of bi-Hamiltonian hierarchies which are defined by a triple of mutually compatible Hamiltonian operators was provided by Lorenzoni, Savoldi, V. (JPA 2017).
Examples: scalar case. We have one third-order operator A_3, two first order operators P_1, Q_1:

$$[A_3, P_1] = [A_3, Q_1] = [P_1, Q_1] = 0$$
$$P_1 = \partial_x, \qquad Q_1 = 2u\partial_x + u_x, \quad A_3 = \partial_x^3.$$

KdV hierarchy (Magri (1978)):

$$\Pi_\lambda = Q_1 + \epsilon^2 A_3 - \lambda P_1 = 2u\partial_x + u_x - \lambda\partial_x + \epsilon^2\partial_x^3$$

Camassa–Holm hierarchy:

$$\tilde{\Pi}_\lambda = Q_1 - \lambda(P_1 + \epsilon^2 A_3) = 2u\partial_x + u_x - \lambda(\partial_x + \epsilon^2\partial_x^3).$$

Example: 2-component case. We have one second-order operator A_2 and two first-order operators P_1, Q_1, all of them mutually compatible:

$$P_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 2u\partial_x + u_x & v\partial_x \\ \partial_x v & -2\partial_x \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & -\partial_x^2 \\ \partial_x^2 & 0 \end{pmatrix}$$

- $\Pi_\lambda = Q_1 + \epsilon^2 A_2 - \lambda P_1$ AKNS (or two-boson) hierarchy;
- $\tilde{\Pi}_\lambda = Q_1 - \lambda(P_1 + \epsilon^2 A_2)$ two-component Camassa-Holm hierarchy.

Aims of the talk

- Give new projective-geometric properties of second-order homogeneous Hamiltonian operators;
- classify second-order homogeneous Hamiltonian operators with a small number of components: $n \leqslant 8$;
- describe the quasilinear systems of first-order PDEs which are Hamiltonian with respect to the above operators;
- recall and compare analogous results on third-order homogeneous Hamiltonian operators.

Canonical forms of homogeneous Hamiltonian operators

In the non-degenerate case ($\det(g^{ij}) \neq 0$) the coefficients $c^{ij}_{2k}(\mathbf{u})$ and $d^{ij}_{3k}(\mathbf{u})$ transform like linear connections; it can be proved that they are symmetric and flat. In *flat coordinates* we have the canonical forms (Potemin '86, '97; Potemin–Balandin, '01; Doyle '95)

$$A_2^{ij} = \partial_x \circ g_2^{ij} \circ \partial_x,$$

$$A_3^{ij} = \partial_x \circ (g_3^{ij}\partial_x + c^{ij}_{3\,k} u^k_x) \circ \partial_x,$$

The flat coordinates are Casimirs of the operators A_2 and A_3.

Second-order operators

Consider the second-order case:

$$A_2^{ij} = \partial_x \circ g^{ij} \circ \partial_x, \tag{1}$$

where $g^{ij} = g^{ij}(\mathbf{u})$, i, $j = 1, \ldots, n$. The skew-symmetry and Jacobi property of the Poisson brackets defined by the homogeneous second-order Hamiltonian operator A are equivalent to:

$$g_{ij} = T_{ijk}u^k + g_{0ij}, \tag{2}$$

where T_{ijk} and g_{0ij} are constant and skew-symmetric (Potemin '86, '97; Doyle '95).

IDEA: try to see if known projective-geometric results from third-order operators (collaboration with E. Ferapontov, M. Pavlov, JGP 2014, IMRN 2016, LMP 2018) can be somehow reproduced.

New results: projective invariance

Theorem 1. Let $A_2^{ij} = \partial_x \circ g^{ij} \circ \partial_x$ be a homogeneous second-order Hamiltonian operator (in canonical form). Then, reciprocal transformations of projective type

$$d\tilde{x} = \Delta dx, \quad \tilde{u}^i = S^i(u^j) = (S^i_j u^j + S^i_0)/\Delta$$

with $\Delta = S^0_i u^i + S^0_0$ preserve the canonical form of second-order homogeneous operators (Vergallo, V., arXiv 2022). The leading coefficient matrix g^{ij} is transformed (in *lower* indices) as

$$g_{ij} \to \frac{\tilde{g}_{2\,ij}}{\Delta^3}.$$

The canonical form of the operators is preserved (as in the third-order case!).

New results: projective interpretation

Let us set

$$T_{n+1\,jk} = -T_{j\,n+1\,k} = T_{jk\,n+1} = g^0_{jk}.$$

We will use Greek indices with the range 1,..., $n+1$. Then, a projective reciprocal transformation $(a^\alpha_\beta) \in SL(n+1, \mathbb{C})$ induces a transformation

$$T_{\lambda\mu\nu} = \frac{1}{\Delta^3}\tilde{T}_{\alpha\beta\gamma} a^\alpha_\lambda a^\beta_\mu a^\gamma_\nu.$$

Let us identify $\omega \in \wedge^3(\mathbb{C}^{n+1})^*$ with maps:

$$i(\omega)\colon \mathbb{C}^{n+1} \to \wedge^2(\mathbb{C}^{n+1})^*, v \mapsto \frac{1}{3} i_v(\omega).$$

In coordinates: $\omega_{\lambda\mu\nu} dv^\lambda \wedge dv^\mu \wedge dv^\nu \mapsto \omega_{\lambda\mu\nu} v^\nu dv^\lambda \wedge dv^\mu$.

New results: projective interpretation

Theorem 2. There is a bijective correspondence between

- homogeneous second-order Hamiltonian operators in canonical form, in dimension n, and
- 3-forms in $\mathbb{C}^{n+1}$.

$$(T_{ijk}u^k + g^0_{ij})du^i \wedge du^j \mapsto \omega_{\lambda\mu\nu} dv^\lambda \wedge dv^\mu \wedge dv^\nu$$

The bijective correspondence is preserved by projective reciprocal transformations up to a conformal factor.

Digression: Plücker's line geometry

Two points $U, V \in \mathbb{P}(\mathbb{C}^{n+1})$,

$$U = [u^1, \dots, u^{n+1}], \qquad V = [v^1, \dots, v^{n+1}]$$

define a line with coordinates $p^{\lambda\mu} = \det \begin{vmatrix} u^\lambda & u^\mu \\ v^\lambda & v^\mu \end{vmatrix}$ inside a new projective space: $\mathbb{P}(\mathbb{C}^{n+1}) \hookrightarrow \mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$ (Plücker's embedding).

Any 3-form $\omega \in \wedge^3 \mathbb{C}^{n+1*}$ defines the following system of linear equations in Plücker's space:

$$i_L(\omega) = 0, \qquad L \in \wedge^2 \mathbb{C}^{n+1};$$

in coordinates, $L = p^{\lambda\mu} \partial_\lambda \wedge \partial_\mu$ and the system is: $\omega_{\lambda\mu\nu} p^{\mu\nu} = 0$.

Digression: Plücker's line geometry

Intersecting the above system with the Grassmannian

$$\mathbb{G}(2, \mathbb{C}^{n+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$$

we obtain, in the generic case, a linear line congruence, an algebraic variety of dimension $n-1$:

$$X_\omega = \mathbb{G}(2, \mathbb{C}^{n+1}) \cap \{i_L \omega = 0\}.$$

In De Poi, Faenzi, Mezzetti, Ranestad, Ann. I. Fourier 2017 there is a detailed study of the geometric properties of such varieties. It is remarkable that they are Fano varieties (of index 3).

Projective classification

3-forms are classified under the action of $SL(n+1,\mathbb{C})$ (Gurevich, 1964). We stress that n *must be even.*

- $\mathbf{n = 2}$: we have only one operator, defined by the 3-form $\omega = dv^1 \wedge dv^2 \wedge dv^3$:

$$g_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A = g^{ij}\partial_x^2$$

- $\mathbf{n = 4}$: only one orbit with representative

$$\omega = dv^5 \wedge (dv^1 \wedge dv^2 + dv^3 \wedge dv^4),$$

defines a non-degenerate operator, namely

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \partial_x^2. \tag{3}$$

Projective classification

- $\mathbf{n = 6}$: the open orbit is generated by

$$\begin{aligned}\omega = {} & dv^1 \wedge dv^2 \wedge dv^3 + dv^4 \wedge dv^5 \wedge dv^6 \\ & + dv^7 \wedge (dv^1 \wedge dv^4 + dv^2 \wedge dv^5 + dv^3 \wedge dv^6). \end{aligned} \tag{4}$$

Then, using the affine projection $v^7 = 1, dv^7 = 0$:

$$i(\omega) = \frac{1}{3}(v^3 dv^1 \wedge dv^2 - v^2 dv^1 \wedge dv^3 + v^3 dv^1 \wedge dv^2 + \tag{5}$$

$$v^4 dv^5 \wedge dv^6 - v^5 dv^4 \wedge dv^6 + v^6 dv^4 \wedge dv^5 + \tag{6}$$

$$dv^1 \wedge dv^4 + dv^2 \wedge dv^5 + dv^3 \wedge dv^6) \tag{7}$$

Projective classification

- $\mathbf{n = 6}$ (continued): the associated 2-form is (up to a factor)

$$g^1_{ij} = \begin{pmatrix} 0 & v^3 & -v^2 & 1 & 0 & 0 \\ -v^3 & 0 & v^1 & 0 & 1 & 0 \\ v^2 & -v^1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & v^6 & -v^5 \\ 0 & -1 & 0 & -v^6 & 0 & v^4 \\ 0 & 0 & -1 & v^5 & -v^4 & 0 \end{pmatrix} \tag{8}$$

and $\det(g^1_{ij}) = (v^1v^4 + v^2v^5 + v^3v^6 - 1)^2$.
There are 4 more closed orbits that lead to non-degenerate Hamiltonian operators; the representatives do not depend on parameters.

Projective classification

$\mathbf{n = 8}$: we use the classification of trivectors in dimension 9 (Vinberg–Elashvili 1988). Every trivector can be represented as a sum $p + e$, where p is *semisimple* and e is *nilpotent*, with $p \wedge e = 0$ (Jordan decomposition). Semisimple trivectors are generated by linear combinations

$$p = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \lambda_4 p_4, \quad \text{with}$$

$$\begin{aligned}
p_1 &= dv^1 \wedge dv^2 \wedge dv^3 + dv^4 \wedge dv^5 \wedge dv^6 + dv^7 \wedge dv^8 \wedge dv^9 \\
p_2 &= dv^1 \wedge dv^4 \wedge dv^7 + dv^2 \wedge dv^5 \wedge dv^8 + dv^3 \wedge dv^6 \wedge dv^9 \\
p_3 &= dv^1 \wedge dv^5 \wedge dv^9 + dv^2 \wedge dv^6 \wedge dv^7 + dv^3 \wedge dv^4 \wedge dv^8 \\
p_4 &= dv^1 \wedge dv^6 \wedge dv^8 + dv^2 \wedge dv^4 \wedge dv^9 + dv^3 \wedge dv^5 \wedge dv^7
\end{aligned}$$

and λ_i satisfy a system of algebraic inequalities.

Projective classification

$\mathbf{n = 8}$ (continued): there are 132 classes of trivectors that yield non-degenerate leading coefficients of second-order homogeneous Hamiltonian operators.

$\mathbf{n \geqslant 10}$ **wild!** Too many parameters.

Projective invariance of compatible triples

Consider a reciprocal transformations of projective type:

$$d\tilde{x} = \Delta dx, \quad \tilde{u}^i = S^i(u^j) = (S^i_j u^j + S^i_0)/\Delta$$

where $\Delta = S^0_j u^j + S^0_0$. Then, in triples/pairs of homogeneous Hamiltonian operators P_1, Q_1, $A_{2/3}$,

- A_2 and A_3 transform into new second-order and third-order homogenous Hamiltonian operators in canonical

$$A_2^{ij} = \partial_x g_2^{ij} \partial_x, \quad A_3^{ij} = \partial_x \circ (g_3^{ij} \partial_x + c_{3\,k}^{ij} u^k_x) \circ \partial_x;$$

- P_1 (or Q_1) transform into new non-local first order homogeneous Hamiltonian operators (Ferapontov 1991).

Problem: projective classification and geometric significance of triples! Initiated in Lorenzoni, Savoldi, V. JPA 2017.

Hamiltonian systems

Problem: find systems that admit a Hamiltonian formulation by a second-order homogeneous Hamiltonian operator.

Inspired by a well-known result: **Theorem** (Tsarev, 1985) A quasilinear first-order system

$$u^i_t = V^i_j(u^k)u^j_x$$

admits a first-order homogeneous Hamiltonian operator P_1 if and only if

$$g_1^{ik}V^j_k = g_1^{jk}V^i_k, \qquad \nabla^i V^j_k = \nabla^j V^i_k$$

we use a technique by Kersten, Krasil'shchik and Verbovetsky (JGP 2004) to generalize it to higher-order homogeneous operators.

Hamiltonian systems for higher-order operators

Theorem A quasilinear first-order system of conservation laws admits

- (Vergallo, V. DGA 2021, `arXiv` 2022) a second-order homogeneous Hamiltonian operator A_2 if and only if

$$
\begin{aligned}
&g_{2\,qj}V^j_{,p} + g_{2\,pj}V^j_{,q} = 0,\\
&g_{2\,qk}V^k_{,pl} + g_{2\,pq,k}V^k_{,l} + g_{2\,qk,l}V^k_{,p} = 0.
\end{aligned}
$$

- (Ferapontov, Pavlov, V. LMP 2018) a second-order homogeneous Hamiltonian operator A_3 if and only if

$$
\begin{aligned}
&g_{3\,im}V^m_{,j} = g_{3\,jm}V^m_{,i}\\
&c_{3\,mkj}V^m_{,i} + c_{3\,mik}V^m_{,j} + c_{3\,mji}V^m_{,k} = 0,\\
&V^k_{,ij} = g_3^{ks}c_{3\,smj}V^m_{,i} + g_3^{ks}c_{3\,smi}V^m_{,j}
\end{aligned}
$$

Hamiltonian systems of conservation laws, singular mechanism

Theorem (Vergallo, V. arXiv 2022) All systems of conservation laws that admit a second-order homogeneous Hamiltonian operator A_2 in canonical form

$$u^i_t = (V^i(u^k))_x = A_2^{ij}\frac{\delta H}{\delta u^j}$$

have the fluxes:

$$V^i(u^k) = g_2^{ij}(W_{jl}u^l + B_j), \qquad W_{jl} = -W_{lj},$$

where W_{jl}, B_j are arbitrary constants. It turns out that the fluxes are rational functions. The eigenvalues of the Jacobian $(V^i_{,j})$ are double. The Hamiltonian is nonlocal: if $b^i_x = u^i$, then

$$H = -\int\left(\frac{1}{2}W_{sl}b^l_x + B_s\right)b^s\,dx.$$

The Haantjes tensor of $V^i_{,j}$ vanishes identically: $H^i_{jk} = 0$.

Further properties of the Hamiltonian systems

- The systems can be identified with a linear line congruence in $\mathbb{P}(\mathbb{C}^{n+2})$: for this reason (Agafonov and Ferapontov, $\sim$1990), the systems are linearly degenerate.
- The systems are: linearizable when $n = 4$, just one non linearizable when $n = 6$, a non-linearizable 4-parameter family (with subfamilies) when $n = 8$.
- Experiments show that random systems in dimension $n = 6$, $n = 8$ are diagonalizable, hence semi-Hamiltonian (Sevennec 1993).

Example

The operator $A_2 = \partial_x g^{ij} \partial_x$ defined by

$$g_{ij} = \begin{pmatrix} 0 & u^3 & -u^2 & 0 \\ -u^3 & 0 & u^1 & 0 \\ u^2 & -u^1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{9}$$

defines the following system:

$$\begin{cases} u^1_t = \left(\dfrac{c_4 \left(u^1\right)^2 + \left(c_1 u^2 + c_2 u^3 + c_8\right) u^1 + c_{10} u^3 - c_1 u^1 - c_2}{u^3} \right)_x \\ u^2_t = \left(\dfrac{c_1 \left(u^2\right)^2 + \left(c_3 u^3 + c_4 u^1 + c_8\right) u^2 + c_9 u^3 + c_4 u^4 + c_6}{u^3} \right)_x \\ u^3 = \left(c_1 u^2 + c_3 u^3 + c_4 u^1 + c_7\right)_x \\ u^4_t = \left(\dfrac{\left(c_1 u^2 + c_3 u^3 + c_4 u^1\right) u^4 + c_2 u^2 + c_5 u^3 + c_6 u^1}{u^3} \right)_x \end{cases} \tag{10}$$

where c_i are parameters, $i = 1, \ldots, 10$.

Comparison with third-order homogeneous Hamiltonian operators

Third-order homogeneous Hamiltonian operators (results by Ferapontov, Pavlov, V.):

- are canonical form-invariant with respect to reciprocal projective transformations;
- are classified up to a number of components $n \leqslant 4$;
- correspond to quadratic line complexes and split as the square of a linear line complex;
- define Hamiltonian systems of first-order conservation laws that are linearly degenerate and non-diagonalizable.

Perspectives

Integrable PDEs	**Projective Geometry**
Third-order Hamiltonian op.	quadratic line complex
Second-order Hamiltonian op.	linear line congruence
Quasilinear system	linear line congruence
First-order compatible Ham. op.	???

Triples P_1, Q_1, $A_{2/3}$ and pairs P_1, $A_{2/3}$ of compatible operators are invariant under projective reciprocal transformations (provided we allow for nonlocal Ferapontov operators in the orbit). The projective-geometric invariance of the corresponding hierarchies has *implications that are yet to be understood.*

Thank you!

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