Bi-Hamiltonian structures of WDVV-type

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Plan of the talk

- Bi-Hamiltonian systems of PDEs: KdV-type and WDVV-type
- Compatible pairs of operators: structure and geometry, from a joint work Lorenzoni – Opanasenko – V. (to appear soon in arXiv).
- A study of bi-Hamiltonian PDEs of WDVV-type, from a joint work Opanasenko – V. PRSA 2024.

Bi-Hamiltonian systems

We consider evolutionary equations with unknowns $u^i = u^i(t,x)$, $i = 1, \ldots, n$.
A wide class of known bi-Hamiltonian systems have their Hamiltonian operators A, B in the form of linear combination of ∂_x-homogeneous Hamiltonian operators with different homogeneity degrees:

$$P = P_1 + \epsilon S_2 + \epsilon^2 S_3 + \ldots$$
$$Q = Q_1 + \epsilon R_2 + \epsilon^2 R_3 + \ldots$$

Frequent combinations:

$$P = P_1, \qquad Q = Q_1 + \epsilon^3 R_3,$$

WDVV-type systems:

$$P = P_1, \qquad Q = R_3.$$

Bi-Hamiltonian systems from combinations of Hamiltonian operators

Two mechanisms for many well-known integrable systems:

- Compatible triples (*regular mechanism*):
  - with third-order operators: KdV, Camassa–Holm, dispersive water waves (Antonowicz–Fordy 1989), coupled Harry–Dym, etc..
  - with second-order operators: AKNS, 2-component Camassa-Holm, Kaup–Broer (Kuperschmidt 1984), etc..
- Compatible pairs (*singular mechanism*):
  - with third-order operators: Monge–Ampère, WDVV, Oriented Associativity (or F-manifolds) equation (as quasilinear systems of the first order);
  - with second-order operators: new systems, no well-known example.

Bi-Hamiltonian systems of KdV-type

(Savoldi, Lorenzoni, V. 2018; Lorenzoni, V. 2024) The KdV equation:

$$u_t = uu_x + \epsilon^2 u_{xxx}$$

We have three compatible Hamiltonian operators:

$$P_1 = \partial_x, \qquad Q_1 = \frac{1}{3}u_x + \frac{2}{3}u\partial_x, \qquad R_3 = \partial_{xxx}$$

The bi-Hamiltonian formalism:

$$A_1 = P_1, \qquad A_2 = Q_1 + \epsilon^2 R_3$$

with Hamiltonians: $H_1 = u^3/6 + u_x^2/2$, $H_2 = u^2/2$.

NOTE: the re-combination $A_1 = Q_1$, $A_2 = P_1 + R_3$ yields the Camassa-Holm hierarchy.

Hamiltonian PDEs

An evolutionary system of PDEs

$$F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \dots) = 0$$

admits a Hamiltonian formulation if there exist A, $\mathcal{H} = \int h\, dx$ such that

$$u^i_t = A^{ij}\left(\frac{\delta \mathcal{H}}{\delta u^j}\right), \quad \text{with} \quad \frac{\delta \mathcal{H}}{\delta u^j} = (-1)^\sigma \partial_\sigma \frac{\partial h}{\partial u^j_\sigma}$$

where $A = (A^{ij})$ is a Hamiltonian operator (Poisson tensor), i.e. a matrix of differential operators $A^{ij} = A^{ij\sigma}\partial_\sigma$, where $\partial_\sigma = \partial_x \circ \cdots \circ \partial_x$ (total x-derivatives σ times), with further properties.

Hamiltonian operators

A is a Hamiltonian operator if and only if

$$\{F,G\}_A = \int \frac{\delta F}{\delta u^i} A^{ij\sigma} \partial_\sigma \frac{\delta G}{\delta u^j}\, dx$$

is a Poisson bracket (skew-symmetric and Jacobi).

$\{,\}_A$ is a Poisson bracket if and only if:

- A is skew-adjoint: $A^* = -A$, where

$$A^*(\psi)^j = (-1)^\sigma \partial_\sigma \left(A^{ij\sigma} \psi_i \right)$$

- The variational Schouten bracket vanishes:

$$[A,A](\psi^1,\psi^2,\psi^3) =$$
$$2\left[\frac{\partial A^{ij\sigma}}{\partial u^l_\tau} \partial_\sigma(\psi^1_j) \partial_\tau (A^{lk\mu} \partial_\mu(\psi^2_k)) \psi^3_i + \text{cyclic}(1,2,3) \right] = 0$$

(the r.h.s. is defined up to total derivatives $\partial_x(B)$).

Homogeneous operators

Homogeneous operators were introduced in 1983-1984 by Dubrovin and Novikov. They are form-invariant under a point transformation of dependent variables $\bar{u}^i = U^i(u^j)$.

First-order local case:

$$\boxed{P_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b_k^{ij}(\mathbf{u})u_x^k}$$

Homogeneity: $\deg \partial_x = 1$.

Ferapontov first-order nonlocal case:

$$\boxed{P_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b_k^{ij}(\mathbf{u})u_x^k + c^{\alpha\beta} w^i_{\alpha k} u_x^k \partial_x^{-1} w^j_{\beta h} u_x^h}$$

where $c^{\alpha\beta} = c^{\beta\alpha}$ is a constant matrix.

Differential geometry and homogeneity

We work in the non-degenerate case $\det(g^{ij}) \neq 0$. Let $(g_{ij}) = (g^{ij})^{-1}$.

After a point transformation $\bar{u}^i = U^i(u^j)$:

- $g^{ij}(\mathbf{u})$ transforms as a contravariant 2-tensor;
- Then
$$\Gamma^i_{jk} = -g_{jp} b^{pi}_k$$
transform as the Christoffel symbols of a linear connection.

Differential geometry and the Hamiltonian property

Skew-adjointness is equivalent to:

- symmetry of g^{ij};
- the condition $g^{ij}_{,k} = \Gamma^{ij}_k + \Gamma^{ji}_k$, where $g^{ij}_{,k} = \partial g^{ij}/\partial u^k$.

Jacobi identity holds iff:

- $g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s$; this implies that Γ is metric: $\nabla[\Gamma]g = 0$;
- $\Gamma^i_{jk} = \Gamma^i_{kj}$ *i.e.* Γ is symmetric, so it is the Levi-Civita connection of g_{ij};
- the following conditions hold:

$$g^{is}w^j_s = g^{js}w^i_s, \quad \nabla[\Gamma]_i w^j_k = \nabla[\Gamma]_k w^j_i, \quad [w_\alpha, w_\beta] = 0;$$

- the curvature condition holds:

$$R[\Gamma]^{ij}_{kh} = c^{\alpha\beta}(w^i_{\alpha h}w^j_{\beta k} - w^i_{\alpha k}w^j_{\beta h}).$$

Higher-order homogeneous operators

Higher order homogeneous operators were introduced in 1984 by Dubrovin and Novikov. We consider here second-order and third-order homogeneous operators:

$$\begin{aligned} R_2^{ij} =& g_2^{ij}(\mathbf{u})\partial_x^2 + b^{ij}_{2\,k}(\mathbf{u})u^k_x\partial_x \\ &+ c^{ij}_{2\,k}(\mathbf{u})u^k_{xx} + c^{ij}_{2\,km}(\mathbf{u})u^k_x u^m_x, \end{aligned}$$

$$\begin{aligned} R_3^{ij} =& g_3^{ij}(\mathbf{u})\partial_x^3 + b^{ij}_{3\,k}(\mathbf{u})u^k_x\partial_x^2 \\ &+ [c^{ij}_{3\,k}(\mathbf{u})u^k_{xx} + c^{ij}_{3\,km}(\mathbf{u})u^k_x u^m_x]\partial_x \\ &+ d^{ij}_{3\,k}(\mathbf{u})u^k_{xxx} + d^{ij}_{3\,km}(\mathbf{u})u^k_x u^m_{xx} + d^{ij}_{3\,kmn}(\mathbf{u})u^k_x u^m_x u^n_x. \end{aligned}$$

Differential geometry and homogeneity

We will work in the non-degenerate case $\det(g_k^{ij}) \neq 0$, $k = 1, 2$.

After a point transformation $\bar{u}^i = U^i(u^j)$:

- g_2^{ij}, g_3^{ij} transform as contravariant 2-tensors;
- $\Gamma^i_{2\ jk} = -g_{jp}c_{2\ k}^{pi}$ and $\Gamma^i_{3\ jk} = -g_{jp}d_{3\ k}^{pi}$ transform as linear connections.

Differential geometry and the Hamiltonian property

It was proved (Potëmin, 1992; Doyle, 1992) that the Hamiltonian property implies that

- $\Gamma_{2}{}^{i}_{\ jk}$ and $\Gamma_{3}{}^{i}_{\ jk}$ are symmetric and flat;
- in flat coordinates, we have

$$R_2 = \partial_x (g_2^{ij}) \partial_x;$$

$$R_3 = \partial_x (g_3^{ij} \partial_x + c_{3}{}^{ij}_{\ \ k} u_x^k) \partial_x.$$

The Hamiltonian property

R_2: $g_{2\ ij} = T_{ijk}u^k + T_{0jk}$, where T is completely skew-symmetric and constant;

R_3: let $c_{ijk} = g_{3iq}g_{3jp}c^{pq}_{3k}$; the following properties hold:

$$c_{nkm} = \frac{1}{3}(g_{3nm,k} - g_{3nk,m}), \quad g_3 \text{ Monge metric};$$

$$g_{3mn,k} + g_{3nk,m} + g_{3km,n} = 0,$$

$$c_{mnk,l} = -g_3^{pq}c_{pml}c_{qnk}.$$

A prototype: WDVV equation

The simplest associativity Witten–Dijkgraaf–Verlinde–Verlinde) equation:

$$\boxed{f_{ttt} = f_{xxt}^2 - f_{xxx}f_{xtt}} \qquad f = f(t,x),$$

can be presented as a system of conservation laws by means of the change of coordinates $u^1 = f_{xxx}$, $u^2 = f_{xxt}$, $u^3 = f_{xtt}$ as

$$\boxed{\begin{cases} u_t^1 = u_x^2, \\ u_t^2 = u_x^3, \\ u_t^3 = ((u^2)^2 - u^1u^3)_x. \end{cases}}$$

Bi-Hamiltonian structure of WDVV equation

Ferapontov, Galvao, Mokhov, Nutku, CMP (1997) found out that the above WDVV first-order system can be rewritten as

$$u^i_t = P_1^{ij} \frac{\delta H_1}{\delta u^j} = R_3^{ij} \frac{\delta H_3}{\delta u^j},$$

$$H_1 = u^3, \quad H_3 = -\frac{1}{2} u^1 (\partial_x^{-1} u^2)^2 - (\partial_x^{-1} u^2)(\partial_x^{-1} u^3)$$

$$P_1 = \begin{pmatrix} -\frac{3}{2}\partial_x & \frac{1}{2}\partial_x a & \partial_x b \\ \frac{1}{2} a \partial_x & \frac{1}{2}(\partial_x b + b\partial_x) & \frac{3}{2} c\partial_x + c_x \\ b\partial_x & \frac{3}{2}\partial_x c - c_x & (b^2 - ac)\partial_x + \partial_x (b^2 - ac) \end{pmatrix},$$

$$R_3 = \partial_x \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & \partial_x & -\partial_x a \\ \partial_x & -a\partial_x & (\partial_x b + b\partial_x + a\partial_x a) \end{pmatrix} \partial_x$$

Bi-Hamiltonian equations of WDVV-type

We introduce the class of systems of conservation laws

$$u^i_t = (V^i(u^j))_x, \qquad i = 1, \ldots, n$$

which are bi-Hamiltonian by a pair of

- a first-order homogeneous operator of Ferapontov type;
- a third-order homogeneous operator in canonical form.

Indeed, third-order operators as above are classified under the action of various groups; the groups keep the form of the first-order operator stable.

Preliminaries: compatibility conditions

Compatibility conditions $[P_1, R_3] = 0$ have been derived in a recent work (Lorenzoni, Opanasenko, V., to appear in arXiv soon). Among the main results we find:

- the compatibility conditions have been integrated to algebraic equations;
- the nonlocal part of P_1 is made by Hamiltonian systems of R_3; it was previously found (Ferapontov, Pavlov, V. 2018) that such systems are determined by linear algebraic equations;
- it has been proved that g_1^{ij} is completely determined by a $n \times n$ matrix $Q^{\alpha\beta}$ of quadratic functions of the field variables.

NOTE: Nijenhuis tensor is not vanishing – no Nijenhuis geometry here!

$n = 2$: affine classification

The affine classification of third-order operators in canonical form is (Ferapnotov, Pavlov, V. 2014)

$$R^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{D}_x^3, \quad R^{(2)} = \mathrm{D}_x \begin{pmatrix} 0 & \mathrm{D}_x \frac{1}{u^1} \\ \frac{1}{u^1}\mathrm{D}_x & \frac{u^2}{(u^1)^2}\mathrm{D}_x + \mathrm{D}_x \frac{u^2}{(u^1)^2} \end{pmatrix} \mathrm{D}_x,$$

$$R^{(3)} = \mathrm{D}_x \begin{pmatrix} \mathrm{D}_x & \mathrm{D}_x \frac{u^2}{u^1} \\ \frac{u^2}{u^1}\mathrm{D}_x & \frac{(u^2)^2+1}{2(u^1)^2}\mathrm{D}_x + \mathrm{D}_x \frac{(u^2)^2+1}{2(u^1)^2} \end{pmatrix} \mathrm{D}_x.$$

$R^{(1)}$ is the Hamiltonian operator of *linear* systems of conservation laws, so we discard it.

$n = 2$: affine classification

$R^{(2)}$ is the Hamiltonian operator of the systems

$$u^1_t = (\alpha u^1 + \beta u^2)_x, \quad u^2_t = \left(\alpha u^2 + \frac{\beta (u^2)^2 + \gamma}{u^1}\right)_x.$$

There are two inequivalent cases: $(\beta, \alpha) = (1, 0)$ and $(\beta, \alpha) = (0, 1)$; we focus on the first case, which is bi-Hamiltonian with respect to three mutually compatible first-order local homogeneous Hamiltonian operators $P^{(2,i)}$ determined by the metrics

$$g^{(2,1)} = \begin{pmatrix} -u^1 & 0 \\ 0 & \frac{(u^2)^2+\gamma}{u^1} \end{pmatrix}, \quad g^{(2,2)} = \begin{pmatrix} 0 & u^1 \\ u^1 & 2u^2 \end{pmatrix},$$

$$g^{(2,3)} = \begin{pmatrix} 2u^2 & \frac{(u^2)^2+\gamma}{u^1} \\ \frac{(u^2)^2+\gamma}{u^1} & 0 \end{pmatrix}.$$

$n = 2$: affine classification

$R^{(3)}$ is the Hamiltonian operator of the systems

$$u^1_t = (\alpha u^1 + \beta u^2)_x, \quad u^2_t = \left(\alpha u^2 + \frac{\beta (u^2)^2 + \gamma u^2 - \beta}{u^1}\right)_x.$$

Two inequivalent cases: $(\beta, \alpha) = (1, 0)$ and $(\beta, \alpha) = (0, 1)$; we focus on the first case, which is bi-Hamiltonian with respect to three mutually compatible first-order local homogeneous Hamiltonian operators $P^{(3,i)}$ determined by the metrics

$$g^{(3,1)} = \begin{pmatrix} -u^1 & 0 \\ 0 & \frac{(u^2)^2 + \gamma u^2 - 1}{u^1} \end{pmatrix}, \quad g^{(3,2)} = \begin{pmatrix} 0 & u^1 \\ u^1 & 2u^2 + \gamma \end{pmatrix},$$

$$g^{(3,3)} = \begin{pmatrix} 2u^2 + \gamma & \frac{(u^2)^2 + \gamma u^2 - 1}{u^1} \\ \frac{(u^2)^2 + \gamma u^2 - 1}{u^1} & 0 \end{pmatrix}.$$

$n = 2$: projective classification

Reciprocal transformations are nonlocal transformations of the independent variable that were introduced to linearize some quasilinear first-order systems in gas dynamics (Rogers, 1968).

We will use projective reciprocal transformations, *i.e.*:

$$\tilde{u}^i = \frac{T^i_j u^j + T^i_0}{\Delta}, \qquad d\tilde{x} = \Delta dx, \quad d\tilde{t} = dt$$
$$\Delta = T^0_j u^j + T^0_0.$$

When $n = 2$, the only equivalence class of third-order operators is represented by $R^{(1)}$; the corresponding Hamiltonian systems are linear, so we do not consider them.

$n = 2$: a well-known example

The Chaplygin gas system:

$$u_t + uu_x + \frac{v_x}{v^3} = 0, \quad v_t + (uv)_x = 0$$

is known (Mokhov, Nutku 1998) to admit three first-order Hamiltonian operators. It can be diagonalized as

$$U_t = VU_x, \qquad V_t = UV_x,$$

to which one of the systems of the classification can be reduced by a nonlinear transformation. So, the above systems also have a compatible third-order homogeneous Hamiltonian operator which is *not* in canonical form.

$n = 2$: another well-known example

The Monge–Ampère equation

$$u_{tt}u_{xx} - u_{tx}^2 = -1.$$

By means of $u^1 = u_{xx}$, $u^2 = u_{tx}$ it can be made into the system

$$u_t^1 = u_x^2, \qquad u_t^2 = \left(\frac{(u^2)^2 - 1}{u^1}\right)_x,$$

which is bi-Hamiltonian by means of $R^{(2)}$ (Mokhov, Nutku 1998) and one of the three first-order operators listed above. Again, a nonlinear transformation brings the above system into the Chaplygin gas system.

$n = 3$: partial projective classification

The WDVV-type systems are bi-Hamiltonian with respect to P_1 and R_3, and have the form

$$u^i_t = (V^i)_x, \qquad i = 1, \ldots, n.$$

We recall (Balandin–Potemin) that the operator

$$R_3^{ij} = \partial_x(f^{ij}\partial_x + c^{ij}_k u^k_x)\partial_x$$

is completely determined by the Monge metric f_{ij}, which splits as

$$f_{ij} = \varphi_{\alpha\beta}\psi^\alpha_i\psi^\beta_j$$

where $\varphi_{\alpha\beta}$ is a constant symmetric matrix and ψ^α_i is a matrix of *linear* functions subject to algebraic constraints.

$n = 3$: partial projective classification

It can be proved that the *compatible* Ferapontov operator P_1 in low dimension has the form

$$P_1^{ij} = g_1^{ij}\mathrm{D}_x + \Gamma^{ij}_s u^s_x + c^{11} V^i_s u^s_x \mathrm{D}_x^{-1} V^j_r u^r_x \\ + c^{12}\left(V^i_s u^s_x \mathrm{D}_x^{-1} u^j_x + u^i_x \mathrm{D}_x^{-1} V^j_s u^s_x\right) + c^{22} u^i_x \mathrm{D}_x^{-1} u^j_x$$

where the metric has the form $g_1^{ij} = \psi^i_\alpha Q^{\alpha\beta}\psi^j_\beta$, with $Q^{\alpha\beta}$ a quadratic function, and $V^i_j = \partial V^i/\partial u^j$ is the Jacobian of the vector of fluxes of the system.

$n = 3$: partial projective classification

The classification of third-order operators under projective reciprocal transformations with $\tilde{t} = t$, Monge metrics:

$$f^{(1)} = \begin{pmatrix} (u^2)^2 + \mu & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 + \mu(u^3)^2 & -\mu u^2u^3 - u^1 \\ 2u^2 & -\mu u^2u^3 - u^1 & \mu(u^2)^2 + 1 \end{pmatrix},$$

$$f^{(2)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 - u^3 & 2u^2 \\ -u^1u^2 - u^3 & (u^1)^2 & -u^1 \\ 2u^2 & -u^1 & 1 \end{pmatrix}, \quad f^{(3)} = \begin{pmatrix} (u^2)^2 + 1 & -u^1u^2 & 0 \\ -u^1u^2 & (u^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$f^{(4)} = \begin{pmatrix} -2u^2 & u^1 & 0 \\ u^1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad f^{(5)} = \begin{pmatrix} -2u^2 & u^1 & 1 \\ u^1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad f^{(6)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$n = 3$: partial projective classification

Case $R^{(6)}$. Linear systems, out of consideration here.
Case $R^{(5)}$. Our prototype WDVV equation.
Case $R^{(4)}$. The following WDVV bi-Hamiltonian system (Kalayci, Nutku, 1998):

$$u^1_t = u^2_x, \quad u^2_t = \left(\frac{(u^2)^2 + u^3}{u^1}\right)_x, \quad u^3_t = u^1_x,$$

P_1 is local: $c^{\alpha\beta} = 0$.

$n = 3$: partial projective classification

Case $R^{(3)}$. Another WDVV bi-Hamiltonian system (Agafonov 1998; Ferapontov, Pavlov, V. 2018; Vašíček, V. 2021):

$$u^1_t = (u^2 + u^3)_x, \quad u^2_t = \left(\frac{u^2(u^2+u^3) - 1}{u^1}\right)_x, \quad u^3_t = u^1_x,$$

In view of compatibility, $g_1^{ij} = \psi^i_\alpha Q^{\alpha\beta} \psi^j_\beta$, where $c^{11} = c^{22} = -1$, $c^{12} = c^{21} = 0$ and

$$Q^{11} = 4(u^1)^2 + (u^2)^2 + 1, \quad Q^{12} = -3u^1, \quad Q^{13} = -2u^2 - u^3,$$
$$Q^{22} = (u^1)^2 + (u^3)^2 + 4, \quad Q^{23} = u^1(u^2 + 2u^3),$$
$$Q^{33} = (u^1)^2 + (u^2 + 2u^3)^2 + 1,$$
$$(\psi^\alpha_i) = \begin{pmatrix} -u^2 & 0 & 1 \\ u^1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (\varphi_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$n = 3$: partial projective classification

Case $R^{(2)}$. The Hamiltonian system

$$u^1_t=(\alpha u^2+\beta u^3)_x,$$
$$u^2_t=\left(\frac{((u^2)^2-1)(\alpha u^2+\beta u^3)-(\gamma+\delta u^1)}{S}\right)_x,$$
$$u^3_t=\left(\frac{(u^2u^3-u^1)(\alpha u^2+\beta u^3)-u^1(\gamma+\delta u^1)}{S}\right)_x,$$

where $S = u^1u^2 - u^3$ and $\alpha, \beta, \gamma, \delta$ are arbitrary constants. We have $g_1^{ij} = \psi^i_\alpha Q^{\alpha\beta}\psi^j_\beta$, where: $c^{11} = 3$, $c^{12} = c^{21} = 0$, $c^{22} = -\beta^2$,

$$Q^{11}=2(A^2+B^2+4BC+2AC),\quad Q^{12}=2(3AD-BC),$$
$$Q^{13}=2B(2A+3C),\quad Q^{22}=-2(2A+C)(2A+3C)$$
$$Q^{23}=8A^2+10AC+2BD\quad Q^{33}=-6A^2+2B^2,$$
$$A=\alpha u^2+\beta u^3,\quad B=\beta u^1+\alpha,\quad C=\delta u^1+\gamma,\quad D=\delta u^3+\gamma u^2,$$
$$(\psi^\alpha_i) = \begin{pmatrix} u^2 & 0 & 1 \\ -u^1 & -u^3 & 0 \\ 1 & u^2 & 0 \end{pmatrix}, \quad (\varphi_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

$n = 3$: partial projective classification

Case $R^{(1)}$. The Hamiltonian system

$$
\begin{aligned}
u^1_t &= (\alpha u^2 + \beta u^3)_x, \\
u^2_t &= \left(\frac{\left((u^2)^2 - \mu\right)(\alpha u^2 + \beta u^3) + \gamma(1 - \mu (u^2)^2) + \delta(u^1 - \mu u^2 u^3)}{S} \right)_x, \\
u^3_t &= \left(\frac{\alpha u^3 \left((u^2)^2 - \mu\right) + \beta u^3 (u^2 u^3 - \mu u^1)}{S} \right. \\
&\qquad \left. + \frac{\gamma(u^1 - \mu u^2 u^3) + \delta\left((u^1)^2 - \mu (u^3)^2\right)}{S} \right)_x,
\end{aligned}
$$

where $S = u^1 u^2 - u^3$ and $\alpha, \beta, \gamma, \delta$ are arbitrary constants. We have $g_1^{ij} = \psi^i_\alpha Q^{\alpha\beta} \psi^j_\beta$, where the nonlocal part has the coefficients:

$$
c^{11} = \mu^2 + 3, \quad c^{12} = c^{21} = -4\mu\delta, \quad c^{22} = \mu^3 \beta^2 + 4\mu^2 \delta^2 - \mu \beta^2,
$$

$n = 3$: partial projective classification

... the matrix $Q^{\alpha\beta}$ is given by:

$$Q^{11} = -(\mu^2-1)\big(\mu^2(A+C)^2+\mu(B^2+D^2)-2BD-4EF\big),$$

$$Q^{12} = -(\mu^2-1)(\mu ED-FB), \quad Q^{13} = -(\mu^2-1)(\mu B(2E+F)-3DE),$$

$$Q^{22} = -F^2\mu^3-\mu^2(4A^2+D^2)+\mu(8BD+F^2)-3D^2,$$

$$Q^{23} = -\mu^2(2BD+(u^1u^2-u^3)(\alpha\delta-\beta\gamma))+4\mu(B^2+D^2)-5BD-EF$$

$$Q^{33} = -\mu^3E^2-\mu^2(B^2+4D^2)+\mu(E^2+8BD)-3B^2,$$

$$A = \beta u^1+\delta u^3, \quad B = \alpha u^2+\beta u^3, \quad C = \gamma u^2+\alpha,$$

$$D = \delta u^1+\gamma, \quad E = \beta u^1+\alpha, \quad F = \delta u^3+\gamma u^2$$

and the Monge metric has the decomposition

$$(\psi_i^\alpha) = \begin{pmatrix} u^2 & 0 & 1 \\ -u^1 & -u^3 & 0 \\ 1 & u^2 & 0 \end{pmatrix}, \quad (\varphi_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 1 & \mu \end{pmatrix}.$$

$n = 3$: projective classification

It was proved (Ferapontov, Pavlov, V. 2018) that third-order homogeneous Hamiltonian operators in canonical form are invariant also with respect to transformations that exchange t and x.
This, together with the previous projective reciprocal transformations generate a larger group of reciprocal transformations of the following type:

$$
\begin{aligned}
\mathrm{d}\tilde{x} &= (A_i u^i + A_0)\mathrm{d}x + (A_i V^i + C_0)\mathrm{d}t, \\
\mathrm{d}\tilde{t} &= (B_i u^i + B_0)\mathrm{d}x + (B_i V^i + D_0)\mathrm{d}t,
\end{aligned}
$$

Under this group, there are two equivalence class, represented by $R_3^{(5)}$, which correspond to our prototype WDVV equation, and $R_3^{(6)}$, which defines linear equations only.

$n = 4$: no-go examples

Not all third-order homogeneous Hamiltonian operators and associated systems admit a compatible first-order local or nonlocal Hamiltonian operator. As an example, consider systems studied by Agafonov in 1998:

$$u^1_t = u^2_x, \quad u^2_t = u^3_x, \quad u^3_t = u^4_x, \quad u^4_t = (f(u))_x. \tag{1}$$

Ferapontov, Pavlov, V. (2018) proved that the above system is Hamiltonian with respect to R_3 only for two values of f:

$$f_1(u) = (u^2)^2 - u^1u^3, \qquad f_2(u) = (u^3)^2 - u^2u^4 + u^1. \tag{2}$$

Proposition. There does not exist a compatible first-order operator for the above systems.

$n = 4$: a bi-Hamiltonian example

It is conjectured (E.V. Ferapontov) that there is a unique integrable case within the class of systems of conservation laws that are Hamiltonian with respect to a R_3:

$$\begin{aligned} u^1_t &= u^3_x, \\ u^2_t &= u^4_x, \\ u^3_t &= \left(\frac{u^1u^2u^4 + u^3((u^3)^2 + (u^4)^2 - (u^2)^2 - 1)}{u^1u^3 + u^2u^4}\right)_x, \\ u^4_t &= \left(\frac{u^1u^2u^3 + u^4((u^3)^2 + (u^4)^2 - (u^1)^2 - 1)}{u^1u^3 + u^2u^4}\right)_x, \end{aligned} \tag{3}$$

$n = 4$: a bi-Hamiltonian example

The system is known (Ferapontov, Pavlov, V. 2018) to possess a Lax pair and a Hamiltonian operator R_3 defined by a Monge metric $f = (f_{ij})$:

$$(f_{ij}) = \begin{pmatrix} (u^2)^2+(u^3)^2+1 & -u^1u^2+u^3u^4 & -u^1u^3+u^2u^4 & -2u^2u^3 \\ -u^1u^2+u^3u^4 & (u^1)^2+(u^4)^2+1 & -2u^1u^4 & u^1u^3-u^2u^4 \\ -u^1u^3+u^2u^4 & -2u^1u^4 & (u^1)^2+(u^4)^2 & u^1u^2-u^3u^4 \\ -2u^2u^3 & u^1u^3-u^2u^4 & u^1u^2-u^3u^4 & (u^2)^2+(u^3)^2 \end{pmatrix}$$

We have $f_{ij} = \varphi_{\alpha\beta}\psi^\alpha_i\psi^\beta_j$ where

$$\Psi = \begin{pmatrix} -u^2 & -u^3 & 1 & 0 \\ u^1 & -u^4 & 0 & 1 \\ -u^4 & u^1 & 0 & 0 \\ u^3 & u^2 & 0 & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$n = 4$: a bi-Hamiltonian example

The system is Hamiltonian with respect to a first-order nonlocal Hamiltonian operator P_1 that is compatible with R and is defined by the metric $g_1^{ij} = \psi^i_\alpha Q^{\alpha\beta}\psi^j_\beta$, where

$$c^{11} = c^{22} = 1, \qquad c^{12} = c^{21} = 0,$$
$$Q^{11} = (u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2, \quad Q^{12} = -2u^1u^4 + 2u^2u^3,$$
$$Q^{13} = -u^2, \quad Q^{14} = u^1,$$
$$Q^{22} = (u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2 + 1, \quad Q^{23} = -2u^3,$$
$$Q^{24} = -2u^4, \quad Q^{33} = (u^1)^2 + (u^3)^2 + 1, \quad Q^{34} = u^1u^2 + u^3u^4,$$
$$Q^{44} = (u^2)^2 + (u^4)^2 + 1.$$

$n = 6$: further examples

- Pavlov and V. in LMP 2015 found a common bi-Hamiltonian pair of WDVV-type for two commuting first-order quasilinear systems of PDEs obtained from $N = 4$ WDVV equations.
- Opanasenko and V. in PRSA 2024 found a common bi-Hamiltonian pair of WDVV-type for two commuting first-order quasilinear systems of PDEs related with integrable Lagrangians of the form

$$\int L(u_{xx}, u_{xy}, u_{yy}) dx \wedge dy.$$

(from the paper *Second-order integrable Lagrangians and WDVV equations* by Ferapontov, Pavlov, Xue, arXiv 2020).

$n = 6$: further examples

- Opanasenko and V. (to appear in arXiv soon) proved that WDVV equations in all dimensions N, once rewritten as $N - 2$ commuting systems of first-order PDEs, admit a Hamiltonian operator of the form of R_3. When $N = 4$, the systems are bi-Hamiltonian of WDVV-type.

Final remarks

We recall the bi-Hamiltonian pencil:

$$P = P_1 + \epsilon R_2 + \epsilon^2 R_3 + \dots$$
$$Q = Q_1 + \epsilon S_2 + \epsilon^2 S_3 + \dots$$

An extension to an infinite formal sum is a building block of Dubrovin–Zhang's perturbative approach to the classification of Integrable Systems. WDVV-type systems are somehow "*singular*" to this classification program.
In principle, extensions to include 0-degree operators are possible (recent studies by Dell'Atti, Oliveri, Rizzo, Sgroi, Vergallo in arXiv), but their application to the study of integrable hierarchies is not known.

Thank you!

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