

Bi-Hamiltonian systems from homogeneous operators: part 2

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Plan of the talk

- ▶ Bi-Hamiltonian systems of PDEs: KdV-type and WDVV-type
- ▶ Compatible pairs of operators: analytic conditions.
- ▶ Compatible pairs of operators: geometric properties.

Bi-Hamiltonian systems

We consider evolutionary bi-Hamiltonian equations with unknowns $u^i = u^i(t, x)$, $i = 1, \dots, n$:

$$u_t^i = P^{ij} \left(\frac{\delta H}{\delta u^j} \right) = Q^{ij} \left(\frac{\delta H}{\delta u^j} \right)$$

A wide class of known bi-Hamiltonian systems have their Hamiltonian operators A , B in the form of linear combination of ∂_x -homogeneous Hamiltonian operators with different homogeneity degrees:

$$P = P_1 + \epsilon S_2 + \epsilon^2 S_3 + \dots$$

$$Q = Q_1 + \epsilon R_2 + \epsilon^2 R_3 + \dots$$

In this talk, we will focus on R_2 .

Kdv-type and WDVV-type systems

KdV-type systems (more frequent):

$$\begin{array}{ll} P = P_1, & Q = Q_1 + \epsilon R_2, \\ & \text{or} \\ & Q = Q_1 + \epsilon^2 R_3, \end{array}$$

WDVV-type systems (more rare):

$$P = P_1, \quad Q = R_3.$$

Bi-Hamiltonian systems from homogeneous Hamiltonian operators

Two mechanisms for many well-known integrable systems:

- ▶ Compatible **triples** (*regular mechanism*):
 - ▶ with **third-order** operators: KdV, Camassa–Holm, dispersive water waves (Antonowicz–Fordy 1989), coupled Harry–Dym, etc..
 - ▶ with **second-order** operators: AKNS, 2-component Camassa–Holm, Kaup–Broer (Kuperschmidt 1984), etc..
- ▶ Compatible **pairs** (*singular mechanism*):
 - ▶ with **third-order** operators: Monge–Ampère, WDVV, Oriented Associativity (or F -manifolds) equation (as quasilinear systems of the first order);
 - ▶ with **second-order** operators: new systems, no well-known example.

Hamiltonian operators

A is a Hamiltonian operator if and only if it defines a **Poisson bracket** on the space of functionals. That amounts to:

- ▶ A is **skew-adjoint**: $A^* = -A$, where

$$A^*(\psi)^j = (-1)^\sigma \partial_\sigma (A^{ij\sigma} \psi_i)$$

- ▶ The **variational Schouten bracket** vanishes:

$$[A, A](\psi^1, \psi^2, \psi^3) = 2 \left[\frac{\partial A^{ij\sigma}}{\partial u_\tau^l} \partial_\sigma (\psi_j^1) \partial_\tau (A^{lk\mu} \partial_\mu (\psi_k^2)) \psi_i^3 + \text{cyclic}(1, 2, 3) \right] = 0$$

(the r.h.s. is defined up to total derivatives $\partial_x(B)$).

Homogeneous Hamiltonian operators: first-order case

Homogeneous operators were introduced in 1983-1984 by Dubrovin and Novikov. They are form-invariant under a point transformation of dependent variables $\bar{u}^i = U^i(u^j)$.

First-order local case:

$$P_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b_k^{ij}(\mathbf{u})u_x^k$$

Homogeneity: $\deg \partial_x = 1$.

Ferapontov first-order nonlocal case:

$$P_1^{ij} = g^{ij}(\mathbf{u})\partial_x + b_k^{ij}(\mathbf{u})u_x^k + c^{\alpha\beta}w_{\alpha k}^i u_x^k \partial_x^{-1} w_{\beta h}^j u_x^h$$

where $\det(g^{ij}) \neq 0$ and $c^{\alpha\beta} = c^{\beta\alpha}$ is a constant matrix.

Homogeneous Hamiltonian operators: higher-order case

Higher order homogeneous operators were introduced in 1984 by Dubrovin and Novikov. We consider here **second-order** and **third-order** homogeneous operators:

$$R_2^{ij} = \eta^{ij}(\mathbf{u})\partial_x^2 + b_{2k}^{ij}(\mathbf{u})u_x^k\partial_x \\ + c_{2k}^{ij}(\mathbf{u})u_{xx}^k + c_{2km}^{ij}(\mathbf{u})u_x^k u_x^m,$$

$$R_3^{ij} = f^{ij}(\mathbf{u})\partial_x^3 + b_{3k}^{ij}(\mathbf{u})u_x^k\partial_x^2 \\ + [c_{3k}^{ij}(\mathbf{u})u_{xx}^k + c_{3km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\ + d_{3k}^{ij}(\mathbf{u})u_{xxx}^k + d_{3km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{3kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n.$$

We work in the *non degenerate case* $\det(f^{ij}) \neq 0$.

Canonical form

It was proved (Potëmin, 1992; Doyle, 1992) that the Hamiltonian property implies that there are coordinates $(\tilde{u}^i)$ in which

$$R_2^{ij} = \partial_x(\eta^{ij})\partial_x;$$

$$R_3^{ij} = \partial_x(\textcolor{red}{f}^{ij}\partial_x + c_k^{ij}u_x^k)\partial_x.$$

Compatibility of Hamiltonian operators

Two Hamiltonian operators A, B are **compatible** iff any linear combination $\lambda A + \mu B$ is still a Hamiltonian operator iff the Schouten bracket vanishes: $[A, B] = 0$ where

$$[A, B](\psi^1, \psi^2, \psi^3) = \left[\frac{\partial A^{ij\sigma}}{\partial u_\tau^l} \partial_\sigma(\psi_j^1) \partial_\tau(B^{lk\mu} \partial_\mu(\psi_k^2)) \psi_i^3 + \frac{\partial B^{ij\sigma}}{\partial u_\tau^l} \partial_\sigma(\psi_j^1) \partial_\tau(A^{lk\mu} \partial_\mu(\psi_k^2)) \psi_i^3 + \text{cyclic}(1, 2, 3) \right] = 0$$

(the r.h.s. is defined up to total derivatives $\partial_x(B)$).

The calculation of a **minimal set of conditions** of compatibility is a nontrivial problem (Lorenzoni, Casati, Valeri, V. 2020-2022).

Compatibility conditions: known cases

- ▶ First-order homogeneous case $[P_1, Q_1] = 0$: well-known (Ferapontov, Mokhov). Nijenhuis tensor of $g_P^{ik} g_{Qkj}$ plays an important role.
- ▶ operators of the same type: the conditions can be ‘easily’ derived from existing conditions, using

$$[\lambda A + \mu B, \lambda A + \mu B] = 0;$$

- ▶ operators of different type: order 1 and order 0 (Mokhov; Dell’Atti, Rizzo, Vergallo);
- ▶ first-order operators with non-local part determined by isometries (Pavlov, Vergallo, V.).

Conditions for $[P_1, R_k] = 0$, $k = 2, 3$

The computation of the Schouten bracket $[P_1, R_k]$, $k = 2, 3$ and the reduction of $[P_1, R_k] = 0$ to a minimal set of conditions is much more difficult. The following results exist:

- ▶ $[P_1, R_3] = 0$, P_1 local, $R_3^{ij} = f^{ij} \partial_x^3$ where f^{ij} constant; Bolsinov, Matveev, Konyaev (2021);
- ▶ $[P_1, R_3] = 0$, P_1 non-local, $R_3^{ij} = f^{ij} \partial_x^3$ where f^{ij} constant; Lorenzoni, V. (2021);
- ▶ $[P_1, R_2] = 0$, P_1 non-local, $R_2^{ij} = \eta^{ij} \partial_x^2$ where η^{ij} constant; Lorenzoni, V. (2024);

Recent advances

- ▶ We integrated all differential conditions, and reduced our problem to algebraic equations only, for $[P_1, R_3] = 0$, P_1 local or nonlocal; published in JPA 2026 (Lorenzoni, Opanasenko and V.)
- ▶ We integrated all differential conditions, and reduced our problem to algebraic equations only, for $[P_1, R_2] = 0$, P_1 local or nonlocal, still unpublished (Lorenzoni, Opanasenko and V.).
- ▶ We provide a set of necessary and sufficient conditions for the compatibility.
- ▶ We provide a geometric interpretation of the compatibility conditions.

$[P_1, R_2] = 0$: nonlocal part

$$P_1^{ij} = g^{ij} \partial_x + \Gamma_k^{ij} u_x^k + c^{\alpha\beta} w_{\alpha k}^i u_x^k \partial_x^{-1} w_{\beta h}^j u_x^h,$$
$$R_2^{ij} = \partial_x \eta^{ij} \partial_x.$$

Theorem 1 The nonlocal part of $[P_1, R_2] = 0$ is equivalent to: $w_{\alpha k}^i u_x^k$ are fluxes of systems of conservation laws which are Hamiltonian with respect to R_2 , *i.e.*

$$w_{\beta k}^p \eta^{kj} + \eta^{kp} w_{\beta k}^j = 0, \quad (1)$$

$$w_{\beta k, l}^j = w_{\beta l, k}^j, \quad (2)$$

$$w_{\beta k, l}^j \eta^{lp} + w_{\beta l}^p \eta_{, k}^{lj} + w_{\beta k}^l \eta_{, l}^{jp} = 0. \quad (3)$$

Exactly the same result (with slightly different equations!) holds for $[P_1, R_3] = 0$.

General solutions for w_α

From Theorem 1, we have

$$w_{\alpha k}^i u_x^k = (w_\alpha^i)_x.$$

Moreover, in Vergallo, V. 2023 Hamiltonian systems of conservation laws of R_2 are shown to have the form:

$$w_\alpha^i = \eta^{ij} (A_{\alpha j k} u^k + B_{\alpha j}), \quad A_{\alpha j k} = -A_{\alpha k j}.$$

$A_{\alpha j k}$ and $B_{\alpha j}$ are arbitrary constants; the space of w_α is **finite-dimensional**.

$[P_1, R_2] = 0$: local part

Theorem 2: The local part of $[P_1, R_2] = 0$ is equivalent to the conditions:

$$\Gamma_k^{ij} \eta^{kp} + \Gamma_k^{pj} \eta^{ki} = 0, \quad (4)$$

$$\Gamma_k^{ij} \eta^{kp} + \Gamma_k^{jp} \eta^{ki} + \Gamma_k^{pi} \eta^{kj} = \eta_{,k}^{ij} g^{kp} + \eta_{,k}^{jp} g^{ki} + \eta_{,k}^{pi} g^{kj}, \quad (5)$$

$$(\Gamma_{k,l}^{ij} + c^{\alpha\beta} w_{\alpha k}^i w_{\beta l}^j) \eta^{kp} = \Gamma_l^{kj} \eta_{,k}^{pi} - \Gamma_k^{pj} \eta_{,l}^{ki}. \quad (6)$$

First consequence: the metric $m_{ij} = \eta_{ik} \eta_{jh} g^{hk}$ is a **Monge metric**:

$$m_{ij,k} + m_{ki,j} + m_{jk,i} = 0,$$

hence it can be identified with a quadratic line complex in the Plücker embedding of a projective space $\mathbb{P}^n$ whose affine chart is (u^i) .

Structure formula for g^{ij}

Theorem 3. If $[P_1, R_2] = 0$ then

$$g^{ij} = \eta^{ik} Z_k^j + \eta^{jk} Z_k^i - c^{\alpha\beta} W_\alpha^i W_\beta^j,$$

where $Z_k^j = A_{kh}^j u^h + B_k^j$ is a matrix of linear functions whose coefficients fulfill $A_{kh}^j = -A_{hk}^j$, and $r^{ij} = \eta^{ik} Z_k^j$ are (fluxes of) Hamiltonian systems of conservation laws with respect to R_2 (indexed by j).

Main consequence: the compatibility conditions $[P_1, R_2] = 0$ are equivalent to a system of algebraic equations.

Hamiltonian property of P_1

Theorem 4. The Hamiltonian property of an operator P_1 which is compatible with a Hamiltonian operator R_2 is equivalent to

- ▶ $g^{ik}w_{\alpha k}^j = g^{jk}w_{\alpha k}^i$;
- ▶ $g^{ik}r_{,k}^{jl} = g^{jk}r_{,k}^{il}$;
- ▶ The two families of vector of fluxes $\{w_{\alpha}^i \mid i = 1, \dots, n\}$ and $\{r^{hk} \mid h = 1, \dots, n\}$ are commuting, both between elements of the same family and between elements of the two distinct families:

$$r_{,p}^{ik}r_{,j}^{ph} = r_{,p}^{ih}r_{,j}^{pk}, \quad r_{,p}^{ik}w_{\alpha,j}^p = w_{\alpha,p}^i r_{,j}^{pk}, \quad w_{\alpha,p}^i w_{\beta,j}^p = w_{\beta,p}^i w_{\alpha,j}^p.$$

Analogies

$[P_1, R_2] = 0$	$[P_1, R_3] = 0$
Nonlocal part by Ham. sys. of R_2	Nonlocal part by Ham. sys. of R_3
Structure formula for g^{ij}	Structure formula for g^{ij}
All Ham. sys. involved are commuting	All Ham. sys. involved are commuting

NOTES:

- ▶ conditions of compatibility were provided with counterexamples generated by Claude that show that they are **necessary and sufficient!**
- ▶ Claude also proved that one condition that was shown in the previous seminar was vanishing on account of others.

Differences

- ▶ $[P_1, R_2] = 0$: remaining condition

$$\Gamma_k^{ij} \eta^{kp} + \Gamma_k^{jp} \eta^{ki} + \Gamma_k^{pi} \eta^{kj} = \eta_{,k}^{ij} g^{kp} + \eta_{,k}^{jp} g^{ki} + \eta_{,k}^{pi} g^{kj},$$

implies, but it is not equivalent, to $m_{ij} = \eta_{ik} \eta_{jh} g^{hk}$ is a **Monge metric**;

- ▶ $[P_1, R_3] = 0$: remaining condition

$$\begin{aligned} (\Gamma_k^{ij} - \Gamma_k^{ji}) f^{kp} + (\Gamma_k^{pj} - \Gamma_k^{jp}) f^{ki} \\ + (c_k^{pi} g^{kj} - c_k^{pj} g^{ki}) + (c_k^{ip} g^{kj} - c_k^{ij} g^{kp}) = 0 \end{aligned}$$

with still unclear interpretation.

Geometric considerations

The compatibility problem is invariant under **projective reciprocal transformations**:

$$\tilde{u}^i = \frac{T_j^i u^j + T_0^i}{\Delta}, \quad d\tilde{x} = \Delta dx, \quad d\tilde{t} = dt$$
$$\Delta = T_j^0 u^j + T_0^0.$$

All Hamiltonian operators and Hamiltonian systems of R_k that we dealt with can be regarded as **algebraic varieties** in the Plücker embedding of $\mathbb{P}^n$, where an affine chart is given by (u^i) , $i = 1, \dots, n$.

NOTE: Nijenhuis tensor of $f^{ik}g_{kj}$ is *not* vanishing!

Projective geometry

Integrable PDEs	Projective Geometry
Third-order Hamiltonian op.	quadratic line complex
Second-order Hamiltonian op.	linear line congruence
Hamiltonian systems (both cases)	linear line congruence
First-order R_2 -compatible Ham. op.	quadratic line complex
First-order R_3 -compatible Ham. op.	system of quadrics

The variety of commuting Hamiltonian systems of R_3 is currently under study (collaboration: R. La Scala, X. Luo, S. Opanasenko, R.V.). It has a relatively high dimension also in low number of components $n = 2, 3$. First studies in Lorenzoni, Savoldi, V. JPA 2017.

Thank you!

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