Homotopy Lie Algebroids

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Introduction

Let $\mathcal{A}$ be a type of algebra. A homotopy $\mathcal{A}$ algebra structure on a cochain complex is a set of operations that satisfy the axioms of $\mathcal{A}$ only *up to homotopy*.

Motivating Fact I

Let (A, d) be a differential algebra of type $\mathcal{A}$ and $f : (A, d) \rightleftarrows (V, \delta) : g$ a pair of homotopy equivalences. The algebra structure in A can be transferred to V along (f, g), but the transferred structure is of the type $\mathcal{A}$ only *up to higher homotopies*. On the other hand *homotopy algebras are homotopy invariant!*

Motivating Fact II

Let $(\mathcal{K}, \delta)$ be a differential algebra. $(\mathcal{K}, \delta)$ is always homotopy equivalent to its cohomology $(H(\mathcal{K}, \delta), 0) \implies H(\mathcal{K}, \delta)$ is a homotopy algebra. *The latter structure characterizes the homotopy type of* $(\mathcal{K}, \delta)$.

Example (Massey products)

Let $(\mathcal{K}, \delta) = (C^\bullet(X), d)$ *be singular cochains of a topological space* X. *The* Massey products *in* $H^\bullet(X)$ *are obtained transferring the cup product along a homotopy.*

Motivations from PDEs

The datum of a PDE is encoded by a *diffiety*, i.e., a countable dimensional manifold with an involutive distribution $\mathcal{C}$. Horizontal cohomologies of $\mathcal{C}$ with suitable local coefficients contain important informations about the PDE (symmetries, conservation laws, etc.). Moreover, they can be interpreted, to some extent, as geometric structures on the space of solutions (i.e., vector fields, differential forms, etc.).

Often, horizontal cohomologies of a diffiety possess canonical algebraic structures. However, the latter does not generically come from algebraic structures on cochains.

Remark

"Usually" algebraic structures on horizontal cohomologies come from homotopy algebraic structures on cochains. In particular, there is a *homotopy Lie algebroid* accounting for the Lie algebra structure on Krasil'shchik cohomologies. The former comes with canonical left/right representations.

Motivations from Differential Geometry

Lie algebroids encode salient features of foliations, complex structures, Poisson structures, Jacobi structures, etc. In some cases (singularieties, ∞ dimensions, etc.) what one really needs is the algebraic counterpart of a Lie algebroid: a Lie-Rinehart algebra.

Of a special interest are representations of Lie algebroids: group actions, $\mathcal{D}$-modules, deformations, etc. Even when higher homotopies are not manifestly involved, they may be relevant: Courant algebroids are Lie algebroids up to homotopy, the adjoint representation of a Lie algebroid is a representation up to homotopy, etc.

Aim

The aim of this talk is to present the first steps of a systematic study of homotopy Lie algebroids/homotopy Lie-Rinehart algebras and their representations, which encompass various results already scattered in literature [Abad & Crainic 09], [Mehta & Zambon 12], *etc. The material presented is preliminary with respect to applications to PDEs.*

Outline

Homotopy Lie Algebras

Let V be a graded vector space.

Definition

An L_∞ algebra structure *in* V *is a family:* $\lambda_k : V^{\otimes k} \longrightarrow V$, *of degree* $2-k$, *skew-symmetric maps such that*

$$\sum_{i+j=k} \sum_{\sigma \in S_{i,j}} \pm \lambda_{j+1}(\lambda_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(i+j)}) = 0,$$

for all $x_1, \ldots, x_k \in V$, $k \in \mathbb{N}$.

Put $\lambda_1 = \delta$, and $\lambda_2 = [-,-]$.

$k=1 \qquad \delta^2(x) = 0$

$k=2 \qquad \delta[x,y] = [\delta x, y] \pm [x, \delta y]$

$k=3 \qquad [x,[y,z]] \pm [y,[z,x]] \pm [z,[x,y]] =$

$$-\delta\lambda_3(x,y,z) - \lambda_3(\delta x, y, z) \mp \lambda_3(x, \delta y, z) \mp \lambda_3(x,y,\delta z)$$

L_∞ modules are defined in a similar way.

One can work with degree 1, symmetric maps instead (use *décalage*): $L_\infty[1]$

Homotopy Lie-Rinehart Algebras

A Lie-Rinehart algebra is a pair (A, L), where

- A is a commutative algebra over a field K, and L is an A-module,
- L is a Lie algebra acting on A by derivations,
- compatibility conditions hold: for $a, b \in A$ and $\xi, \zeta \in L$

$$(a\xi).b = a(\xi.b) \quad \text{and} \quad [\xi, a\zeta] = a[\xi, \zeta] + (\xi.a)\zeta.$$

Definition [Kjeseth 01, up to *décalage*]

An $LR_\infty[1]$ algebra *is a pair* (A, L) *where*

- A *is a graded commutative* K*-algebra, and* L *is an* A*-module,*
- L *is an* $L_\infty[1]$ *algebra acting on* A *by derivations,*
- *compatibility conditions hold: for* $a, b \in A$ *and* $\xi_1, \ldots, \xi_k \in L$

$$\{a\xi_1, \xi_2, \ldots, \xi_{k-1}|b\} = \pm a\{\xi_1, \ldots, \xi_{k-1}|b\},$$
$$\{\xi_1, \ldots, \xi_{k-1}, a\xi_k\} = \pm a\{\xi_1, \ldots, \xi_k\} + \{\xi_1, \ldots, \xi_{k-1}|a\}\xi_k.$$

Homotopy Lie-Rinehart Algebras

Remark

Kjeseth's definition is actually realized in "nature": examples come from BRST, Lie algebroids, complex geometry, Poisson geometry, *PDEs*, ...

A systematic investigation shows that many standard constructions with Lie algebroids have an analogue in terms of LR_∞ algebras:

- associated CE and Gerstenhaber algebras,
- cohomologies of left/right modules,
- transformation Lie algebroid,
- Schouten-Nijenhuis calculus,
- derivative representations of Lie algebroids,
- right actions and BV algebras,
- BV algebras from Poisson manifolds,
- BV algebras from Jacobi manifold.

The CE and Schouten Algebras of an LR_∞ Algebra

An LR algebra (A, L) determines:

- a homological derivation of $\mathrm{Alt}_A(L, A)$,
- a Gerstenhaber algebra structure on $\Lambda_A^\bullet L$.

Proposition

An $LR_\infty[1]$ algebra (A, L) determines:

- *a homological derivation $D = D_1 + D_2 + \cdots$ in $\mathrm{Sym}_A(L, A)$ via:*

$$
\begin{aligned}
&(D_k\omega)(\xi_1, \ldots, \xi_{r+k}) \\
&:= \sum \pm\{\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)} \,|\, \omega(\xi_{\sigma(k+1)}, \ldots, \xi_{\sigma(k+r)})\} \\
&\quad - \sum \pm\omega(\{\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k+1)}\}, \xi_{\sigma(k+2)}, \ldots, \xi_{\sigma(k+r)}),
\end{aligned}
$$

  $\omega \in \mathrm{Sym}_A^r(L, A)$, $\xi_1, \ldots, \xi_{r+k} \in L$;
- *a homotopy Gerstenhaber algebra structure on $S_A^\bullet L$.*

More General L_∞ Algebroids

Remark

When L is projective and finitely generated, an $LR_\infty[1]$ algebra structure on (A, L) is the same as a homological derivation of $\mathrm{Sym}_A(L, A)$ descending to a derivation of A along the projection $\mathrm{Sym}_A(L, A) \to A$.

Example (L_∞ algebroids over graded manifolds)

- $A = C^\infty(\mathcal{M})$, *with* $\mathcal{M}$ *a graded manifold,*
- $L = \Gamma(\mathcal{E})$, *with* $\mathcal{E} \to \mathcal{M}$ *a graded vector bundle.*

An $LR_\infty[1]$ *algebra structure on* (A, L) *is the same as a homological vector field* D *on* $\mathcal{E}$ *tangent to the* 0 *section.*

Higher Left Linear Connections

Let (A, L) be an LR algebra and P an A-module. A left (A, L)-connection in P is a K-linear map $\nabla : L \to \mathrm{End}_K P$ such that: for $a \in A$, $\xi \in L$, $p \in P$

$$\nabla_{a\xi} p = a \nabla_\xi p \quad \text{and} \quad \nabla_\xi (ap) = a\nabla_\xi p + (\xi.a)p.$$

∇ is flat, and (P, ∇) is a left (A, L)-module if it is a left Lie module over L.

Definition

Let (A, L) be a $LR_\infty[1]$ algebra and P an A-module. A left (A, L) connection in P *is a family of K-multilinear, graded symmetric, degree 1 maps $\nabla : L^{\otimes(k-1)} \to \mathrm{End}_K P$ such that: for $a \in A$, $\xi_1, \ldots, \xi_{k-1} \in L$, $p \in P$*

$$\nabla(a\xi_1, \xi_2, \ldots, \xi_{k-1}|p) = \pm a\nabla(\xi_1, \ldots, \xi_{k-1}|p),$$
$$\nabla(\xi_1, \ldots, \xi_{k-1}|ap) = \pm a\nabla(\xi_1, \ldots, \xi_{k-1}|p) + \{\xi_1, \ldots, \xi_{k-1}|a\}p.$$

∇ *is* flat, *and* (P, ∇) *is a left* (A, L)-module, *if* ∇ *is a left* $L_\infty[1]$ *module structure.*

Cohomologies of Left LR_∞ Modules

A left connection ∇ in P along an LR algebra determines an operator D^∇ in $\mathrm{Alt}_A(L,P)$, and $(D^\nabla)^2 = 0$ if ∇ is flat.

Proposition

A left connection ∇ in P along an $LR_\infty[1]$ algebra (A, L) determines an operator $D^\nabla = D_1^\nabla + D_2^\nabla + \cdots$ in $\mathrm{Sym}_A(L,P)$ via higher CE formulas:

$$
\begin{aligned}
&(D_k^\nabla \Omega)(\xi_1,\ldots,\xi_{r+k-1}) \\
&:= \sum \pm \nabla(\xi_{\sigma(1)},\ldots,\xi_{\sigma(k-1)} \,|\, \Omega(\xi_{\sigma(k)},\ldots,\xi_{\sigma(r+k-1)})) \\
&\quad - \sum \pm \Omega(\{\xi_{\sigma(1)},\ldots,\xi_{\sigma(k)}\}, \xi_{\sigma(k+1)},\ldots,\xi_{\sigma(r+k-1)}),
\end{aligned}
$$

$\Omega \in \mathrm{Sym}_A^r(L,P)$, $\xi_1,\ldots,\xi_{r+k-1} \in L$, *and* $(D^\nabla)^2 = 0$ *if* ∇ *is flat.*

Connections Along L_∞ Algebroids

Remark

A (flat) left connection in P along a $LR_\infty[1]$ algebra (A, L) is the same as a (homological) derivation D^∇ of $\mathrm{Sym}_A(L, P)$

1. subordinate to D, and
2. descending to a derivation of P along $\mathrm{Sym}_A(L, P) \to P$.

Example

- $(A, L) = (C^\infty(\mathcal{M}), \Gamma(\mathcal{E}))$, *with* $\mathcal{E} \to \mathcal{M}$ *an* L_∞ *algebroid,*
- $P = \Gamma(\mathcal{V})$, *with* $\mathcal{V} \to \mathcal{M}$ *a vector bundle.*

A (flat) left (A, L)*-connection in* P *is the same as a fiber-wise linear (homological) vector field* D^∇ *on* $\mathcal{E} \times_{\mathcal{M}} \mathcal{V}^*$ *(or, equivalently, on* $\mathcal{E} \times_{\mathcal{M}} \mathcal{V}$*) which is*

1. *compatible with* D *along* $\mathcal{E} \times_{\mathcal{M}} \mathcal{V}^* \to \mathcal{E}$*, and*
2. *tangent to the* 0 *section of* $\mathcal{E} \times_{\mathcal{M}} \mathcal{V}^* \to \mathcal{V}^*$.

Higher Left Schouten-Nijenhuis Calculus

Let (A, L) be an LR algebra. For $u, v \in \Lambda_A^\bullet L$, let i_u be the *insertion operator* in $\mathrm{Alt}_A(L, A)$, and $L_u = [i_u, d]$ the *Lie derivative*. Then

$$[L_u, i_v] = i_{[u,v]} \quad \text{and} \quad [L_u, L_v] = L_{[u,v]}$$

Proposition

Let (A, L) be an $LR_\infty[1]$ algebra, and (P, ∇) a left (A, L)-module. For $u, u_1, \ldots, u_k \in S_A^\bullet L$, let i_u be the insertion operator *in $\mathrm{Sym}_A(L, P)$, and*

$$L^\nabla(u_1, \ldots, u_{k-1}) = [[\cdots [D_k^\nabla, i_{u_1}] \cdots], i_{u_{k-1}}]$$

a higher Lie derivative. *Then*

$$[L^\nabla(u_1, \ldots, u_{k-1}), i_{u_k}] = \pm i_{\{u_1,\ldots,u_k\}},$$

and L^∇ is an $L_\infty[1]$-module structure on $\mathrm{Sym}_A(L, P)$.

Higher Right Linear Connections

Let (A, L) be an LR algebra and Q an A-module. A right (A, L)-connection in Q is a K-linear map $\Delta : L \to \mathrm{End}_K Q$ such that: for $a \in A$, $\xi \in L$, $q \in Q$

$$\Delta_{a\xi} q = \Delta_\xi (aq) \quad \text{and} \quad \Delta_\xi (aq) = a\Delta_\xi q - (\xi.a)q.$$

Δ is flat, and (Q, Δ) is a right (A, L)-module if it is a right Lie module over L.

Definition

Let (A, L) be a $LR_\infty[1]$ algebra and Q and A-module. A right (A, L) connection in Q *is a family of K-multilinear, graded symmetric, degree 1 maps $\Delta : L^{\otimes(k-1)} \to \mathrm{End}_K Q$ such that: for $a \in A$, $\xi_1, \ldots, \xi_{k-1} \in L$, $q \in Q$*

$$\Delta(\xi_1, \ldots, \xi_{k-2}, a\xi_{k-1}|q) = \pm\Delta(\xi_1, \ldots, \xi_{k-1}|aq)$$
$$\Delta(\xi_1, \ldots, \xi_{k-1}|aq) = \pm a\Delta(\xi_1, \ldots, \xi_{k_1}|q) - \{\xi_1, \ldots, \xi_{k-1}|a\}q.$$

Δ is flat, *and (Q, Δ) is a left (A, L)*-module, *if Δ is a right $L_\infty[1]$ module structure.*

Cohomologies of Right LR_∞ Modules

A right connection Δ in Q along an LR algebra determines an operator D^Δ in $\Lambda_A^\bullet L \otimes_A Q$, and $(D^\Delta)^2 = 0$ if Δ is flat.

Proposition

A right connection Δ in Q along an $LR_\infty[1]$ algebra (A, L) determines an operator $D^\Delta = D_1^\Delta + D_2^\Delta + \cdots$ in $S_A^\bullet L \otimes_A Q$ via higher Rinehart formulas:

$$
\begin{aligned}
& D_k^\Delta(\xi_1 \cdots \xi_{r+k-1} \otimes q) \\
& := - \sum \pm \xi_{\sigma(1)} \cdots \xi_{\sigma(r)} \otimes \Delta(\xi_{\sigma(r+1)}, \ldots, \xi_{\sigma(k+r-1)} | q) \\
& \quad + \sum \pm \{\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)}\} \xi_{\sigma(k+1)} \cdots \xi_{\sigma(k+r)} \otimes q,
\end{aligned}
$$

$\xi_1, \ldots, \xi_{r+k-1} \in L$, $q \in Q$, and $(D^\Delta)^2 = 0$ if Δ is flat.

Higher Right Schouten-Nijenhuis Calculus

Let (A, L) be an LR algebra. A right connection along (A, L) determines a *right version* of the Schouten-Nijenhuis calculus.

Proposition

Let (A, L) be an $LR_\infty[1]$ algebra, and (Q, Δ) a right (A, L)-module. For $u, u_1, \ldots, u_k \in S_A^\bullet L$, let μ_u be the multiplication operator *in $S_A^\bullet L \otimes_A Q$, and*

$$R^\Delta(u_1, \ldots, u_{k-1}) = [[\cdots[D_k^\Delta, \mu_{u_1}]\cdots], \mu_{u_{k-1}}]$$

a higher right Lie derivative. *Then*

$$[R^\Delta(u_1, \ldots, u_{k-1}), \mu_{u_k}] = \mu_{\{u_1, \ldots, u_k\}},$$

and R^Δ is a right $L_\infty[1]$-module structure on $S_A^\bullet L \otimes_A Q$.

BV Algebras from Right Lie-Rinehart Modules

A BV algebra is a graded commutative, unital algebra B with a degree 1 homological differential operator $\square$ of order 2.

$$(a, b) := [[\square, a], b](1)$$

equips B with a Gerstenhaber algebra structure (up to *décalage*).

Remark

Let (A, L) be an LR algebra. There is *no* canonical right connection in A.

Proposition [Huebschmann 99], [Xu 99]

Let (A, L) be a Lie-Rinehart algebra, and Δ an (A, L)-module structure on A. The homological operator D^Δ equips $\Lambda_A^\bullet L$ with a BV algebra structure. The BV bracket coincides with the canonical Gerstenhaber brackets in $\Lambda_A^\bullet L$.

BV_∞ Algebras from Right LR_∞ Modules

Definition [Kravchenko 00]

A BV_∞ algebra is a graded commutative, unital algebra B with a degree 1 homological operator $\square$.

$$(a_1, a_2, \ldots, a_k) := [\cdots[[\square, a_1], a_2]\cdots, a_k](1)$$

equip B with a homotopy Gerstenhaber algebra structure.

Remark

Let (A, L) be an LR_∞ algebra. There is *no* canonical right connection in A.

Proposition

Let (A, L) be an $LR_\infty[1]$ algebra, and Δ an (A, L)-module structure on A. The homological operator D^Δ equips $S^\bullet_A L$ with a BV_∞ algebra structure. The higher Koszul brackets coincide with the canonical homotopy Gerstenhaber brackets in $S^\bullet_A L$.

BV_∞ Algebras in Higher Poisson Geometry

Let M be a Poisson manifold. Then $T^*M \to M$ is a Lie algebroid, and there is a canonical, flat, right connection Δ along the LR algebra $(C^\infty(M), \Omega^1(M))$ in $C^\infty(M)$: for $f, g \in C^\infty(M)$

$$[df, dg] = d\{f, g\} \quad \text{and} \quad \Delta_{df} g = -\{f, g\}$$

Example

Let $\mathcal{M}$ be a higher Poisson manifold, *i.e., a graded manifold with a degree -2, homological multivector field P. Then $T^*[-1]\mathcal{M} \to \mathcal{M}$ is an $L_\infty[1]$ algebroid, and there is a canonical, flat, right connection Δ along the LR_∞ algebra $(C^\infty(\mathcal{M}), \Omega^1(\mathcal{M}))$ in $C^\infty(M)$: for $f_1, \ldots, f_k, g \in C^\infty(\mathcal{M})$*

$$\{df_1, \ldots, df_k\} = \pm d\{f_1, \ldots, f_k\}_P,$$
$$\Delta(df_1, \ldots, df_{k-1}|g) = \mp\{f_1, \ldots, f_{k-1}, g\}_P.$$

Consequently, $\Omega(\mathcal{M})$ is a BV_∞ algebra!

A Bibliographic Reference

- L. V., Representations of Homotopy Lie-Rinehart Algebras, e-print: arXiv:1304.4353.

Thank you!