

Hamilton-Jacobi Field Theory

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Introduction: Hamilton-Jacobi Theory

(Q, L) : a Lagrangian system, $L \in C^\infty(TQ)$

$$TQ \xrightarrow{\text{Leg}} T^*Q$$

Remark: If Leg is a diffeomorphism

then $H = E \circ \text{Leg}^{-1} \in C^\infty(T^*Q)$, $E = \Delta L - L$ and

$$(HE) \quad i_{\dot{\sigma}} \omega|_{\sigma} - dH|_{\sigma} = 0, \quad \sigma : \mathbb{R} \rightarrow T^*Q$$

Theorem: (if L is regular) the HE cover the ELE via $T^*Q \rightarrow Q$

One wants to find solutions of the HE/ELE!

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(Q, L) : a regular Lagrangian system, $H \in C^\infty(T^*Q)$: the Hamiltonian

Hamilton-Jacobi Theorem *



$$(Leg^{-1})^*(\dot{q}) = \frac{\partial H}{\partial p}$$

Let $T^*(\omega) = T^*(H) = 0$, then

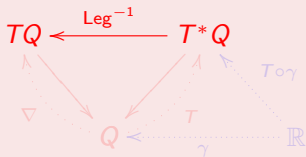
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 - Q is foliated by solutions of the ELE
- 2 for any integral curve γ of ∇ , $T \circ \gamma$ is a solution of the HE.

$im T$ is a Lagrangian submanifold preserved by X_H !

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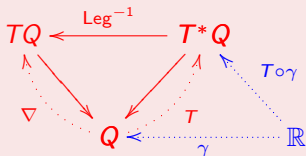
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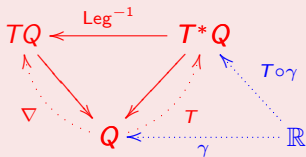
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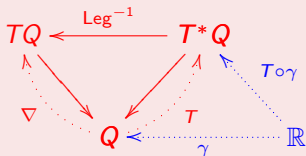
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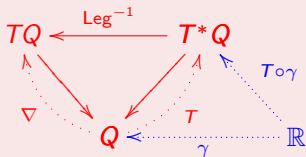
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Introduction: Hamilton-Jacobi Theory

Locally, conditions $T^*(\omega) = T^*(H) = 0$ read

$$T = dS, \quad \frac{\partial}{\partial q} (H(q, \partial S / \partial q)) = 0$$

This is not the whole story: complete solutions, integrability, ...

The Hamilton-Jacobi (HJ) formalism is a cornerstone of the calculus of variations and the theory of Hamiltonian systems. Moreover, it is a first, important step through the quantization of a mechanical system.

The aim of the talk is twofold:

- 1 to present a higher derivative, field theoretic analogue of HJ theory
- 2 to show that, when framed within Secondary Calculus, HJ field theory is just Secondary HJ theory

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Multimomentum Bundles

$\alpha : P \rightarrow M$: a fiber bundle $\implies J^1 \rightarrow P$: 1st jet bundle \star

Definition (extended multimomentum bundle $\mathcal{M}\alpha \rightarrow P$)

Sections of $\mathcal{M}\alpha \rightarrow P =$ affine bundle morphisms $J^1 \rightarrow P \times_M \wedge^n T^*M$

Definition (multimomentum bundle $J^\dagger\alpha \rightarrow P$)

Sections of $J^\dagger\alpha \rightarrow P =$ linear parts of sections of $\mathcal{M}\alpha \rightarrow P$.

Main Properties

- coordinates $\star (x^i, y^a)$ on $P \implies$ coordinates (x^i, y^a, p_a^i, p) \star on $\mathcal{M}\alpha$
- $\mathcal{M}\alpha \rightarrow J^\dagger\alpha$ is a 1-dim. affine bundle with no distinguished section \star
- \exists a tautological n -form on $\mathcal{M}\alpha$: $\Theta_0 = p_a^i dy^a \wedge d^{n-1}x_i - pd^n x \star$

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Hamiltonian Formalism: Ingredients

- $\pi : E \rightarrow M$: a fiber bundle $\implies \pi_k : J^k \rightarrow M$: k th jet bundle,
coordinates (x^i, u) on P \implies coordinates (x^i, u_I) on J^k .
- $\mathbf{S} = \int \mathcal{L}$: a $(k+1)$ th derivative Lagrangian field theory on π :

$$\mathcal{L} = L[x, u] d^n x, \quad L[x, u] \in C^\infty(J^{k+1}).$$

- Θ_0 : the tautological n -form on $\mathcal{M}\pi_k$

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Assumption: $(\partial^2 L[x, u] / \partial u_{k+1} \partial u_{k+1})$ is non-degenerate. \star

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Hamiltonian Formalism: Main Constructions and Results

- Θ : a (dynamical) Hamiltonian n -form on $J^\dagger\pi_k$:

$$\Theta = p^{l,i} du_l \wedge d^{n-1}x_i - H[x, u, p] d^n x,$$

$$H[x, u, p] = p^{l,i} u_{li} - L[x, u] \mid_{u_{k+1} = u_{k+1}[x, p, u_1, \dots, u_k]}$$

- Hamilton-like equations

$$i_{\dot{\sigma}} d\Theta|_{\sigma} = 0, \quad \sigma : M \rightarrow J^\dagger\pi_k$$

locally,

$$(dDE) \quad \begin{cases} p^{l,i},_{i} = -\frac{\partial H}{\partial u_l} \\ u_{l,i} = \frac{\partial H}{\partial p^{l,i}} \end{cases} \quad *$$

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Hamiltonian Formalism: Mechanics vs Field Theory

Mechanics	Field Theory
$TQ \times \mathbb{R} \rightarrow Q \times \mathbb{R}$	$J^{k+1} \rightarrow J^k$
t, q, \dot{q}	x^i, u_I
$T^*Q \times \mathbb{R} \rightarrow Q \times \mathbb{R}$	$J^\dagger \pi_k \rightarrow J^k$
t, q, p	$x^i, u_I, p^{I \cdot i}$
$pdq - Hdt \star$	Θ
$i_{\dot{\sigma}} \omega _{\sigma} - dH _{\sigma} = 0$	$i_{\dot{\sigma}} d\Theta _{\sigma} = 0$

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(π, \mathbf{S}) : a regular Lagrangian field theory, Θ : the Hamiltonian n -form

(Higher Derivative, Field Theoretic) Hamilton-Jacobi Theorem * [LV]



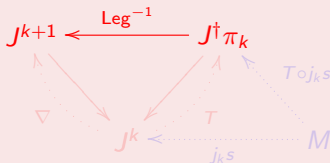
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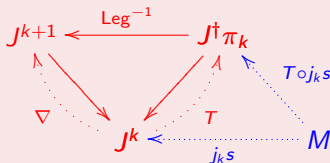
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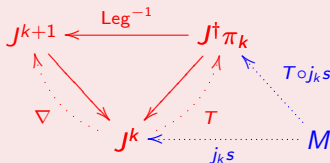
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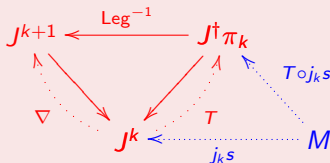
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An Example: the Biharmonic Equation

Locally the (field theoretic) Hamilton-Jacobi equation $T^*(d\Theta) = 0$ reads

$$T : p^{l,i} = \frac{\partial}{\partial u_l} S^i, \quad \frac{\partial}{\partial u_l} \left(\frac{\partial}{\partial x^l} S^i + H[x, u, \partial S / \partial u] \right) = 0.$$

A Toy Example

The biharmonic equation \star

$$\nabla^4 u = 0$$

is EL with second derivative action: $\mathbf{S} = \int \frac{1}{2} u_{ij} u^{ij} d^n x$.

Remark: $p^{ij} = u^{ij}$ and $H = p^i u_i + \frac{1}{2} p^{ij} p_{ij}$. dDE read

$$\left\{ \begin{array}{l} p^i{}_{,i} = 0 \\ p^{ij}{}_{,i} = p^j \\ u_{,i} = u_i \\ u_{j,i} = p_{ij} \end{array} \right.$$

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$$T : p^{l,i} = \frac{\partial}{\partial u_l} S^i, \quad \frac{\partial}{\partial u_l} \left(\frac{\partial}{\partial x^l} S^i + H[x, u, \partial S / \partial u] \right) = 0.$$

A Toy Example

The biharmonic equation \star

$$\nabla^4 u = 0$$

is EL with second derivative action: $\mathbf{S} = \int \frac{1}{2} u_{ij} u^{ij} d^n x$.

Remark: $p^{ij} = u^{ij}$ and $H = p^i u_i + \frac{1}{2} p^{ij} p_{ij}$. dDE read

$$\left\{ \begin{array}{l} p^i{}_{,i} = 0 \\ p^{ij}{}_{,i} = p^j \\ u_{,i} = u_i \\ u_{j,i} = p_{ij} \end{array} \right.$$

which covers $\nabla^4 u = 0$.

An Example: the Biharmonic Equation

The HJE reads

$$\partial_i S^i + u_i \frac{\partial}{\partial u} S^i + \frac{1}{4} \left(\frac{\partial}{\partial u^i} S_j + \frac{\partial}{\partial u^j} S_i \right) \frac{\partial}{\partial u^i} S^j = f(x).$$

If $\phi = \phi(x)$ is a biharmonic function, then

$$S^i := u^j \phi_{,j}{}^i - u \phi_{,j}{}^{j i} + G^i(x)$$

is a solution of the HJE determining the connection $\nabla : J^1 \rightarrow J^2$ given by $\nabla^*(u_{ij}) = \phi_{,ij}$. ∇ is flat with flat sections (foliating J^1) being 1st jets of the biharmonic functions:

$$u = \phi(x) + A_i x^i + B.$$

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Outline

- 1 Hamiltonian Field Theory
- 2 Hamilton-Jacobi Field Theory
- 3 Secondary Hamilton-Jacobi Theory

Secondary Calculus: a Sketch

$\mathcal{E} \subset J^r$ a PDE $\implies \mathcal{E} \subset J^\infty$ its ∞ th prolongation.

\mathcal{E} is a diffeity: i.e., it is endowed with an involutive distribution

$$\mathcal{C} = \langle \dots, D_i, \dots \rangle, \quad D_i = \frac{\partial}{\partial x^i} + u_{li} \frac{\partial}{\partial u_l}.$$

Solutions of \mathcal{E} are \star n -dimensional integral manifolds of \mathcal{C} .

Secondary calculus is a homological formalization of the idea of *differential calculus on the “manifold” of solutions of \mathcal{E}*

Example: secondary vector fields are higher symmetries of \mathcal{E} \star
more precisely, they are cohomologies of a suitable Spencer complex

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$$\dots \rightarrow \mathcal{C}^\bullet \wedge^\bullet \otimes \bar{\Lambda}^q \xrightarrow{\bar{d}} \mathcal{C}^\bullet \wedge^\bullet \otimes \bar{\Lambda}^{q+1} \xrightarrow{\bar{d}} \dots$$

- $\mathcal{C}^\bullet \wedge^\bullet$: Cartan form algebra, generated by $\dots, du_I - u_{Ij} dx^j, \dots$
- $\bar{\Lambda}^\bullet$: horizontal form algebra, generated by \dots, dx^i, \dots
- \bar{d} : horizontal de Rham differential, $\bar{d} = dx^i D_i$

Cartan calculus have a secondary analogue!

Remark: interpretation of secondary functions

- secondary functions of degree n are actions constrained by \mathcal{E}
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(π, \mathbf{S}) : a Lagrangian field theory, $\mathcal{E}_{\text{EL}} \subset J^\infty$: the ∞ th prolong. of ELE

Theorem [Zuckerman]: \mathcal{E}_{EL} possesses a canonical (depending on \mathbf{S} only) secondary (pre)symplectic structure ω (in horizontal degree $n - 1$),

Definition: $(\mathcal{E}_{\text{EL}}, \omega)$ is called the *covariant phase space* (CPS)

Main Properties

- \mathbf{X} a Noether symmetry and \mathbf{f} the corr. conserv. law $\Rightarrow i_{\mathbf{X}}\omega = \mathbf{d}\mathbf{f}$
- There is a “Poisson” bracket among conservation laws
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Secondary Hamilton-Jacobi Problem

(π, \mathbf{S}) : a (regular) $(k + 1)$ th derivative Lagrangian field theory,



$$T^*(d\theta) = 0 \quad (\text{HJE})$$

$\text{im } \nabla \subset J^{k+1}$ may be interpreted as a PDE \mathcal{O} . Let $\mathcal{O} \subset J^\infty$ be its infinite prolongation. \mathcal{O} is finite-dimensional.

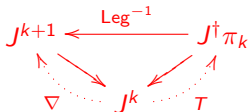
Theorem [LV]: if \exists a solution T of the HJE such that $\nabla = \text{Leg}^{-1} \circ T$ then \mathcal{O} is a (finite-dimensional) isotropic subdiffiety of $(\mathcal{E}_{\text{EL}}, \omega)$. \star

It is natural to formulate the HJ problem in secondary terms

The Secondary Hamilton Jacobi Problem: consists in searching for (finite-dimensional) isotropic subdiffieties of the CPS.

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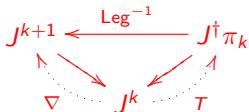
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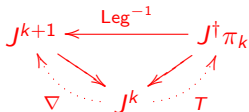
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Conclusions and Perspectives

There exists a natural, geometric, higher derivative, field theoretic version of (part of) the standard HJ formalism in Hamiltonian mechanics.

A solution of the field theoretic HJE determines an isotropic subdiffiety of the CPS \implies secondary Hamilton-Jacobi theory of the CPS.

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HJ theory is a door through *quantization*. (*Complete*) solutions of the HJE determines approximate solutions of the Shroedinger equation. A complete solution maybe understood as a family of solutions parameterized by initial data.

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Secondary HJ theory (semi-classical quantization?) on \mathcal{N} .

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