Hamilton-Jacobi Field Theory

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Introduction: Hamilton-Jacobi Theory

(Q, L): a Lagrangian system, $L \in C^\infty(TQ)$

$$TQ \xrightarrow{\mathrm{Leg}} T^*Q$$

Remark: If Leg is a diffeomorphism

then $H = E \circ \mathrm{Leg}^{-1} \in C^\infty(T^*Q)$, $E = \Delta L - L$ and

$$(HE) \qquad i_{\dot\sigma}\omega|_\sigma - dH|_\sigma = 0, \qquad \sigma : \mathbb{R} \to T^*Q$$

Theorem: (if L is regular) the HE cover the ELE via $T^*Q \to Q$

One wants to find solutions of the HE/ELE!

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(Q, L): a regular Lagrangian system, $H \in C^\infty(T^*Q)$: the Hamiltonian

Hamilton-Jacobi Theorem ⋆

$$\begin{tikzcd}
TQ \arrow[dr] & & T^*Q \arrow[ll, "\mathrm{Leg}^{-1}"'] \arrow[dl] & \\
& Q \arrow[ul, dotted, "\nabla"] \arrow[ur, "T"'] & & \mathbb{R} \arrow[ll, dotted, "\gamma"] \arrow[ul, dotted, "T\circ\gamma"']
\end{tikzcd}$$

$$(\mathrm{Leg}^{-1})^*(\dot q) = \frac{\partial H}{\partial p}$$

Let $T^*(\omega) = T^*(H) = 0$, then

1. integral curves γ of $\nabla := \mathrm{Leg}^{-1} \circ T$ are solutions of the ELE,
    - Q is foliated by solutions of the ELE
2. for any integral curve γ of ∇, $T \circ \gamma$ is a solution of the HE.

im T is a Lagrangian submanifold preserved by X_H!

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Introduction: Hamilton-Jacobi Theory

Locally, conditions $T^*(\omega) = T^*(H) = 0$ read

$$T = dS, \qquad \frac{\partial}{\partial q}\left(H(q, \partial S/\partial q)\right) = 0$$

This is not the whole story: complete solutions, integrability, ...

The Hamilton-Jacobi (HJ) formalism is a cornerstone of the calculus of variations and the theory of Hamiltonian systems. Moreover, it is a first, important step through the quantization of a mechanical system.

The aim of the talk is twofold:

1. to present a higher derivative, field theoretic analogue of HJ theory
2. to show that, when framed within Secondary Calculus, HJ field theory is just Secondary HJ theory

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Multimomentum Bundles

$\alpha : P \to M$: a fiber bundle $\Longrightarrow$ $J^1 \to P$: 1st jet bundle ⋆

Definition (extended multimomentum bundle $\mathscr{M}\alpha \to P$)

Sections of $\mathscr{M}\alpha \to P$ = affine bundle morphisms $J^1 \to P \times_M \wedge^n T^*M$

Definition (multimomentum bundle $J^\dagger\alpha \to P$)

Sections of $J^\dagger\alpha \to P$ = linear parts of sections of $\mathscr{M}\alpha \to P$.

Main Properties

- coordinates ⋆ (x^i, y^a) on $P \Longrightarrow$ coordinates (x^i, y^a, p^i_a, p) ⋆ on $\mathscr{M}\alpha$
- $\mathscr{M}\alpha \to J^\dagger\alpha$ is a 1-dim. affine bundle with no distinguished section ⋆
- $\exists$ a tautological n–form on $\mathscr{M}\alpha$: $\Theta_0 = p^i_a dy^a \wedge d^{n-1}x_i - pd^n x$ ⋆

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Hamiltonian Formalism: Ingredients

- $\pi : E \to M$: a fiber bundle $\Longrightarrow$ $\pi_k : J^k \to M$: kth jet bundle,

  coordinates (x^i, u) on P $\Longrightarrow$ coordinates (x^i, u_I) on J^k.

- $\mathbf{S} = \int \mathscr{L}$: a $(k+1)$th derivative Lagrangian field theory on π:

$$\mathscr{L} = L[x,u]d^n x, \quad L[x,u] \in C^\infty(J^{k+1}).$$

- Θ_0: the tautological n–form on $\mathscr{M}\pi_k$

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Hamiltonian Formalism: Main Constructions and Results

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- Hamilton-like equations

$$i_{\dot\sigma} d\Theta|_\sigma = 0, \qquad \sigma : M \to J^\dagger\pi_k$$

locally,

$$(dDE) \qquad \begin{cases} p^{I.i}{}_{,i} = -\frac{\partial H}{\partial u_I} \\ u_{I,i} = \frac{\partial H}{\partial p^{I.i}} \end{cases} \qquad \star$$

Theorem: the dDE cover the ELE via $J^\dagger\pi_k \to E$

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Hamiltonian Formalism: Mechanics vs Field Theory

Mechanics	**Field Theory**
$TQ \times \mathbb{R} \to Q \times \mathbb{R}$	$J^{k+1} \to J^k$
$t, q, \dot{q}$	x^i, u_I
$T^*Q \times \mathbb{R} \to Q \times \mathbb{R}$	$J^\dagger \pi_k \to J^k$
t, q, p	$x^i, u_I, p^{I.i}$
$pdq - Hdt$ $\star$	Θ
$i_{\dot{\sigma}}\omega\vert_\sigma - dH\vert_\sigma = 0$	$i_{\dot{\sigma}} d\Theta\vert_\sigma = 0$

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Hamilton-Jacobi Theorem

$(\pi, \mathbf{S})$: a regular Lagrangian field theory, Θ: the Hamiltonian n-form

(Higher Derivative, Field Theoretic) Hamilton-Jacobi Theorem ⋆ [LV]

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\qquad (\mathrm{Leg}^{-1})^*(u_{Ii}) = \frac{\partial H}{\partial p^{I \cdot i}}$$

(Remark: ∇ is a connection in π_k) Let $T^*(d\Theta) = 0$, then

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An Example: the Biharmonic Equation

Locally the (field theoretic) Hamilton-Jacobi equation $T^*(d\Theta) = 0$ reads

$$T : p^{I.i} = \frac{\partial}{\partial u_I} S^i, \qquad \frac{\partial}{\partial u_I}\left(\frac{\partial}{\partial x^i} S^i + H[x, u, \partial S/\partial u]\right) = 0.$$

A Toy Example

The biharmonic equation ⋆

$$\nabla^4 u = 0$$

is EL with second derivative action: $\mathbf{S} = \int \frac{1}{2} u_{ij} u^{ij} d^n x$.

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If $\phi = \phi(x)$ is a biharmonic function, then

$$S^i := u^j \phi_{,j}{}^{i} - u\phi_{,j}{}^{ji} + G^i(x)$$

is a solution of the HJE determining the connection $\nabla : J^1 \to J^2$ given by $\nabla^*(u_{ij}) = \phi_{,ij}$. ∇ is flat with flat sections (foliating J^1) being 1st jets of the biharmonic functions:

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Outline

Secondary Calculus: a Sketch

$\mathcal{E} \subset J^r$ a PDE $\Longrightarrow \mathscr{E} \subset J^\infty$ its ∞th prolongation.

$\mathscr{E}$ is a diffiety: i.e., it is endowed with an involutive distribution

$$\mathscr{C} = \langle \ldots, D_i, \ldots \rangle, \quad D_i = \frac{\partial}{\partial x^i} + u_{Ii} \frac{\partial}{\partial u_I}.$$

Solutions of $\mathcal{E}$ are ⋆ n–dimensional integral manifolds of $\mathscr{C}$.

Secondary calculus is a homological formalization of the idea of *differential calculus on the "manifold" of solutions of* $\mathcal{E}$

Example: secondary vector fields are higher symmetries of $\mathcal{E}$ ⋆

more precisely, they are cohomologies of a suitable Spencer complex

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$$\cdots \to \mathscr{C}^\bullet\Lambda^\bullet \otimes \overline{\Lambda}^q \xrightarrow{\overline{d}} \mathscr{C}^\bullet\Lambda^\bullet \otimes \overline{\Lambda}^{q+1} \xrightarrow{\overline{d}} \cdots$$

- $\mathscr{C}^\bullet\Lambda^\bullet$: Cartan form algebra, generated by $\ldots, du_I - u_{Ii}dx^i, \ldots$
- $\overline{\Lambda}^\bullet$: horizontal form algebra, generated by $\ldots, dx^i, \ldots$
- $\overline{d}$: horizontal de Rham differential, $\overline{d} = dx^i D_i$

Cartan calculus have a secondary analogue!

Remark: interpretation of secondary functions

- secondary functions of degree n are actions constrained by $\mathcal{E}$
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Covariant Phase Space

$(\pi, \mathbf{S})$: a Lagrangian field theory, $\mathscr{E}_{\mathrm{EL}} \subset J^{\infty}$: the ∞th prolong. of ELE

Theorem [Zuckerman]: $\mathscr{E}_{\mathrm{EL}}$ possesses a canonical (depending on $\mathbf{S}$ only) secondary (pre)symplectic structure ω (in horizontal degree $n-1$),

Definition: $(\mathscr{E}_{\mathrm{EL}}, \omega)$ is called the *covariant phase space* (CPS)

Main Properties

- $\mathbf{X}$ a Noether symmetry and $\mathbf{f}$ the corr. conserv. law $\Rightarrow i_{\mathbf{X}}\omega = \mathbf{df}$
- There is a "Poisson" bracket among conservation laws
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$$\begin{tikzcd} J^{k+1} \arrow[dr] & & J^{\dagger}\pi_k \arrow[ll, "\mathrm{Leg}^{-1}"'] \arrow[dl] \\ & J^k \arrow[ul, bend left, dotted, "\nabla"] \arrow[ur, bend right, dotted, "T"'] & \end{tikzcd} \qquad T^*(d\Theta) = 0 \quad \text{(HJE)}$$

$\mathrm{im}\,\nabla \subset J^{k+1}$ may be interpreted as a PDE $\mathcal{O}$. Let $\mathscr{O} \subset J^\infty$ be its infinite prolongation. $\mathscr{O}$ is finite-dimensional.

Theorem [LV]: if $\exists$ a solution T of the HJE such that $\nabla = \mathrm{Leg}^{-1} \circ T$

then $\mathscr{O}$ is a (finite-dimensional) isotropic subdiffiety of $(\mathscr{E}_{\mathrm{EL}}, \omega)$. $\star$

It is natural to formulate the HJ problem in secondary terms

The Secondary Hamilton Jacobi Problem: consists in searching for

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There exists a natural, geometric, higher derivative, field theoretic version of (part of) the standard HJ formalism in Hamiltonian mechanics.

A solution of the field theoretic HJE determines an isotropic subdiffiety of the CPS $\Longrightarrow$ secondary Hamilton-Jacobi theory of the CPS.

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- Can one define the concept of Lagrangian subdiffiety of the CPS?
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