Dirac-Jacobi Bundles

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- presymplectic geometry,
- Poisson geometry.

*Dirac geometry* is a common extension of both!

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\mathbb{T}M := TM \oplus T^*M.
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The main structures on $TM = TM \oplus T^*M$ are:
- the projection $\text{pr}_T : TM \to TM$,
- the symmetric bilinear form $\langle \langle - , - \rangle \rangle : TM \otimes TM \to \mathbb{R}_M$:
  \[
  \langle \langle (X, \sigma), (Y, \tau) \rangle \rangle := \tau(X) + \sigma(Y),
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Definition

A Dirac manifold is a manifold $M + a$ Dirac structure, i.e. a maximally isotropic subbundle $\mathcal{L} \subset TM$ such that $\llbracket \Gamma(\mathcal{L}), \Gamma(\mathcal{L}) \rrbracket \subset \Gamma(\mathcal{L})$.

Examples

- graphs of presymplectic forms $\omega : TM \to T^*M$,
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Luca Vitagliano  Dirac-Jacobi Bundles  3 / 14
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- *precontact geometry*,
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**Definition**

A *precontact manifold* is a manifold + an hyperplane distribution.

**Definition**

A *Jacobi manifold* is a manifold \( M \) + a *Jacobi bundle*, i.e. a line bundle \( L \to M \) equipped with a Lie bracket on sections

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J : \Gamma(L) \times \Gamma(L) \to \Gamma(L)
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which is a 1\(^{st}\) order DO in each entry.

Every contact manifold is both a precontact and a Jacobi manifold.

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*There is a common extension of both precontact and Jacobi geometry.*
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Contact Geometry Revisited

A contact manifold is a manifold $M$ + a maximally non-integrable hyperplane distribution $H \subset TM$. Dually $H = \ker(\theta : TM \to L)$.

**Definition**

Sections of the Atiyah algebroid $DE \to M$ of a vector bundle $E \to M$ are $\mathbb{R}$-linear operators $\Delta : \Gamma(E) \to \Gamma(E)$ such that

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Atiyah forms are cochains in $(\Omega^*_E := \Gamma(\bigwedge^\bullet(DE)^* \otimes E), d_{DE})$.

**Proposition**

Precontact structures $H$ with $TM/H = L$ are in 1-1 correspondence with (nowhere vanishing) $d_{DL}$-closed Atiyah 2-forms on $L$. $H$ corresponds to $\omega := d_{DL}(\theta \circ \sigma)$. $H$ is contact iff $\omega$ is non-degenerate.

**Symplectic to Contact Dictionary Principle**

A contact analogue of a construction in symplectic geometry can be defined replacing the tangent bundle with the Atiyah algebroid of $L \to M$. 
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A contact analogue of a construction in symplectic geometry can be defined replacing the tangent bundle with the Atiyah algebroid of $L \to M$. 
The arena for Dirac-Jacobi geometry is the \textit{omni-Lie algebroid}:

\[ \mathcal{DL} := DL \oplus J^1L \quad \text{(notice that } J^1L = (DL)^* \otimes L). \]

The main structures on \( \mathcal{DL} \) are:

- the projection \( \text{pr}_D : \mathcal{DL} \to DL \),
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A \textit{Dirac-Jacobi bundle} is a line bundle \( L \to M \) + a \textit{Dirac-Jacobi structure}, i.e. a maximally isotropic subbundle \( \mathcal{L} \subset \mathcal{DL} \) such that \( [[\Gamma(\mathcal{L}), \Gamma(\mathcal{L})]] \subset \Gamma(\mathcal{L}). \)
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A *Dirac-Jacobi bundle* is a line bundle \( L \to M \) + a *Dirac-Jacobi structure*, i.e. a maximally isotropic subbundle \( \mathcal{L} \subset IDL \) such that \( \lbrack \Gamma(\mathcal{L}),\Gamma(\mathcal{L}) \rbrack \subset \Gamma(\mathcal{L}) \).
Characteristic Foliation

Examples

- graphs of Atiyah forms \( \omega : DL \to J^1L \) of precontact structures,
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Let \( \mathcal{L} \subset DL \) be a Dirac-Jacobi structure

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A submanifold $S$ of a Jacobi manifold $(M, L)$ is **coisotropic** if sections of $L$ vanishing on $M$ are closed under the Jacobi bracket.

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Let $S \subset M$ be a coisotropic submanifold. (Under clean intersection) the restricted line bundle $L|_S \rightarrow S$ carries an induced Dirac-Jacobi structure. To see this, restrict to $S$ the lcs/contact foliation of $M$.

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A submanifold $S$ of a Jacobi manifold $(M, L)$ is coisotropic if sections of $L$ vanishing on $M$ are closed under the Jacobi bracket.

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Let $S \subset M$ be a coisotropic submanifold. (Under clean intersection) the restricted line bundle $L|_S \to S$ carries an induced Dirac-Jacobi structure. To see this, restrict to $S$ the lcs/contact foliation of $M$.

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Let $(L \to S, \xi)$ be a Dirac-Jacobi bundle. $S$ can be coisotropically embedded in a manifold equipped with a Jacobi bundle iff $\text{rank } \ker \omega_L = \text{const.}$
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- a Lie algebroid $A$ may integrate to a Lie groupoid $G$,  
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$(\mathcal{L}, [[-,-]], \sigma_{pr_D})$ is a Lie algebroid, and $L$ carries a representation of $\mathcal{L}$.

**Definition**

A precontact groupoid is a triple $(\mathcal{G}, L, \theta)$ where

1. $\mathcal{G} \rightarrow M$ is a Lie groupoid with $\dim \mathcal{G} = 2 \dim M + 1$,
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Luca Vitagliano

Dirac-Jacobi Bundles

11 / 14
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Luca Vitagliano  
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11 / 14
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<table>
<thead>
<tr>
<th></th>
<th>( \mathcal{E}^1(M) )-Dirac</th>
<th>Dirac-Jacobi in ( \mathbb{D}L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>definition</td>
<td>[Wade 2000]</td>
<td>[V 2015]</td>
</tr>
<tr>
<td>characteristic foliation</td>
<td>[Iglesias &amp; Marrero 2002]</td>
<td>[V 2015]</td>
</tr>
<tr>
<td>Jacobi reduction</td>
<td>—</td>
<td>[V 2015]</td>
</tr>
<tr>
<td>coisotropic embeddings</td>
<td>—</td>
<td>[V 2015]</td>
</tr>
<tr>
<td>groupoid counterpart</td>
<td>[Iglesias &amp; Wade 2006]</td>
<td>[V 2015]</td>
</tr>
<tr>
<td>gauge transformations</td>
<td>—</td>
<td>[V 2015]</td>
</tr>
<tr>
<td>local structure</td>
<td>—</td>
<td>[V 2015]</td>
</tr>
<tr>
<td>backward-forward maps</td>
<td>—</td>
<td>[V 2015]</td>
</tr>
<tr>
<td>\textit{Dirac-ization}</td>
<td>[Iglesias &amp; Marrero 2002]</td>
<td>[V 2015]</td>
</tr>
<tr>
<td>generalized geometry</td>
<td>[Iglesias &amp; Wade 2005]</td>
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</tr>
</tbody>
</table>
D. Iglesias-Ponte, and J.C. Marrero, 
*Lie algebroid foliations and $\mathcal{E}^1(M)$-Dirac structures*, 

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*Contact manifolds and generalized complex structures*, 

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*Integration of Dirac-Jacobi structures*, 

L. V.,  
*Dirac-Jacobi bundles*, 

A. Wade, 
*Conformal Dirac structures*, 
Thank you!