A.M. Vinogradov's Cohomological Geometry of PDEs

to my teacher

Luca Vitagliano

University of Salerno, Italy

Diffieties, Cohomological Physics, and Other Animals
Alexandre Vinogradov Memorial Conference

December 14, 2021

ALEXANDRE M. VINOGRADOV : MY TEACHER

he taught me how to think of

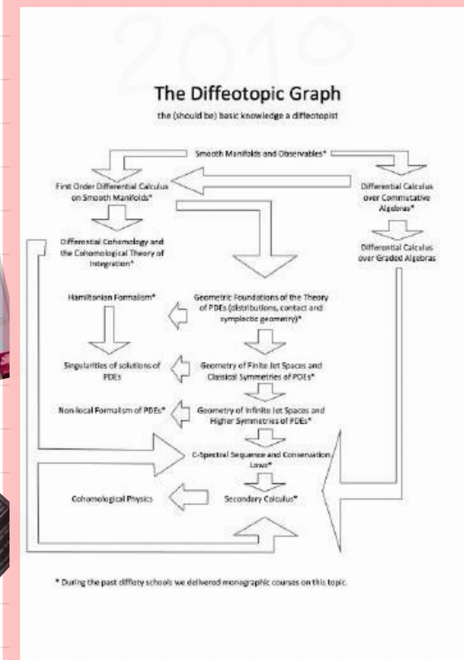
- SPACES
- GEOMETRIC STRUCTURES

VINOGRADOV'S
SECONDARY CALCULUS
=
GEOMETRY ON THE SPACE OF
SOLUTIONS OF A PDE

A 21 YEARS LONG STORY 1998 → 2019

> 20 DIFFIETY SCHOOLS

- dozens classes given
- several different topics
- 3 different countries



DIFFIETY SCHOOL

International School on

Geometry of PDEs & Secondary Calculus

FIRST ITALIAN EDITION: Summer 1998

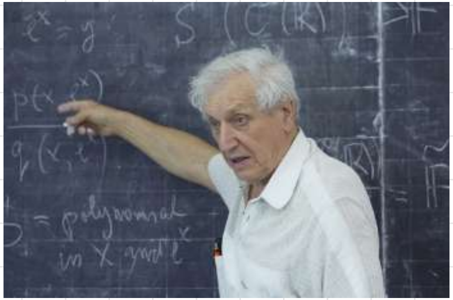
<u>not an ordinary school</u>

<u>but a mission</u>

MAIN AIM: CREATING A TEAM

Mathematical Industrialization

A PERSONAL VIEW-POINT



LEARNING FROM VINOGRADOV

DOING RESEARCH WITH VINOGRADOV

LEARNING SECONDARY CALCULUS

t

1998

2000

2004

2008

2010

2019

2021

Vinogradov's Seminar

PostDoc under Vinogradov's supervision

Today

my first Diffiety School

Vinogradov's retirement

organizing Diffiety Schools

VINOGRADOV'S SECONDARY CALCULUS

(an informal introduction)

∞ - Prolongation

$x = (x^1, \dots, x^n)$ independent variables

$u = (u^1, \dots, u^m)$ dependent variables

$\mathcal{S} : \{ F(x, \dots, u_I, \dots) = 0$ system of PDEs

$\Downarrow$ $\quad D_i = \frac{\partial}{\partial x^i} + \sum_I u_{Ii} \frac{\partial}{\partial u_I}$ <u>total derivatives</u>

$$\mathcal{S}_\infty : \begin{cases} F = 0 \\ D_i F = 0 \\ D_i D_j F = 0 \\ \vdots \end{cases} \rightleftharpoons \mathcal{E} \subseteq J^\infty$$

∞-jet space coordinatized by $(x, \dots, u_I, \dots)$

VINOGRADOV'S SECONDARY CALCULUS

(an informal introduction)

Diffieties

$\mathcal{E} \subseteq J^\infty$

$\mathcal{C} = \langle \ldots, D_i|_{\mathcal{E}}, \ldots \rangle_{i=1,\ldots,n}$ CARTAN DISTRIBUTION on $\mathcal{E}$

Def. $(\mathcal{E}, \mathcal{C})$ is a (elementary) diffiety. $\Rightarrow$ Diffiety School!

Rem. $\mathcal{C}$ is involutive and

$\{\text{solutions of } \mathcal{S}\} \rightleftarrows \{n\text{-dim int. submfds of } (\mathcal{E}, \mathcal{C})\}$

(but Frobenius fails!)

VINOGRADOV'S SECONDARY CALCULUS

(an informal introduction)

Horizontal Cohomology

IDEA: functions, vector fields, differential forms, etc on the space of solutions of $\mathcal{S}$ are horizontal cohomology!

Def The horizontal De Rham complex of $\mathcal{E}$ is the "leaf-wise" De Rham complex of the involutive distribution $\mathcal{C}$ on $\mathcal{E}$.

$$\left(\bar{\Omega}^{\bullet}(\mathcal{E}) = \Gamma(\Lambda^{\bullet}\mathcal{C}^{*}) \, , \; \bar{d} = dx^i D_i \right)$$

$$\bar{H}^{\bullet}(\mathcal{E}) := H^{\bullet}\left(\bar{\Omega}(\mathcal{E}), \bar{d}\right)$$

horizontal De Rham cohomology

VINOGRADOV'S SECONDARY CALCULUS

(an informal introduction)

Secondary Functions I

$\mathbb{S} = \int L(x_1, \dots, u_I, \dots)\, d^n x$

variational principle imposed on $u = u(x)$

$\mathcal{L} := L\, d^n x$ LAGRANGIAN DENSITY, $\mathcal{L} \in \overline{\Omega}^n(\mathcal{E})$

STOKES THEOREM $\Longrightarrow$ $\mathbb{S} \rightleftarrows [\mathcal{L}] \in \overline{H}^n(\mathcal{E})$

$\overline{H}^n(\mathcal{E})$ $\rightleftarrows$ variational principles constrained by $\mathcal{S}$

$\overline{H}^{n-1}(\mathcal{E})$ $\rightleftarrows$ conservation laws for $\mathcal{S}$

$\overline{H}^{n-2}(\mathcal{E})$ $\rightleftarrows$ gauge charges

⋮

VINOGRADOV'S SECONDARY CALCULUS

(an informal introduction)

Secondary Functions II

VINOGRADOV'S SECONDARY FUNCTION PRINCIPLE

$\overline{H}^{\circ}(\mathcal{E}) :=$ { smooth functions on the space of solutions of $\mathcal{S}$ }

Secondary Functions $C^{\infty}(\mathcal{E})$

VINOGRADOV'S SECONDARY CALCULUS PRINCIPLE

Differential calculus on the space of solutions of $\mathcal{S}$ is differential calculus up to homotopy on $(\overline{\Omega}^{\bullet}(\mathcal{E}), \overline{d})$

VINOGRADOV'S SECONDARY CALCULUS

(an informal introduction)

Secondary Vector Fields

Rem. There are local coefficients for $\overline{H}^{\cdot}(\mathcal{E})$.

Exmpl. $V\mathfrak{X} := \Gamma(T\mathcal{E}/\mathcal{C})$ BOTT CONNECTION

$\overline{\Omega}^{\cdot}(\mathcal{E}) \otimes V\mathfrak{X}$ is a DG module over $(\overline{\Omega}^{\cdot}(\mathcal{E}), \overline{d})$

$$\Rightarrow \quad \overline{H}^{\cdot}(\mathcal{E}, V\mathfrak{X}) := H^{\cdot}(\overline{\Omega}(\mathcal{E}) \otimes V\mathfrak{X}, \overline{d})$$

$\overline{H}^{0}(\mathcal{E}, V\mathfrak{X}) \rightleftarrows$ nontrivial infinitesimal symmetries of $\mathcal{S}$

VINOGRADOV'S SECONDARY VECTOR FIELD PRINCIPLE

$\overline{H}^{\cdot}(\mathcal{E}, V\mathfrak{X}) := \{$vector fields on the space of solutions of $\mathcal{S}\}$

Secondary Vector Fields $\mathfrak{X}(\mathcal{E})$

VINOGRADOV'S SECONDARY CALCULUS

(an informal introduction)

Secondary Differential Forms

<u>Exmpl.</u> $V\Omega^1 := \Gamma((T\mathcal{E}/\mathcal{C})^* = \mathrm{Ann}\,\mathcal{C})$ DUAL BOTT CONNECTION

$\overline{\Omega}^\bullet(\mathcal{E}) \otimes V\Omega^1$ is a DG module over $(\overline{\Omega}^\bullet(\mathcal{E}), \overline{d})$

$$\Longrightarrow \quad \overline{H}^\bullet(\mathcal{E}, V\Omega^1) := H^\bullet(\overline{\Omega}(\mathcal{E}) \otimes V\Omega^1, \overline{d})$$

VINOGRADOV'S SECONDARY 1-FORM PRINCIPLE

$\overline{H}^\bullet(\mathcal{E}, V\Omega^1) := \{$ 1-forms on the space of solutions of $\mathcal{S}$ $\}$

Secondary 1-Forms $\Omega^1(\mathcal{E})$

Vinogradov's $\mathcal{C}$-spectral sequence

Rem. There is $d\ell : \mathcal{C}^\infty(\mathcal{E}) \longrightarrow \Omega^1(\mathcal{E})$ such that

$$d\ell\,[\mathcal{L}] = EL(\mathcal{L})$$

variational principle

lhs of Euler-Lagrange eqs.

Vinogradov's $\mathcal{C}$-spectral sequence

0

$$\begin{array}{c}
\bar{\Omega}^n(\mathcal{E}) \\
\uparrow \\
\bar{\Omega}^{n-1}(\mathcal{E}) \\
\uparrow \\
\vdots
\end{array}$$

1

$$\begin{array}{ccccccc}
\bar{H}^n(\mathcal{E}) & \longrightarrow & \bar{H}^n(\mathcal{E}, V\Omega^1) & \longrightarrow & \bar{H}^n(\mathcal{E}, V\Omega^2) & \longrightarrow & \cdots \\
\bar{H}^{n-1}(\mathcal{E}) & \longrightarrow & \bar{H}^{n-1}(\mathcal{E}, V\Omega^1) & \longrightarrow & \bar{H}^{n-1}(\mathcal{E}, V\Omega^2) & \longrightarrow & \cdots \\
\vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

$$\longrightarrow H_{dR}(\mathcal{E})$$

$$0 \longrightarrow \mathcal{C}^\infty(\mathcal{E}) \xrightarrow{d\ell} \Omega^1(\mathcal{E}) \xrightarrow{d\ell} \Omega^2(\mathcal{E}) \longrightarrow \cdots$$

2004-2008: SEMINARS IN THE MOUNTAINS

- Weekly Seminars
- Exercise Sessions
- Reading Groups

⋮

DOING RESEARCH with VINOGRADOV

Differential Calculus over Graded Algebras

$\mathcal{A} \longmapsto \Omega^{\bullet}(\mathcal{A})$ ↳ Supergeometry

graded algebra

Graded Geometry

$M \longmapsto \Omega^{\bullet}(M) \longmapsto \Omega^{\bullet}(\Omega^{\bullet}(M)) \longmapsto \cdots \longmapsto \Omega_{\infty}$

manifold — DeRham Algebra — Iterated DFs

Vinogradov's Conjecture:

Iterated differential forms can be used to build GRAND UNIFICATION SCHEMES in Particle Physics

2010 → ∞ : RETIREMENT & BEYOND

A Silver Caravel for New Trips

- More Diffiety Schools (Poland, Italy)
- Networking Projects (COST, ...)
- A New Book

AN UNFINISHED BOOK:
DE RHAM COHOMOLOGY & INTEGRATION

PLAN **Algebraic / Cohomological Integration**

<u>without Riemann/Lebesgue</u>

(except for existence & uniqueness of solutions of $\frac{df}{dx}(x) = g(x)$)

Algebraic Topology with Differential Forms

<u>no Singular Homology, Sheaf Cohomology, De Rham Thm</u>

VINOGRADOV'S PRINCIPLE Differential Calculus over Comm. Algebras is the main thing (even for topological issues).

2019: AN IMPRESSIVE SPEECH

Diffiety School 2017, Lizzano in Belvedere

2015

2006

1998

2005

A.M. VINOGRADOV

The limits of my language are the limits of my world

L. WITTGENSTEIN

2007

2004

2017

2010

2011