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n-ary Batalin–Vilkovysky brackets

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The first very short work by Alexandre Vinogradov on differential topics appeared in 1972. It was called "The logic algebra for the theory of linear differential operators" and contained the construction of basic functors of differential calculus in commutative algebras.

His latest work ([4], 2016) on this topic was similarly titled "Logic of differential calculus and the zoo of geometric structures" and contained, in his words, "zoo" of geometrical structures having a common source in the calculus of functors of differential calculus over commutative algebras.

Among these structures, the Batalin-Vilkovysky brackets were mentioned but not described.
The purpose of my talk is to fill this gap.

Graded commutative algebras

I will have to start by repeating the basic definitions.

Recall that a *G-graded commutative algebra* over a field $\Bbbk$, $\operatorname{char}\Bbbk \neq 2$, is a triple $(\mathcal{A}, G, \langle\cdot,\cdot\rangle)$ where:

1. $\mathcal{A}$ is an associative $\Bbbk$-algebra;
2. G is a commutative (Abelian) semigroup with unit (which may sometimes be a group);
3. $\mathcal{A} = \oplus_{g\in G}\mathcal{A}^g$, where all $\mathcal{A}^g$ are $\Bbbk$-vector spaces and $\mathcal{A}^g \cdot \mathcal{A}^q \subset \mathcal{A}^{g+q}$ for all $g, q \in G$. If $a \in \mathcal{A}^g$, then g is called *grading* of a and is denoted by $\omega(a)$, while a is called a *homogeneous element of grading g*;
4. The semigroup G is supplied with a *parity form* $\langle\cdot,\cdot\rangle$, i.e., a $\mathbb{Z}$-bilinear symmetric map $\langle\cdot,\cdot\rangle\colon G \times G \to \mathbb{Z}_2$, and $ab = (-1)^{\langle\omega(a),\omega(b)\rangle} ba$ for any homogeneous elements $a,\ b \in \mathcal{A}$.

An important and often arising particular case of a parity form is the form that can be constructed from *parity homomorphism* $\rho\colon G \to \mathbb{Z}_2$, $\langle g_1, g_2\rangle_\rho = \rho(g_1)\cdot\rho(g_2)$. Parity forms constructed via the parity homomorphism will be called *decomposable*.

In what follows, when dealing with elements of graded algebras or of the other graded objects defined below, we shall assume that they are homogeneous.

Concepts such as left and right $\mathcal{A}$-modules, $\mathcal{A}$-modules homomorphisms, and so on can be defined in the usual way. I will omit the details.

If $\mathcal{U}, \mathcal{V}$ are arbitrary G-graded objects, then to any homogeneous elements $u \in \mathcal{U},\ v \in \mathcal{V}$, we can assign the element $\langle u, v\rangle \in \mathbb{Z}_2$ by setting

$$\langle u, v\rangle \stackrel{\text{def}}{=} \langle \omega(u), \omega(v)\rangle.$$

If the parity form was constructed from the corresponding homomorphism, then it is sometimes convenient to use the notation $|u|$ instead of $\rho(\omega(u))$. In that case $\langle u, v\rangle = |u|\cdot|v|$.

The operators of left and right multiplication

Let $\mathcal{A}$ be a G-graded commutative algebra over $\Bbbk$, $\mathcal{P}$ and $\mathcal{Q}$ be G-graded $\mathcal{A}$-modules. To each element $a \in \mathcal{A}$, we assign the operators

$$l_a, \;\; r_a \text{ и } \delta_a \colon \; \mathrm{Hom}_{\Bbbk}(\mathcal{P}, \mathcal{Q}) \to \mathrm{Hom}_{\Bbbk}(\mathcal{P}, \mathcal{Q}),$$

defined as follows:

$$\begin{aligned} l_a(\varphi)(p) &= a\,\varphi(p), \quad \varphi \in \mathrm{Hom}_{\Bbbk}(\mathcal{P}, \mathcal{Q}), \quad p \in \mathcal{P}, \\ r_a(\varphi)(p) &= (-1)^{\langle \varphi, a\rangle} \varphi(ap), \\ \delta_a(\varphi) &= r_a(\varphi) - l_a(\varphi). \end{aligned}$$

The operators l_a and r_a of left and right multiplication of elements of the $\Bbbk$-module $\mathrm{Hom}_{\Bbbk}(\mathcal{P}, \mathcal{Q})$ by elements of the algebra $\mathcal{A}$ obviously commute. This allows us to define, in $\mathrm{Hom}_{\Bbbk}(\mathcal{P}, \mathcal{Q})$, the structure of a bimodule over the algebra $\mathcal{A}$. Let $a_0, \ldots, a_s \in \mathcal{A}$; then we put

$$\delta_{a_0, \ldots, a_s} = \delta_{a_0} \circ \ldots \circ \delta_{a_s}.$$

An element $\Delta \in \mathrm{Hom}_{\Bbbk}(\mathcal{P}, \mathcal{Q})$ will be called a *$\Bbbk$-linear differential operator* (DO) of order $\leqslant s$ over $\mathcal{A}$ if, for any tuple of elements $a_0, a_1, \ldots, a_s \in \mathcal{A}$,

$$\delta_{a_0,\ldots,a_s}(\Delta) = 0.$$

The set of all DO's of order $\leqslant s$ from $\mathcal{P}$ to $\mathcal{Q}$ is stable with respect to the left as well as to the right multiplication by elements of the algebra $\mathcal{A}$ and is therefore supplied with two natural G-graded $\mathcal{A}$-module structures. The G-graded $\mathcal{A}$-module defined by the left structure will be denoted by $\mathrm{Diff}_s(\mathcal{P}, \mathcal{Q})$.
By definition, $\mathrm{Hom}_{\mathcal{A}}(\mathcal{P}, \mathcal{Q}) = \mathrm{Diff}_0(\mathcal{P}, \mathcal{Q})$.
Obviously, we have the sequence of natural embeddings

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{P}, \mathcal{Q}) = \mathrm{Diff}_0(\mathcal{P}, \mathcal{Q}) \subset \ldots \subset \mathrm{Diff}_l(\mathcal{P}, \mathcal{Q}) \subset \mathrm{Diff}_{l+1}(\mathcal{P}, \mathcal{Q}) \subset \ldots$$

The map $\Delta \colon \mathcal{A} \to \mathcal{P}$ is called a derivation with values in the A-module $\mathcal{P}$ if it satisfies the graded Leibnitz rule $\Delta(ab) = \Delta(a)b + (-1)^{\langle a,\Delta\rangle} a\Delta(b)$.

The set of all derivations from $\mathcal{A}$ to $\mathcal{P}$ is denoted by $D(\mathcal{P})$.
Obviously, $D(\mathcal{P})$ is a submodule of $\mathrm{Diff}_1(\mathcal{A}, \mathcal{P})$.

Symbols

From now we will simply write $\operatorname{Diff}_s \mathcal{Q}$ instead of $\operatorname{Diff}_s(\mathcal{A}, \mathcal{Q})$.

The obvious embedding of $\mathcal{A}$-modules $\operatorname{Diff}_{k-1} \mathcal{A} \subset \operatorname{Diff}_k \mathcal{A}$ allows us to define the quotient module

$$\mathcal{S}_k(\mathcal{A}) \stackrel{\text{def}}{=} \operatorname{Diff}_k \mathcal{A} / \operatorname{Diff}_{k-1} \mathcal{A},$$

which is called the *module of symbols of order k* (or the module of *k-symbols*). The coset of an operator $\Delta \in \operatorname{Diff}_k \mathcal{A}$ modulo $\operatorname{Diff}_{k-1} \mathcal{A}$ will be denoted by $\operatorname{smbl}_k \Delta$ and called the *symbol* of Δ. Let us define the *algebra of symbols* $\mathcal{S}_*(\mathcal{A})$ for the algebra $\mathcal{A}$ by setting

$$\mathcal{S}_*(\mathcal{A}) = \bigoplus_{n=0}^{\infty} \mathcal{S}_n(\mathcal{A}).$$

Putting $\operatorname{smbl}_k \Delta \cdot \operatorname{smbl}_n \nabla \stackrel{\text{def}}{=} \operatorname{smbl}_{k+n}(\Delta \circ \nabla)$ we equip $\mathcal{S}_*(\mathcal{A})$ with the structure of the $\mathcal{A}$-algebra. It is easy to check that $\operatorname{smbl}_k \Delta \cdot \operatorname{smbl}_n \nabla = (-1)^{\langle \Delta, \nabla \rangle} \operatorname{smbl}_n \nabla \cdot \operatorname{smbl}_k \Delta$, therefore $\mathcal{S}_*(\mathcal{A})$ is a commutative graded algebra.

Brackets defined by symbols

Let $a_1, \dots, a_n \in \mathcal{A}$ and $\Delta \in \operatorname{Diff}_n \mathcal{A}$. It is easy to prove that the map $\operatorname{smbl}_n(\Delta)\colon \mathcal{A} \times \dots \times \mathcal{A} \to \mathcal{A}$ given by

$$\big(\operatorname{smbl}_n(\Delta)\big)(a_1, \dots, a_n) \mapsto \delta_{a_1,\dots,a_n}(\Delta)$$

is well defined, depends only on the symbol of the operator Δ, and is linear in each argument.
Set

$$[a_1, \dots, a_n]_\Delta \overset{\text{def}}{=} \big(\operatorname{smbl}_n(\Delta)\big)(a_1, \dots, a_n).$$

(I stress that $[a_1, \dots, a_n]_\Delta$ depends not on the operator Δ itself, but only on its symbol.)

Multiindex notations

We put $I^n \stackrel{\text{def}}{=} (1, \dots, n)$, by the letters $I,\ J$ we will always denote ordered multiindices

$$I = (i_1, \dots, i_k),\ 1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n,$$
$$J = (j_1, \dots, j_l),\ 1 \leqslant j_1 < j_2 < \dots < j_l \leqslant n.$$

Then, we put $|I| = k,\ |J| = l$. If $i_r \neq j_s$ for all $r \leqslant k,\ s \leqslant l$, then we can consider the nonordered multiindex $(I, J) \stackrel{\text{def}}{=} (i_1, \dots, i_k, j_1, \dots, j_l)$.
The sum $I + J$ will be defined as the multiindex obtained by ordering the multiindex (I, J).
Let $a_1, \dots, a_n \in \mathcal{A}$ be homogeneous elements and let $I = (i_1, \dots, i_n)$ be an ordered multiindex; the ordered set $a_{i_1}, \dots, a_{i_n}$ will be denoted by $a(I)$.
Set $\omega(a(I)) = \sum_{i \in I} \omega(a_i)$.

Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ be a permutation of the natural numbers $1, 2, \dots, n$, let

$$a(\sigma) = (a_{\sigma_1}, a_{\sigma_2}, \dots, a_{\sigma_n})$$

be a set of homogeneous elements from $\mathcal{A}$. Denote by $\Pi_{a(\sigma)}$ the symmetric product of elements $(a_{\sigma_1}, a_{\sigma_2}, \dots, a_{\sigma_n})$:

$$\Pi_{a(\sigma)} \stackrel{\text{def}}{=} a_{\sigma_1} \odot a_{\sigma_2} \odot \dots \odot a_{\sigma_n}.$$

Products $\Pi_{a(\sigma)}$ and $\Pi_{a(I^n)}$ may differ in sign:

$$\Pi_{a(I^n)} = (-1)^r \Pi_{a(\sigma)}$$

Set, by definition, $|a(\sigma)|_s \stackrel{\text{def}}{=} r$. Now let the multiindices I and J be such that $I + J = I^n$. It is easy to check, that

$$|a(I, J)|_s = |a(J, I)|_s + \langle a(I), a(J) \rangle. \tag{1}$$

Action of an operator δ_a

The operator δ_a acts on the composition of DO's as a derivation:

$$\delta_a(\Delta \circ \nabla) = \delta_a(\Delta) \circ \nabla + (-1)^{\langle a, \Delta \rangle} \Delta \circ \delta_a(\nabla). \tag{2}$$

Iterating the last equality, we get

$$\delta_{a(I^m)}(\Delta \circ \nabla) = \sum_{\substack{I+J=I^m \\ 0 \leqslant |I| \leqslant m}} (-1)^{|a(I,J)|_s + \langle a(J), \Delta \rangle} \delta_{a(I)}(\Delta) \circ \delta_{a(J)}(\nabla) \tag{3}$$

All the required bracket structures are obtained from this basic formula.

Now assume that $\nabla = \Delta \in \mathrm{Diff}_n \mathcal{A}$ and $m = 2n - 1$; then in the decomposition (3) only the terms with $|I| = n, |J| = n - 1$, and $|I| = n - 1, |J| = n$ will be nonzero, and so in this case the equality (3) can be given the following form:

$$\delta_{a(I^{2n-1})}(\Delta \circ \Delta) = \sum_{\substack{I+J=I^{2n-1} \\ n-1 \leqslant |I| \leqslant n}} (-1)^{|a(I,J)|_s + \langle a(J), \Delta \rangle} \delta_{a(I)}(\Delta) \circ \delta_{a(J)}(\Delta) \tag{4}$$

In this sum, to each index I, $|I| = n$, corresponds two summands:

$$
\begin{aligned}
&(-1)^{|a(I,J)|_s+\langle a(J),\Delta\rangle}\delta_{a(I)}(\Delta)\circ\delta_{a(J)}(\Delta)\\
+&(-1)^{|a(J,I)|_s+\langle a(I),\Delta\rangle}\delta_{a(J)}(\Delta)\circ\delta_{a(I)}(\Delta)\\
=&(-1)^{|a(I,J)|_s+\langle a(J),\Delta\rangle}\delta_{a(I)}(\Delta)\circ\delta_{a(J)}(\Delta)\\
+&(-1)^{|a(I,J)|_s+\langle a(I),a(J)\rangle+\langle a(I),\Delta\rangle}\delta_{a(J)}(\Delta)\circ\delta_{a(I)}(\Delta)\\
=&(-1)^{|a(I,J)|_s+\langle a(J),\Delta\rangle+\langle\Delta,\Delta\rangle}\left[(-1)^{\langle\Delta,\Delta\rangle}\delta_{a(I)}(\Delta)\circ\delta_{a(J)}(\Delta)\right.\\
+&\left.(-1)^{\langle a(J),\Delta\rangle+\langle\Delta,\Delta\rangle+\langle a(I),a(J)\rangle+\langle a(I),\Delta\rangle}\delta_{a(J)}(\Delta)\circ\delta_{a(I)}(\Delta)\right]\\
=&(-1)^{|a(I,J)|_s+\langle a(J),\Delta\rangle+\langle\Delta,\Delta\rangle}\left[(-1)^{\langle\Delta,\Delta\rangle}\delta_{a(I)}(\Delta)\circ\delta_{a(J)}(\Delta)\right.\\
+&\left.(-1)^{\langle\delta_{a(I)}(\Delta),\delta_{a(J)}(\Delta)\rangle}\delta_{a(J)}(\Delta)\circ\delta_{a(I)}(\Delta)\right]
\end{aligned}
\tag{5}
$$

Note, that $\delta_{a(I)}(\Delta)\in \mathrm{Hom}_{\mathcal{A}}(\mathcal{A},\mathcal{A})=\mathcal{A}$. Suppose now that the operator Δ is odd, that is, $\langle\Delta,\Delta\rangle = 1$. Then the expression in square brackets is nothing more than

$$(-1)^{\langle\delta_{a(I)}(\Delta),\delta_{a(J)}(\Delta)\rangle}\delta_{a(J)}(\Delta)\circ\delta_{a(I)}(\Delta)-\delta_{a(I)}(\Delta)\circ\delta_{a(J)}(\Delta)=\delta_{\delta_{a(I)}(\Delta)}(\delta_{a(J)}(\Delta))$$

Now equality (4) can be rewritten as

$$\delta_{a(I^{2n-1})}(\Delta\circ\Delta) = -\sum_{\substack{I+J=I^{2n-1}\\ |I|=n}} (-1)^{|a(I,J)|_s+\langle a(J),\Delta\rangle}\delta_{\delta_{a(I)}(\Delta)}(\delta_{a(J)}(\Delta))$$

Since, by definition, $\delta_{a(I)}(\Delta)=[a(I)]_\Delta$ and

$$\delta_{[a(I)]_\Delta}(\delta_{a(J)}(\Delta)) = [[a(I)]_\Delta, a(J)]_\Delta,$$

we obtain the following equality

$$\delta_{a(I^{2n-1})}(\Delta\circ\Delta) = -\sum_{\substack{I+J=I^{2n-1}\\ |I|=n}} (-1)^{|a(I,J)|_s+\langle a(J),\Delta\rangle}[[a(I)]_\Delta, a(J)]_\Delta. \tag{6}$$

A priori $\Delta\circ\Delta\in\mathrm{Diff}_{2n}\mathcal{A}$, but if Δ satisfies the condition $\Delta\circ\Delta\in\mathrm{Diff}_{2n-2}\mathcal{A}$, then from (6) follows that

$$\sum_{\substack{I+J=I^{(2n-1)}\\ |I|=n}} (-1)^{|a(I,J)|_s+\langle a(J),\Delta\rangle}[[a(I)]_\Delta, a(J)]_\Delta = 0. \tag{7}$$

This is the *n-ary Jacobi identity* for the bracket $[\cdot,\ldots,\cdot]_\Delta$.

n-ary Batalin–Vilkovysky brackets

Now it is natural to give the following definitions:

A DO of odd grading $\Delta \in \mathrm{Diff}_n \mathcal{A}$ satisfies to the condition

$$\Delta \circ \Delta \in \mathrm{Diff}_{2n-2} \mathcal{A}. \tag{8}$$

is called a *Batalin-Vilkovysky operator* (BVO).
Further, $\mathrm{smbl}_n \Delta$ is called the *Batalin-Vilkovysky symbol*. The bracket

$$[\,\cdot, \ldots, \cdot\,]_\Delta \colon \mathcal{A} \otimes \ldots \otimes \mathcal{A} \to \mathcal{A},$$

is called the *Batalin-Vilkovysky bracket* (BVB).

It is easy to check that if Δ is BVO and $\mathrm{smbl}_n \Delta = \mathrm{smbl}_n \Delta'$, then Δ' is BVO too. That is, BVB depends only on the symbol of BVO.

Formula (7) means that the Batalin-Vilkovysky operator defines the structure of a n-ary Lie algebra on $\mathcal{A}$.

Recall that Lie structures $\mathcal{L}$ and $\mathcal{N}$ are said to be compatible, if $\alpha\mathcal{L} + \beta\mathcal{N}$ is a Lie structure again for all $\alpha,\ \beta \in \mathcal{A}_0$.

Proposition

Let $\Delta,\ \nabla \in \mathrm{Diff}_n\,\mathcal{A}$ be Batalin-Vilkovysky operators.
Lie structures $[\quad]_\Delta$ and $[\quad]_\nabla$ are compatible $\Leftrightarrow$ $[\Delta, \nabla] \in \mathrm{Diff}_{2n-2}\,\mathcal{A}$.

Proposition

If Δ is DO of order n such that $\Delta \circ \Delta \in \mathrm{Diff}_{2n-2}\,\mathcal{A}$ and ∇ is DO of order k such that $\nabla \circ \nabla \in \mathrm{Diff}_{2k-2}\,\mathcal{A}$, then the composition $\Delta \circ \nabla \in \mathrm{Diff}_{2n+2k-2}\,\mathcal{A}$.

Corollary

1. If Δ is a Batalin-Vilkovysky operator of order n and ∇ is an even DO of order k such that $\nabla \circ \nabla \in \mathrm{Diff}_{2k-2}\,\mathcal{A}$, then the composition $\Delta \circ \nabla$ is a Batalin-Vilkovysky operator of order $n + k$.
2. If $\Delta,\ \Delta',\ \Delta''$ are Batalin-Vilkovysky operators, then the composition $\Delta \circ \Delta' \circ \Delta''$ is a Batalin-Vilkovysky operator too.

Remark. Since $\mathrm{smbl}_k\,(\Delta \circ \nabla) = (-1)^{\langle \Delta, \nabla \rangle}\,\mathrm{smbl}_k\,(\nabla \circ \Delta)$, the order of the operators in compositions mention above is not important.

Batalin–Vilkovysky brackets, $n = 2$

Consider the case $n = 2$. Let $\langle\ ,\ \rangle$ be a decomposable parity form, then the Jakobi identity (7) takes the following form

$$\begin{aligned}(-1)^{|a_3|}[[a_1, a_2]_\Delta, a_3]_\Delta + (-1)^{|a_2|\cdot|a_3|+|a_2|}[[a_1, a_3]_\Delta, a_2]_\Delta \\ +(-1)^{|a_1|\cdot|a_2|+|a_1|\cdot|a_3|+|a_1|}[[a_2, a_3]_\Delta, a_1]_\Delta = 0\end{aligned} \tag{9}$$

It is a classical form of Jacobi identity for Batalin–Vilkovysky bracket. We stress, however, that the operator Δ must satisfy the condition $\Delta \circ \Delta \in \mathrm{Diff}_2\,\mathcal{A}$, which is weaker than the original Batalin-Vilkovysky condition $\Delta \circ \Delta = 0$.

Strong homotopy Lie algebra

For $\Delta \in \operatorname{Diff}_n \mathcal{A}$, we define the hierarchy of brackets Φ^k_Δ by setting

$$\Phi^k_\Delta(a(I^k)) \stackrel{\text{def}}{=} \big(\delta_{a(I^k)}(\Delta)\big)(1).$$

Notice, that $\Phi^n_\Delta = [\quad]_\Delta$ and $\Phi^k_\Delta = 0$ for $k > n$.
Unlike Akman [1], we consider here arbitrary differential operators, and not only those that

$$\delta_{a_1,\dots,a_{n-1}}(\Delta) \in D(\mathcal{A}) \quad \text{for any tuple of elements} \quad a_1,\dots,a_{n-1} \in \mathcal{A}.$$

Now, set $\Omega^m = \Omega^m(a(I^m)) \stackrel{\text{def}}{=} \delta_{a(I^m)}(\Delta \circ \Delta)$, $\Delta_{a(I)} \stackrel{\text{def}}{=} \delta_{a(I)}(\Delta)$ and $\Delta_{a(J)} \stackrel{\text{def}}{=} \delta_{a(J)}(\Delta)$.
For $\nabla = \Delta$, the basic formula (3) can be rewritten as follows

$$\Omega^m = \delta_{a(I^m)}(\Delta \circ \Delta) = \sum_{k=0}^{m} \sum_{\substack{I+J=I^m \\ |J|=k}} (-1)^{|a(I,J)|_s + \langle a(J),\Delta\rangle} \Delta_{a(I)} \circ \Delta_{a(J)}. \quad (10)$$

It is easy to obtain by direct calculation that

$$\Omega^m(1) = -\sum_{k=0}^{m} \sum_{\substack{I+J=I^m \\ |J|=k}} (-1)^{|a(J,I)|_s + \langle a(I),\Delta\rangle} \Big(\delta_{\Delta_{a(J)}(1)}(\Delta_{a(I)})\Big)(1),$$

if $\langle \Delta, \Delta \rangle = 1$. Recall, that

$$\Delta_{a(J)}(1) = \Phi_\Delta^k(a(J)) \quad \text{and}$$
$$\Big(\delta_{\Phi_\Delta^k(a(J))}(\Delta_{a(I)})\Big)(1) = \Phi_\Delta^{m-k+1}\Big(\Phi_\Delta^k(a(J)), a(I)\Big),$$

therefore we have

$$\sum_{k=0}^{m} \sum_{\substack{I+J=I^m \\ |J|=k}} (-1)^{|a(J,I)|_s + \langle a(I),\Delta\rangle} \Phi_\Delta^{m-k+1}\Big(\Phi_\Delta^k(a(J)), a(I)\Big) = -\Omega(1)$$
$$= -\Big(\delta_{a(I^m)}(\Delta \circ \Delta)\Big)(1)$$

If $\Delta \circ \Delta \in \mathrm{Diff}_{m-1} \mathcal{A}$ then the right-hand side of this equality is zero. In particular, if $\Delta \circ \Delta = 0$ then the right-hand side of this equality is zero for all m.

So, if the odd operator $\Delta \in \mathrm{Diff}_n \mathcal{A}$ is such that $\Delta \circ \Delta = 0$, then it defines the structure of an "ordinary" n-ary Lie algebra and the structure of a strong homotopy Lie algebra:

$$\sum_{\substack{I+J=I^{(2n-1)} \\ |I|=n}} (-1)^{|a(I,J)|_s + \langle a(J),\Delta\rangle} [[a(I)]_\Delta, a(J)]_\Delta$$

$$\equiv \sum_{\substack{I+J=I^{(2n-1)} \\ |I|=n}} (-1)^{|a(I,J)|_s + \langle a(J),\Delta\rangle} \Phi^n_\Delta\Big(\Phi^n_\Delta\big(a(I)\big), a(J)\Big) = 0,$$

$$\sum_{k=0}^{m} \sum_{\substack{I+J=I^m \\ |I|=k}} (-1)^{|a(J,I)|_s + \langle a(I),\Delta\rangle} \Phi^{m-k+1}_\Delta(\Phi^k_\Delta(a(I)), (a(J)) = 0,$$

where $m = 1, \ldots, 2n-2$.

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