

Workshop on Geometry of Differential Equations and Integrability
11–15 October 2010, Hradec nad Moravicí, Czech Republic

On the tangent and cotangent coverings
over differential equations.
Part II: invariance

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This is joint work in progress with Sergey Igonin, Paul Kersten,
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The Problem

$$\begin{aligned} \mathcal{E} &\subset J^\infty & F = 0, \dots, D_i(F) = 0, \dots, D_{ij}(F) = 0, \dots \\ F &\in P \end{aligned}$$

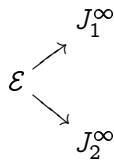
$$D_i = \partial_{x^i} + \sum_{j,\sigma} u_{\sigma i}^j \partial_{u_\sigma^j}$$

$$\text{Cartan distribution: } \mathcal{C}D = \langle D_1, \dots, D_n \rangle$$

$$\text{Tangent covering } \mathcal{T}\mathcal{E}: \quad F = 0, \quad \ell_{\mathcal{E}}(\mathbf{q}) = 0$$

$$\text{Cotangent covering } \mathcal{T}^*\mathcal{E}: \quad F = 0, \quad \ell_{\mathcal{E}}^*(\mathbf{p}) = 0$$

$$\ell_{\mathcal{E}} = \ell_F|_{\mathcal{E}}: \kappa \rightarrow P, \quad \ell_F = \left\| \sum_{\sigma} \partial_{u_\sigma^j} (F^i) D_\sigma \right\|$$



$$u_{tt}u_{xx} - u_{tx}^2 + u_{xz} + u_{ty} = 0$$

$$u_t = q, \quad q_t = \frac{1}{u_{xx}} (q_x^2 - q_y - u_{xz})$$

Tangent covering and the linearizations

$$\mathcal{T}\mathcal{E} = T(\mathcal{E})/CD, \quad \tilde{X}(\omega) = L_X(\omega),$$

$$X \in CD, \quad \omega \in \Lambda_C^1(\mathcal{E}) = \{\chi \in \Lambda^1(\mathcal{E}) \mid \chi|_{CD} = 0\}$$

$$0 \rightarrow \mathcal{C}(P, \mathcal{F}) \xrightarrow{\ell_{\mathcal{E}}^+} \mathcal{C}(\mathcal{X}, \mathcal{F}) \xrightarrow{\mu} \Lambda_C^1(\mathcal{E}) \rightarrow 0$$

$$\mathcal{F} = C^\infty(\mathcal{E})$$

$\mathcal{C}(Q_1, Q_2)$ is the set of \mathcal{C} -differential operators $Q_1 \rightarrow Q_2$

$$\mu(\nabla)(E_\varphi) = \nabla(\varphi), \quad \varphi \in \mathcal{X}, \quad E_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j) \partial_{u_\sigma^j}$$

$$\mu(0, \dots, \underset{j\text{th place}}{D_\sigma}, \dots, 0) = du_\sigma^j - \sum_i u_{\sigma i}^j dx^i$$

$$\ell_{\mathcal{E}}^+(\Delta) = \Delta \circ \ell_{\mathcal{E}}$$

$$\begin{array}{ccc}
 & & 0 \longrightarrow \mathcal{C}(P_1, \mathcal{F}) \xrightarrow{\ell_{\mathcal{E}}^{1+}} \mathcal{C}(\mathcal{X}_1, \mathcal{F}) \longrightarrow 0 \\
 \mathcal{E} & \begin{array}{l} \nearrow \\ \searrow \end{array} & \\
 & & 0 \longrightarrow \mathcal{C}(P_2, \mathcal{F}) \xrightarrow{\ell_{\mathcal{E}}^{2+}} \mathcal{C}(\mathcal{X}_2, \mathcal{F}) \longrightarrow 0 \\
 & & \begin{array}{l} J_1^\infty \\ J_2^\infty \end{array}
 \end{array}$$

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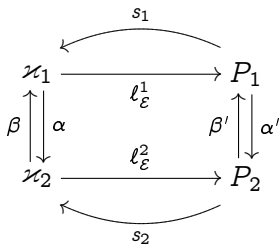
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$$\ell_{\mathcal{E}}^+(\Delta) = \Delta \circ \ell_{\mathcal{E}}$$

$$\mathcal{E} \begin{cases} \nearrow J_1^\infty \\ \searrow J_2^\infty \end{cases}$$

$$\begin{array}{ccccccc} & & & \xleftarrow{s_1^+} & & & \\ & & & \swarrow & & & \\ 0 & \longrightarrow & \mathcal{C}(P_1, \mathcal{F}) & \xrightarrow{\ell_{\mathcal{E}}^{1+}} & \mathcal{C}(\mathcal{X}_1, \mathcal{F}) & \longrightarrow & 0 \\ & & \updownarrow \alpha'^+ \beta'^+ & & \updownarrow \alpha^+ \beta^+ & & \\ 0 & \longrightarrow & \mathcal{C}(P_2, \mathcal{F}) & \xrightarrow{\ell_{\mathcal{E}}^{2+}} & \mathcal{C}(\mathcal{X}_2, \mathcal{F}) & \longrightarrow & 0 \\ & & & \swarrow & & & \\ & & & \xleftarrow{s_2^+} & & & \end{array}$$



$$l_{\mathcal{E}}^2 \alpha = \alpha' l_{\mathcal{E}}^1$$

$$l_{\mathcal{E}}^1 \beta = \beta' l_{\mathcal{E}}^2$$

$$\beta \alpha = \text{id} + s_1 l_{\mathcal{E}}^1$$

$$\alpha \beta = \text{id} + s_2 l_{\mathcal{E}}^2$$

$$\beta' \alpha' = \text{id} + l_{\mathcal{E}}^1 s_1$$

$$\alpha' \beta' = \text{id} + l_{\mathcal{E}}^2 s_2$$

$$\mathcal{E} \begin{matrix} \nearrow J_1^\infty \\ \searrow J_2^\infty \end{matrix} \quad u_{tt} u_{xx} - u_{tx}^2 + u_{xz} + u_{ty} = 0$$

$$u_t = q, \quad q_t = \frac{1}{u_{xx}} (q_x^2 - q_y - u_{xz})$$

$$l_{\mathcal{E}}^1 = u_{xx} D_{tt} + u_{tt} D_{xx} - 2u_{tx} D_{tx} + D_{xz} + D_{ty}$$

$$l_{\mathcal{E}}^2 = \begin{pmatrix} D_t & -1 \\ \frac{q_x^2 - q_y - u_{xz}}{u_{xx}^2} D_{xx} + \frac{1}{u_{xx}} D_{xz} & D_t - \frac{2q_x}{u_{xx}} D_x + \frac{1}{u_{xx}} D_y \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 1 \\ D_t \end{pmatrix}, \quad \beta = (1, 0)$$

$$\alpha' = \begin{pmatrix} 0 \\ \frac{1}{u_{xx}} \end{pmatrix}, \quad \beta' = (u_{xx} D_t - 2u_{tx} D_x + D_y, u_{xx}), \quad s_1 = 0, \quad s_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{array}{ccc}
 & J_h^\infty(\mathcal{X}_1) \xrightarrow{\Phi_{\ell^1_\mathcal{E}}} J_h^\infty(P_1) & \\
 \mathcal{T}\mathcal{E} \nearrow & \updownarrow \begin{array}{c} \Phi_\beta \\ \Phi_\alpha \end{array} & \updownarrow \begin{array}{c} \Phi_{\beta'} \\ \Phi_{\alpha'} \end{array} \\
 & J_h^\infty(\mathcal{X}_2) \xrightarrow{\Phi_{\ell^2_\mathcal{E}}} J_h^\infty(P_2) & \\
 \\
 & & J_h^\infty(\hat{P}_1) \xrightarrow{\Phi_{\ell^1_\mathcal{E}^*}} J_h^\infty(\hat{\mathcal{X}}_1) \\
 \mathcal{T}^*\mathcal{E} \nearrow & \updownarrow \begin{array}{c} \Phi_{\alpha'^*} \\ \Phi_{\beta'^*} \end{array} & \updownarrow \begin{array}{c} \Phi_{\alpha^*} \\ \Phi_{\beta^*} \end{array} \\
 & J_h^\infty(\hat{P}_2) \xrightarrow{\Phi_{\ell^2_\mathcal{E}^*}} J_h^\infty(\hat{\mathcal{X}}_2) &
 \end{array}$$

$F = 0, \quad \ell_\mathcal{E}^*(p) = 0$

$$\mathcal{T}^*\mathcal{E} \subset J_h^\infty(\hat{P})$$

Fibers of the cotangent covering $\mathcal{T}^*\mathcal{E} \rightarrow \mathcal{E}$ are considered *odd*

Functions on $\mathcal{T}^*\mathcal{E}$ are skew-symmetric multi-linear \mathcal{C} -differential operators

$$\langle \psi, \ell_\mathcal{E}(\varphi) \rangle - \langle \varphi, \ell_\mathcal{E}^*(\psi) \rangle = d_h \gamma(\varphi, \psi)$$

$$\varphi \in \mathcal{X}, \psi \in \hat{P}, \gamma(\varphi, \psi) \in \Lambda_h^{n-1}(\mathcal{E}) = \{ \sum_\sigma f_\sigma dx^\sigma \}, \Lambda_h^1 = \Lambda^1 / \Lambda_C^1$$

$$\text{Sym } \mathcal{E} \ni \varphi \mapsto \gamma(\varphi, \cdot) \in \mathcal{C}(\hat{P}, \Lambda_h^{n-1}) \quad \text{Sym } \mathcal{E} \rightarrow \text{CL}(\mathcal{T}^*\mathcal{E})$$

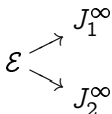
Variational multivectors on \mathcal{E} are conservation laws on $\mathcal{T}^*\mathcal{E}$.

$$A: \hat{P} \rightarrow \varkappa \quad \ell_{\mathcal{E}} A - A^* \ell_{\mathcal{E}}^* = 0$$

$$\ell_F \tilde{A} - \tilde{A}^* \ell_F^* = \tilde{B}(F, \cdot) \quad \text{on } J^\infty$$

$$B^*: \hat{P} \times \hat{P} \rightarrow \hat{P} \quad B^*(\psi_1, \psi_2) = \tilde{B}^{*1}(\psi_1, \psi_2)|_{\mathcal{E}}$$

(A, B^*) is the generating function of the conservation law on $\mathcal{T}^*\mathcal{E}$



$$A^2 = \alpha A^1 \alpha'^*$$

$$A^1 = \beta A^2 \beta'^*$$

$$u_{tt} u_{xx} - u_{tx}^2 + u_{xz} + u_{ty} = 0$$

$$u_t = q$$

$$q_t = \frac{1}{u_{xx}} (q_x^2 - q_y - u_{xz})$$

$$A^2 = \begin{pmatrix} 0 & \frac{1}{u_{xx}} \\ -\frac{1}{u_{xx}} & \frac{q_x}{u_{xx}^2} D_x + D_x \frac{q_x}{u_{xx}^2} - \frac{1}{u_{xx}} D_y \frac{1}{u_{xx}} \end{pmatrix}$$

Symplectic structure on $\mathcal{T}^*\mathcal{E}$

$$F = 0 \quad \ell_{\mathcal{E}}^*(\mathbf{p}) = 0, \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\ell_{\mathcal{E}}^*(\mathbf{p}) = 0 \quad F = 0 \quad \longleftrightarrow \quad L = \langle F, \mathbf{p} \rangle, \quad \Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} & [[A_1, A_2]](\psi_1, \psi_2) \\ &= \ell_{A_1, \psi_1}(A_2(\psi_2)) - \ell_{A_1, \psi_2}(A_2(\psi_1)) \\ &+ \ell_{A_2, \psi_1}(A_1(\psi_2)) - \ell_{A_2, \psi_2}(A_1(\psi_1)) \\ &\quad - A_1(B_2^*(\psi_1, \psi_2)) - A_2(B_1^*(\psi_1, \psi_2)) \end{aligned}$$

$$\ell_{A, \psi}(\varphi) = E_{\varphi}(A)(\psi)$$

If \mathcal{E} is in evolution form then $B^*(\psi_1, \psi_2) = \ell_{A, \psi_2}^*(\psi_1)$

$$\Lambda_{\mathcal{C}}^1(\mathcal{E}) \text{ is a } \mathcal{C}\text{-module} \quad \mathcal{T}\mathcal{E} \quad \longleftrightarrow \quad \Lambda_{\mathcal{C}}^1(\mathcal{E})$$

$$\mathcal{T}^*\mathcal{E} \quad \longleftrightarrow \quad \tau^*(\mathcal{E}) = ?$$

$$\tau^*(\mathcal{E}) \stackrel{?}{=} \cancel{\text{Hom}_{\mathcal{C}}(\Lambda_{\mathcal{C}}^1, \mathcal{C}(\mathcal{F}, \mathcal{F}))}$$

$$\tau^*(\mathcal{E}) = \text{Ext}_{\mathcal{C}}^1(\Lambda_{\mathcal{C}}^1, \mathcal{C}(\Lambda_h^n, \mathcal{F}))$$

$$(1) \quad \nabla \cdot \Delta = \nabla \circ \Delta, \quad \nabla \in \mathcal{C}(\mathcal{F}, \mathcal{F}), \quad \Delta \in \mathcal{C}(\Lambda_h^n, \mathcal{F})$$

$$(2) \quad \nabla \cdot \Delta = \Delta \circ \nabla^*$$

$$0 \rightarrow \mathcal{C}(P, \mathcal{F}) \xrightarrow{\ell_{\mathcal{E}}^+} \mathcal{C}(\mathcal{X}, \mathcal{F}) \xrightarrow{\mu} \Lambda_{\mathcal{C}}^1(\mathcal{E}) \rightarrow 0$$

$$\tau^*(\mathcal{E}) = \text{Hom}_{\mathcal{C}}(\mathcal{C}(P, \mathcal{F}), \mathcal{C}(\Lambda_h^n, \mathcal{F})) / \text{im}(\text{Hom}_{\mathcal{C}}(\ell_{\mathcal{E}}^+, \mathcal{C}(\Lambda_h^n, \mathcal{F})))$$

$$\quad \quad \quad \parallel$$

$$\quad \quad \quad \mathcal{C}(\Lambda_h^n, P)$$

$$\tau^*(\mathcal{E}) = \mathcal{C}(\hat{P}, \mathcal{F}) / \{ \square \in \mathcal{C}(\hat{P}, \mathcal{F}) \mid \square = \square' \ell_{\mathcal{E}}^* \}$$

$p dq$ on the cotangent covering

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{C}(\mathcal{F}, \Lambda_C^1) \rightarrow \Lambda_C^1 \rightarrow 0$$
$$\Delta \mapsto \Delta(1)$$

$$“p dq” \in \text{Ext}_C^1(\Lambda_C^1, \mathcal{K})$$

$$\mathcal{C}(\Lambda_h^n, \Lambda_C^1 \otimes_{\mathcal{F}} \Lambda_h^{n-1}) = \mathcal{C}(\Lambda_h^1, \Lambda_C^1) \rightarrow \mathcal{K} \rightarrow 0$$
$$\Delta \mapsto \Delta \circ d_h$$

$$\text{Ext}_C^1(\Lambda_C^1, \cdot): \quad \text{Ext}_C^1(\Lambda_C^1, \mathcal{C}(\Lambda_h^n, \Lambda_C^1 \otimes_{\mathcal{F}} \Lambda_h^{n-1})) \rightarrow \text{Ext}_C^1(\Lambda_C^1, \mathcal{K}) \rightarrow 0$$

$$\text{Ext}_C^2(\Lambda_C^1, \cdot) = 0 \quad \parallel$$
$$\tau^*(\mathcal{E}) \otimes_{\mathcal{F}} \Lambda_C^1 \otimes_{\mathcal{F}} \Lambda_h^{n-1}$$

$$“p dq” \in E_1^{1, n-1}(\tau^*(\mathcal{E}))$$

$$\Omega = d_1^{1, n-1}(“p dq”) \in E_1^{2, n-1}(\tau^*(\mathcal{E}))$$