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On the tangent and cotangent coverings over differential equations.
Part II: invariance

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The Problem

$\mathcal{E} \subset J^\infty \qquad F = 0,\ \ldots,\ D_i(F) = 0,\ \ldots,\ D_{ij}(F) = 0,\ \ldots$

$F \in P$

$D_i = \partial_{x^i} + \sum_{j,\sigma} u^j_{\sigma i} \partial_{u^j_\sigma}$

Cartan distribution: $\mathcal{C}D = \langle D_1, \ldots, D_n \rangle$

Tangent covering $\mathcal{T}\mathcal{E}$: $\qquad F = 0, \qquad \ell_{\mathcal{E}}(\boldsymbol{q}) = 0$

Cotangent covering $\mathcal{T}^*\mathcal{E}$: $\qquad F = 0, \qquad \ell^*_{\mathcal{E}}(\boldsymbol{p}) = 0$

$\ell_{\mathcal{E}} = \ell_F\big|_{\mathcal{E}} \colon \varkappa \to P, \qquad \ell_F = \left\| \sum_\sigma \partial_{u^j_\sigma}(F^i) D_\sigma \right\|$

$$\begin{tikzcd}
 & J^\infty_1 & u_{tt} u_{xx} - u_{tx}^2 + u_{xz} + u_{ty} = 0 \\
\mathcal{E} \arrow[ur] \arrow[dr] & & \\
 & J^\infty_2 & u_t = q, \quad q_t = \frac{1}{u_{xx}} \left(q_x^2 - q_y - u_{xz} \right)
\end{tikzcd}$$

Tangent covering and the linearizations

$$\mathcal{T}\mathcal{E} = T(\mathcal{E})/\mathcal{C}D, \qquad \tilde{X}(\omega) = L_X(\omega),$$

$$X \in \mathcal{C}D,\ \ \omega \in \Lambda^1_{\mathcal{C}}(\mathcal{E}) = \{\,\chi \in \Lambda^1(\mathcal{E}) \mid \chi\big|_{\mathcal{C}D} = 0\,\}$$

$$0 \to \mathcal{C}(P, \mathcal{F}) \xrightarrow{\ell_{\mathcal{E}}^{+}} \mathcal{C}(\varkappa, \mathcal{F}) \xrightarrow{\mu} \Lambda^1_{\mathcal{C}}(\mathcal{E}) \to 0$$

$$\mathcal{F} = C^\infty(\mathcal{E})$$

$\mathcal{C}(Q_1, Q_2)$ is the set of $\mathcal{C}$-differential operators $Q_1 \to Q_2$

$$\mu(\nabla)(E_\varphi) = \nabla(\varphi),\ \varphi \in \varkappa, \quad E_\varphi = \textstyle\sum_{j,\sigma} D_\sigma(\varphi^j)\partial_{u^j_\sigma}$$

$$\mu(0, \dots, \underbrace{D_\sigma}_{j\text{th place}}, \dots, 0) = du^j_\sigma - \textstyle\sum_i u^j_{\sigma i}\, dx^i$$

$$\ell_{\mathcal{E}}^{+}(\Delta) = \Delta \circ \ell_{\mathcal{E}}$$

$$\begin{tikzcd}
& J_1^\infty & 0 \arrow[r] & \mathcal{C}(P_1, \mathcal{F}) \arrow[r, "\ell_{\mathcal{E}}^{1+}"] & \mathcal{C}(\varkappa_1, \mathcal{F}) \arrow[r] & 0 \\
\mathcal{E} \arrow[ur] \arrow[dr] & & & & & \\
& J_2^\infty & 0 \arrow[r] & \mathcal{C}(P_2, \mathcal{F}) \arrow[r, "\ell_{\mathcal{E}}^{2+}"] & \mathcal{C}(\varkappa_2, \mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

Tangent covering and the linearizations

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$$0 \to \mathcal{C}(P,\mathcal{F}) \xrightarrow{\ell^+_{\mathcal{E}}} \mathcal{C}(\varkappa,\mathcal{F}) \xrightarrow{\mu} \Lambda^1_{\mathcal{C}}(\mathcal{E}) \to 0$$

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$\mu(0,\underset{j\text{th place}}{\ldots, D_\sigma,}\ldots,0) = du^j_\sigma - \sum_i u^j_{\sigma i}\,dx^i$

$\ell^+_{\mathcal{E}}(\Delta) = \Delta \circ \ell_{\mathcal{E}}$

$$\begin{tikzcd}
 & J^\infty_1 \\
\mathcal{E} \arrow[ur] \arrow[dr] & \\
 & J^\infty_2
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}(P_1,\mathcal{F}) \arrow[r, "\ell^{1+}_{\mathcal{E}}"] \arrow[d, shift left, "\beta'^+"] & \mathcal{C}(\varkappa_1,\mathcal{F}) \arrow[r] \arrow[l, bend right, "s^+_1"'] \arrow[d, shift left, "\beta^+"] & 0 \\
0 \arrow[r] & \mathcal{C}(P_2,\mathcal{F}) \arrow[r, "\ell^{2+}_{\mathcal{E}}"] \arrow[u, shift left, "\alpha'^+"] & \mathcal{C}(\varkappa_2,\mathcal{F}) \arrow[r] \arrow[l, bend left, "s^+_2"] \arrow[u, shift left, "\alpha^+"] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\varkappa_1 \arrow[r, "\ell^1_{\mathcal{E}}"'] \arrow[d, shift left, "\alpha"] & P_1 \arrow[l, bend right, "s_1"'] \arrow[d, shift left, "\alpha'"] \\
\varkappa_2 \arrow[r, "\ell^2_{\mathcal{E}}"] \arrow[u, shift left, "\beta"] & P_2 \arrow[l, bend left, "s_2"] \arrow[u, shift left, "\beta'"]
\end{tikzcd}$$

$$\begin{aligned}
\ell^2_{\mathcal{E}}\,\alpha &= \alpha'\,\ell^1_{\mathcal{E}} \\
\ell^1_{\mathcal{E}}\,\beta &= \beta'\,\ell^2_{\mathcal{E}} \\
\beta\,\alpha &= \mathrm{id} + s_1\,\ell^1_{\mathcal{E}} \\
\alpha\,\beta &= \mathrm{id} + s_2\,\ell^2_{\mathcal{E}} \\
\beta'\,\alpha' &= \mathrm{id} + \ell^1_{\mathcal{E}}\,s_1 \\
\alpha'\,\beta' &= \mathrm{id} + \ell^2_{\mathcal{E}}\,s_2
\end{aligned}$$

$$\begin{tikzcd}
& J^\infty_1 \\
\mathcal{E} \arrow[ur] \arrow[dr] & \\
& J^\infty_2
\end{tikzcd}$$

$$u_{tt}\,u_{xx} - u_{tx}^2 + u_{xz} + u_{ty} = 0$$

$$u_t = q, \quad q_t = \frac{1}{u_{xx}}\left(q_x^2 - q_y - u_{xz}\right)$$

$$\ell^1_{\mathcal{E}} = u_{xx}\,D_{tt} + u_{tt}\,D_{xx} - 2u_{tx}\,D_{tx} + D_{xz} + D_{ty}$$

$$\ell^2_{\mathcal{E}} = \begin{pmatrix} D_t & -1 \\ \frac{q_x^2 - q_y - u_{xz}}{u_{xx}^2}\,D_{xx} + \frac{1}{u_{xx}}\,D_{xz} & D_t - \frac{2\,q_x}{u_{xx}}\,D_x + \frac{1}{u_{xx}}\,D_y \end{pmatrix}$$

$$\alpha = \left(\begin{smallmatrix} 1 \\ D_t \end{smallmatrix}\right),\ \beta = (1, 0)$$

$$\alpha' = \begin{pmatrix} 0 \\ \frac{1}{u_{xx}} \end{pmatrix},\ \beta' = (u_{xx}\,D_t - 2u_{tx}\,D_x + D_y,\, u_{xx}),\ \ s_1 = 0,\ s_2 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$$

$$\begin{tikzcd}
 & J_h^\infty(\varkappa_1) \arrow[r, "\Phi_{\ell_{\mathcal{E}}^1}"] \arrow[dd, shift left, "\Phi_\alpha"] & J_h^\infty(P_1) \arrow[dd, shift left, "\Phi_{\alpha'}"] \\
\mathcal{T}\mathcal{E} \arrow[ur] \arrow[dr] & & \\
 & J_h^\infty(\varkappa_2) \arrow[r, "\Phi_{\ell_{\mathcal{E}}^2}"] \arrow[uu, shift left, "\Phi_\beta"] & J_h^\infty(P_2) \arrow[uu, shift left, "\Phi_{\beta'}"]
\end{tikzcd}$$

$$F = 0, \quad \boldsymbol{\ell}_{\mathcal{E}}^*(\boldsymbol{p}) = 0$$

$$\begin{tikzcd}
 & J_h^\infty(\hat{P}_1) \arrow[r, "\Phi_{\ell_{\mathcal{E}}^{1*}}"] \arrow[dd, shift left, "\Phi_{\beta'*}"] & J_h^\infty(\hat{\varkappa}_1) \arrow[dd, shift left, "\Phi_{\beta*}"] \\
\mathcal{T}^*\mathcal{E} \arrow[ur] \arrow[dr] & & \\
 & J_h^\infty(\hat{P}_2) \arrow[r, "\Phi_{\ell_{\mathcal{E}}^{2*}}"] \arrow[uu, shift left, "\Phi_{\alpha'*}"] & J_h^\infty(\hat{\varkappa}_2) \arrow[uu, shift left, "\Phi_{\alpha*}"]
\end{tikzcd}$$

$\mathcal{T}^*\mathcal{E} \subset J_h^\infty(\hat{P})$

Fibers of the cotangent covering $\mathcal{T}^*\mathcal{E} \to \mathcal{E}$ are considered *odd*

Functions on $\mathcal{T}^*\mathcal{E}$ are skew-symmetric multi-linear $\mathcal{C}$-differential operators

$$\langle \psi, \boldsymbol{\ell}_{\mathcal{E}}(\varphi) \rangle - \langle \varphi, \boldsymbol{\ell}_{\mathcal{E}}^*(\psi) \rangle = d_h \gamma(\varphi, \psi)$$

$\varphi \in \varkappa$, $\psi \in \hat{P}$, $\gamma(\varphi, \psi) \in \Lambda_h^{n-1}(\mathcal{E}) = \{ \sum_\sigma f_\sigma \, dx^\sigma \}$, $\Lambda_h^1 = \Lambda^1 / \Lambda_{\mathcal{C}}^1$

$\operatorname{Sym} \mathcal{E} \ni \varphi \mapsto \gamma(\varphi, \,\cdot\,) \in \mathcal{C}(\hat{P}, \Lambda_h^{n-1})$ $\qquad$ $\operatorname{Sym} \mathcal{E} \to \mathrm{CL}(\mathcal{T}^*\mathcal{E})$

Variational multivectors on $\mathcal{E}$ are conservation laws on $\mathcal{T}^*\mathcal{E}$.

$$A\colon\ \hat{P} \to \varkappa \qquad \boldsymbol{\ell}_{\mathcal{E}} A - A^* \boldsymbol{\ell}_{\mathcal{E}}^* = 0$$

$$\boldsymbol{\ell}_F \tilde{A} - \tilde{A}^* \boldsymbol{\ell}_F^* = \tilde{B}(F, \cdot\,) \quad \text{on } J^\infty$$

$$B^*\colon\ \hat{P} \times \hat{P} \to \hat{P} \qquad B^*(\psi_1, \psi_2) = \tilde{B}^{*_1}(\psi_1, \psi_2)\big|_{\mathcal{E}}$$

(A, B^*) is the generating function of the conservation law on $\mathcal{T}^*\mathcal{E}$

$$\begin{tikzcd} & J_1^\infty \\ \mathcal{E} \arrow[ur] \arrow[dr] & \\ & J_2^\infty \end{tikzcd}$$

$$A^2 = \alpha A^1 \alpha'^*$$

$$A^1 = \beta A^2 \beta'^*$$

$$u_{tt} u_{xx} - u_{tx}^2 + u_{xz} + u_{ty} = 0$$

$$u_t = q$$
$$q_t = \frac{1}{u_{xx}} \left(q_x^2 - q_y - u_{xz} \right)$$

$$A^2 = \begin{pmatrix} 0 & \dfrac{1}{u_{xx}} \\ -\dfrac{1}{u_{xx}} & \dfrac{q_x}{u_{xx}^2} D_x + D_x \dfrac{q_x}{u_{xx}^2} - \dfrac{1}{u_{xx}} D_y \dfrac{1}{u_{xx}} \end{pmatrix}$$

F. Neyzi, Y. Nutku, M.B. Sheftel, Multi-Hamiltonian structure of Plebanski's second heavenly equation, J. Phys. A **38** (2005), 8473–8485, arXiv:nlin/0505030

Symplectic structure on $\mathcal{T}^*\mathcal{E}$

$$F = 0 \quad \boldsymbol{\ell}^*_{\mathcal{E}}(\boldsymbol{p}) = 0, \qquad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\boldsymbol{\ell}^*_{\mathcal{E}}(\boldsymbol{p}) = 0 \quad F = 0 \qquad \longleftarrow\!\!\mid \qquad L = \langle F, \boldsymbol{p} \rangle, \qquad \Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} &[\![A_1, A_2]\!](\psi_1, \psi_2) \\ &\qquad = \boldsymbol{\ell}_{A_1,\psi_1}(A_2(\psi_2)) - \boldsymbol{\ell}_{A_1,\psi_2}(A_2(\psi_1)) \\ &\qquad + \boldsymbol{\ell}_{A_2,\psi_1}(A_1(\psi_2)) - \boldsymbol{\ell}_{A_2,\psi_2}(A_1(\psi_1)) \\ &\qquad\qquad - A_1(B_2^*(\psi_1, \psi_2)) - A_2(B_1^*(\psi_1, \psi_2)) \end{aligned}$$

$\boldsymbol{\ell}_{A,\psi}(\varphi) = E_\varphi(A)(\psi)$

If $\mathcal{E}$ is in evolution form then $B^*(\psi_1, \psi_2) = \boldsymbol{\ell}^*_{A,\psi_2}(\psi_1)$

$\Lambda^1_{\mathcal{C}}(\mathcal{E})$ is a $\mathcal{C}$-module

$$\begin{aligned} \mathcal{T}\mathcal{E} &\quad\longleftrightarrow\quad \Lambda^1_{\mathcal{C}}(\mathcal{E}) \\ \mathcal{T}^*\mathcal{E} &\quad\longleftrightarrow\quad \tau^*(\mathcal{E}) = ? \end{aligned}$$

$$\tau^*(\mathcal{E}) \overset{?}{=} \xcancel{\mathrm{Hom}_{\mathcal{C}}(\Lambda^1_{\mathcal{C}}, \mathcal{C}(\mathcal{F}, \mathcal{F}))}$$

$$\tau^*(\mathcal{E}) = \mathrm{Ext}^1_{\mathcal{C}}(\Lambda^1_{\mathcal{C}}, \mathcal{C}(\Lambda^n_h, \mathcal{F}))$$

$$\begin{aligned} &(1) \qquad \nabla \cdot \Delta = \nabla \circ \Delta, \qquad \nabla \in \mathcal{C}(\mathcal{F}, \mathcal{F}), \quad \Delta \in \mathcal{C}(\Lambda^n_h, \mathcal{F})) \\ &(2) \qquad \nabla \cdot \Delta = \Delta \circ \nabla^* \end{aligned}$$

$$0 \to \mathcal{C}(P, \mathcal{F}) \xrightarrow{\ell^+_{\mathcal{E}}} \mathcal{C}(\varkappa, \mathcal{F}) \xrightarrow{\mu} \Lambda^1_{\mathcal{C}}(\mathcal{E}) \to 0$$

$$\tau^*(\mathcal{E}) = \mathrm{Hom}_{\mathcal{C}}(\underset{\underset{\mathcal{C}(\Lambda^n_h, P)}{\shortparallel}}{\mathcal{C}(P, \mathcal{F})}, \mathcal{C}(\Lambda^n_h, \mathcal{F})) \Big/ \mathrm{im}(\mathrm{Hom}_{\mathcal{C}}(\ell^+_{\mathcal{E}}, \mathcal{C}(\Lambda^n_h, \mathcal{F})))$$

$$\tau^*(\mathcal{E}) = \mathcal{C}(\hat{P}, \mathcal{F}) \Big/ \{ \Box \in \mathcal{C}(\hat{P}, \mathcal{F}) \mid \Box = \Box' \ell^*_{\mathcal{E}} \}$$

$p\ dq$ on the cotangent covering

$$0 \to \mathcal{K} \to \mathcal{C}(\mathcal{F}, \Lambda^1_{\mathcal{C}}) \to \Lambda^1_{\mathcal{C}} \to 0$$

$$\Delta \longmapsto \Delta(1)$$

$$\text{“}p\ dq\text{”} \in \mathrm{Ext}^1_{\mathcal{C}}(\Lambda^1_{\mathcal{C}}, \mathcal{K})$$

$$\mathcal{C}(\Lambda^n_h, \Lambda^1_{\mathcal{C}} \otimes_{\mathcal{F}} \Lambda^{n-1}_h) = \mathcal{C}(\Lambda^1_h, \Lambda^1_{\mathcal{C}}) \to \mathcal{K} \to 0$$

$$\Delta \longmapsto \Delta \circ d_h$$

$$\mathrm{Ext}^1_{\mathcal{C}}(\Lambda^1_{\mathcal{C}}, \,\cdot\,)\colon \qquad \mathrm{Ext}^1_{\mathcal{C}}(\Lambda^1_{\mathcal{C}}, \underset{\substack{\| \\ \tau^*(\mathcal{E}) \otimes_{\mathcal{F}} \Lambda^1_{\mathcal{C}} \otimes_{\mathcal{F}} \Lambda^{n-1}_h}}{\mathcal{C}(\Lambda^n_h, \Lambda^1_{\mathcal{C}} \otimes_{\mathcal{F}} \Lambda^{n-1}_h)}) \to \mathrm{Ext}^1_{\mathcal{C}}(\Lambda^1_{\mathcal{C}}, \mathcal{K}) \to 0$$

$$\mathrm{Ext}^2_{\mathcal{C}}(\Lambda^1_{\mathcal{C}}, \,\cdot\,) = 0$$

$$\text{“}p\ dq\text{”} \in E_1^{1,n-1}(\tau^*(\mathcal{E}))$$

$$\Omega = d_1^{1,n-1}(\text{“}p\ dq\text{”}) \in E_1^{2,n-1}(\tau^*(\mathcal{E}))$$