Local and Nonlocal Geometry of PDEs and Integrability
8–12 October 2018, SISSA, Trieste, Italy
In honor of the 70th birthday of Joseph Krasil'shchik

Nonlocal Jacobi identity: a geometric approach

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Joint work with Joseph Krasil'shchik

A recursion for the sine-Gordon equation

$u_{xy} = \sin u$

Recursion: $D_x^2 + u_x^2 - u_x D_x^{-1} u_{xx} = B \circ A^{-1}$

$$\begin{aligned} u_{xy} - \sin u &= 0 \\ w_y - u_x (u_{xx}^{-1} w_x)_y &= 0 \end{aligned}$$

$q = u_{xx}^{-1} w_x = A(w)$

$q = (D_x^2 + u_x^2) u_{xx}^{-1} w_x - u_x w = B(w)$

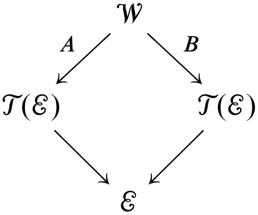
$$\begin{aligned} u_{xy} - \sin u &= 0 \\ q_{xy} - (\cos u)\, q &= 0 \end{aligned}$$

$$\begin{aligned} u_{xy} - \sin u &= 0 \\ q_{xy} - (\cos u)\, q &= 0 \end{aligned}$$

$$u_{xy} - \sin u = 0$$

Toward nonlocal Hamiltonian structure

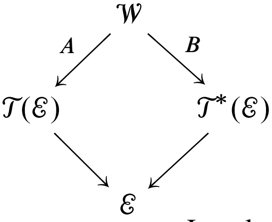
Recursion:



$$\mathcal{E} : F = u_{xy} - \sin u = 0$$

$$\mathcal{T}(\mathcal{E}) : \begin{array}{c} F = u_{xy} - \sin u = 0 \\ \ell_F(q) = q_{xy} - (\cos u)\, q = 0 \end{array}$$

Nonlocal Hamiltonian structure:



$$\mathcal{T}^*(\mathcal{E}) : \begin{array}{c} F = u_{xy} - \sin u = 0 \\ \ell_F^*(p) = p_{xy} - (\cos u)\, p = 0 \end{array}$$

Local structure: $\begin{cases} \text{Hamiltonian} & \longleftrightarrow & B = \text{id} \\ \text{(pre)symplectic} & \longleftrightarrow & A = \text{id} \end{cases}$

Finite-dimensional toy model

$$\begin{array}{c} \mathcal{W} \\ \lambda \big\downarrow \\ \mathcal{E} \end{array} \qquad Z \in \mathrm{Vect}(\mathcal{W}), \quad \Omega \in \Lambda^2(\mathcal{W}), \quad |Z| = 1, \quad |\Omega| = 1$$

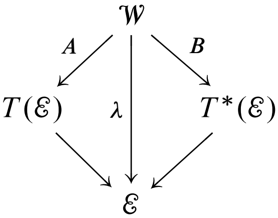
$$\boxed{d\Omega = 0, \quad Z^2 = \tfrac{1}{2}[Z, Z] = 0, \quad L_Z(\Omega) = 0}$$

$$C^\infty(\mathcal{E}) \xrightarrow{Z} C_1^\infty(\mathcal{W}), \quad \Lambda^1(\mathcal{E}) \to C_1^\infty(\mathcal{W}), \quad \mathcal{W} \xrightarrow{A} T(\mathcal{E})$$

$$R \in \mathrm{Vect}(\mathcal{W}), \quad R(f) = |f| f, \quad R = w\partial_w$$

$\rho = R \lrcorner \Omega, \quad \Omega = d\rho, \quad \rho(V) = 0$ if $V \in \mathrm{Vect}(\mathcal{W})$ is λ-vertical

$$\mathrm{Vect}(\mathcal{E}) \to C_1^\infty(\mathcal{W}), \quad X \mapsto \rho(X), \quad \mathcal{W} \xrightarrow{B} T^*(\mathcal{E})$$



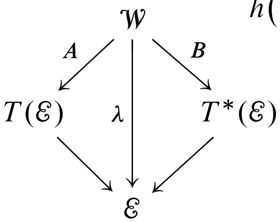
$u^1, \ldots, u^m$ are coordinated on $\mathcal{E}$

$w^1, \ldots, w^l$ are linear fiber coordinates

$$Z = A(w)\,\partial_u + C\,\partial_w + \cdots$$

$$\rho = B(w)\,du, \quad \Omega = dB(w) \wedge du$$

The toy Poisson bracket



$$h\big([s_1, s_2]\big) = As_1(h(s_2)) - As_2(h(s_1)) - Z(h)(s_1, s_2)$$

$$s_1, s_2 \in \Gamma(\lambda)$$

$$\forall h \in C_1^\infty(\mathcal{W}), \quad h\colon \Gamma(\lambda) \to C^\infty(\mathcal{E})$$

$$As_i \in \mathrm{Vect}(\mathcal{E}),$$

$$Z(h)\colon \Gamma(\lambda) \times \Gamma(\lambda) \to C^\infty(\mathcal{E})$$

$$\omega\big([s_1, s_2]\big) = L_{As_1}(\omega(s_2)) - L_{As_2}(\omega(s_1)) - L_Z(\omega)(s_1, s_2)$$

$$\omega \in \Lambda^1(\mathcal{W}), \quad |\omega| = 1. \quad \text{Take } \omega = \rho, \quad \rho(s) = Bs$$

$$B[s_1, s_2] = L_{As_1}(Bs_2) - L_{As_2}(Bs_1) - L_Z(\rho)(s_1, s_2)$$

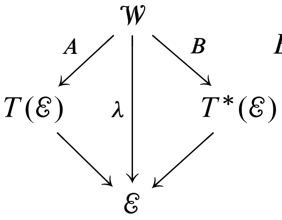
$$L_Z(\Omega) = 0 \quad \Rightarrow \quad L_Z(\rho) = \tfrac{1}{2} d(\rho(Z))$$

$$L_Z(\rho)(s_1, s_2) + L_Z(\rho)(s_2, s_1) = 0$$

$$\boxed{Bs_1(As_2) + Bs_2(As_1) = 0}$$

$$\boxed{B[s_1, s_2] = L_{As_1}(Bs_2) - L_{As_2}(Bs_1) - d(Bs_2(As_1))}$$

The toy Poisson bracket and Dirac structures



$$Bs_1(As_2) + Bs_2(As_1) = 0$$

$$B[s_1, s_2] = L_{As_1}(Bs_2) - L_{As_2}(Bs_1) - d(Bs_2(As_1))$$

$$A[s_1, s_2] = [As_1, As_2]$$

$$Bs = df\,, \quad f \in C^\infty(\mathcal{E}) \text{ is } \textit{admissible}$$

$$Bs_i = df_i, \quad B[s_1, s_2] = d(As_1(f_2))$$

$$Bs = df\,, \quad X_f = As, \quad \{f_1, f_2\} = X_{f_1}(f_2) = -X_{f_2}(f_1)$$

$$X_{\{f_1, f_2\}} = [X_{f_1}, X_{f_2}]$$

$$B^* \circ A + A^* \circ B = 0$$

$$\langle L_{As_1}(Bs_2), As_3\rangle + \langle L_{As_2}(Bs_3), As_1\rangle + \langle L_{As_3}(Bs_1), As_2\rangle = 0$$

$$0 \to \mathcal{W} \xrightarrow{\binom{A}{B}} T(\mathcal{E}) \oplus T^*(\mathcal{E}) \xrightarrow{(\,B^*\ \ A^*\,)} \mathcal{W}^* \to 0$$

Differential equations

The jet space J^∞ with coordinates x_i, $i = 1, \ldots, n$, $\quad u^j_\sigma$

$D_i = \partial_{x_i} + \sum_{j,\sigma} u^j_{\sigma i} \partial_{u^j_\sigma}$ are total derivatives

D_i span the Cartan distribution $\mathcal{C}$, $\quad [\mathcal{C}, \mathcal{C}] \subset \mathcal{C}$

Let $F_k(x_i, u^j_\sigma) = 0$ be a system of equations

Relations $F = 0$, $D_\sigma(F) = 0$ define its infinite prolongation $\mathcal{E} \subset J^\infty$

vector fields $\rightsquigarrow$ symmetries

$$\begin{array}{ccc} \mathcal{T}(\mathcal{E}) & & F = 0, \quad \ell_F(q) = 0 \\ \downarrow & & \\ \mathcal{E} & & F = 0 \end{array}$$

$$\mathcal{T}(\mathcal{E}) = T(\mathcal{E}) / \mathcal{C}$$

Conservation laws and cosymmetries

Cartan forms: $\mathcal{C}\Lambda^*(\mathcal{E}) = \{\, \omega \in \Lambda^*(\mathcal{E}) \mid \omega|_{\mathcal{C}} = 0 \,\}$

$$\Lambda^*(\mathcal{E}) \supset \mathcal{C}\Lambda^*(\mathcal{E}) \supset \mathcal{C}^2\Lambda^*(\mathcal{E}) \supset \mathcal{C}^3\Lambda^*(\mathcal{E}) \supset \cdots$$

$\Delta(F) = 0 \implies \Delta|_{\mathcal{E}} = 0$ for any $\mathcal{C}$-differential operator Δ

$$\begin{array}{ccccccccc} E_1^{0,n-1} & \xrightarrow{d_1^{0,n-1}} & E_1^{1,n-1} & \xrightarrow{d_1^{1,n-1}} & E_1^{2,n-1} & \xrightarrow{d_1^{2,n-1}} & E_1^{3,n-1} & \xrightarrow{d_1^{3,n-1}} & \cdots \\ \| & & \| & & & & & & \\ \mathrm{CL}(\mathcal{E}) & & \mathrm{Cosym}(\mathcal{E}) & & & & & & \end{array}$$

$$E_0^{0,q} = \bar{\Lambda}^q(\mathcal{E}) = \left\{ \sum f_{i_1 \dots i_q} dx_{i_1} \wedge \cdots \wedge dx_{i_q} \right\}, \quad \bar{d} = \sum dx_i \wedge D_i$$

$$0 \longrightarrow C^\infty(\mathcal{E}) \xrightarrow{\bar{d}} \bar{\Lambda}^1(\mathcal{E}) \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \bar{\Lambda}^{n-1}(\mathcal{E}) \xrightarrow{\bar{d}} \bar{\Lambda}^n(\mathcal{E}) \longrightarrow 0$$

$$\mathrm{CL}(\mathcal{E}) = \bar{H}^{n-1}(\mathcal{E}) \qquad \mathrm{Cosym}(\mathcal{E}) = \ker \ell_F^*\big|_{\mathcal{E}}$$

| functions $\rightsquigarrow$ conservation laws |

| 1-forms $\rightsquigarrow$ cosymmetries |

Cotangent covering

$$\begin{array}{ccl} \mathcal{T}^*(\mathcal{E}) & \quad & F=0, \quad \ell_F^*(p)=0 \\ \downarrow & & \\ \mathcal{E} & & F=0 \end{array}$$

$$0 \to \mathcal{C}\Lambda^1 \to \mathcal{C}\Lambda^1 \otimes \bar{\Lambda}^1 \to \cdots \to \mathcal{C}\Lambda^1 \otimes \bar{\Lambda}^{n-1} \to \mathcal{C}\Lambda^1 \otimes \bar{\Lambda}^n \to 0$$

$$\mathcal{J}^\infty(S) = \{[s]_\theta^\infty\}, \quad s \in S, \quad \theta \in \mathcal{E}$$

$$s_1 \sim s_2 \iff D_\sigma(s_1^j) = D_\sigma(s_2^j)$$

$$\cdots \to \mathcal{J}^\infty\left(\mathcal{C}\Lambda^1 \otimes \bar{\Lambda}^{n-1}\right) \to \mathcal{J}^\infty\left(\mathcal{C}\Lambda^1 \otimes \bar{\Lambda}^n\right) \to 0 \qquad (\checkmark)$$

$$\boxed{\mathcal{T}^*(\mathcal{E}) = H^{n-1}(\checkmark)}$$

$$\begin{array}{lcl} F=0 & \dashleftarrow & \ker \mathcal{J}^\infty(\ell_F^*\big|_{\mathcal{E}}) \\ \ell_F^*(p)=0 & & \end{array}$$

Sine-Gordon equation: the covering A

$$\mathcal{W}: \quad \begin{aligned} u_{xy} - \sin u &= 0 \\ w_y - u_x (u_{xx}^{-1} w_x)_y &= 0 \end{aligned}$$

$$u_{xy} = \sin u$$

$$q_{xy} = (\cos u)\, q$$

$$(u_{xx} q)\, dx + (u_x q_y)\, dy$$

$$\downarrow \; q = u_{xx}^{-1} w_x = A(w)$$

$$w_x = u_{xx} q$$

$$w_y = u_x q_y$$

$$\mathcal{T}(\mathcal{E}): \quad \begin{aligned} u_{xy} - \sin u &= 0 \\ q_{xy} - (\cos u)\, q &= 0 \end{aligned}$$

$$\downarrow$$

$$\ell_F^*(\psi)\big|_{\mathcal{W}} = 0$$

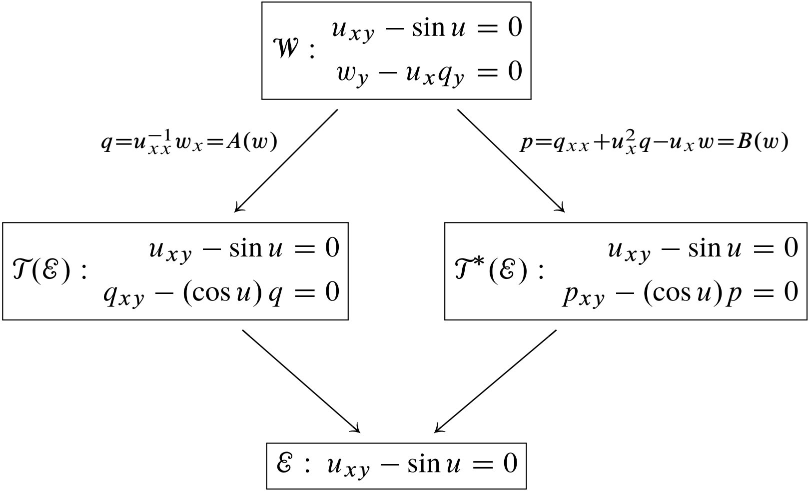
$$\mathcal{E}: \; u_{xy} - \sin u = 0$$

$$F = u_{xy} - \sin u = 0$$

$$\psi = A_3 w_{xxx} + A_2 w_{xx} + A_1 w_x + A_0 w, \quad A_i \in C^\infty(\mathcal{E})$$

Sine-Gordon equation: the covering B

$$\psi = q_{xx} + u_x^2 q - u_x w, \quad q = u_{xx}^{-1} w_x$$



Sine-Gordon equation: the symmetry

$$Z = q\partial_u + C\partial_w + \cdots, \quad Z(q) = 0$$

$$w_x = u_{xx}q \qquad C_x = q_{xx}q$$
$$C = q_x q$$
$$w_y = u_x q_y \qquad C_y = q_x q_y$$

$$Z = q\partial_u + q_x q\partial_w + \cdots$$

$$Z\begin{pmatrix} u_{xy} - \sin u \\ w_y - u_x q_y \end{pmatrix} = \begin{pmatrix} q_{xy} - (\cos u)\, q \\ 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} u_x^{-1} q D_x & -u_x^{-1} D_x \\ 0 & 0 \end{pmatrix}}_{\Delta} \begin{pmatrix} u_{xy} - \sin u \\ w_y - u_x q_y \end{pmatrix}$$

$$\boxed{Z^2 = 0}$$

$$\rho = \begin{pmatrix} p \\ 0 \end{pmatrix} = \begin{pmatrix} B(w) \\ 0 \end{pmatrix} = \begin{pmatrix} q_{xx} + u_x^2 q - u_x w \\ 0 \end{pmatrix}$$

Sine-Gordon equation: the conservation law

$$L_Z(\rho) = Z(\rho) + \Delta^*(\rho) = \begin{pmatrix} u_x q_x q - (u_x^{-1} q q_{xx})_x \\ (u_x^{-1} q_{xx})_x + u_x q_x \end{pmatrix}$$

$$\langle F_{\mathcal{W}}, L_Z(\rho)\rangle = \bar{d}\eta, \quad \eta \in \bar{\Lambda}^{n-1}(\mathcal{W}) \qquad F_{\mathcal{W}}\colon \begin{array}{r} u_{xy} - \sin u = 0 \\ w_y - u_x q_y = 0 \end{array}$$

$$(u_{xy} - \sin u)(u_x q_x q - (u_x^{-1} q q_{xx})_x) + (w_y - u_x q_y)((u_x^{-1} q_{xx})_x + u_x q_x)$$

$$= D_x\big[((w - u_x q)_y + (\sin u)\, q) u_x^{-1} q_{xx} + (\cos u)\, q_x q + \tfrac{1}{2}(q_{xy} q_x - (w - u_x q)_y (w - u_x q))\big] + D_y\big[\tfrac{1}{2}((w - u_x q)_x (w - u_x q) - q_{xx} q_x)\big]$$

$$\boxed{L_Z(\Omega) = 0}$$

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Software: Reduce
  the CDE package by Raffaele Vitolo
  the CDIFF package by Paul Kersten et al.
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Thank you!