Sub-Riemannian geodesic equations on the Virasoro-Bott group

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Geometry of PDEs and Integrability
October 18^{th}, 2013

Shape morphing

Morphing between Bush and Obama

Shape morphing

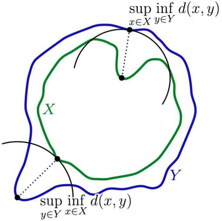
How does one compare a rabbit and a girl? How do we transform a rabbit to a girl?

Factorization

Rabbits...

Comparing shapes

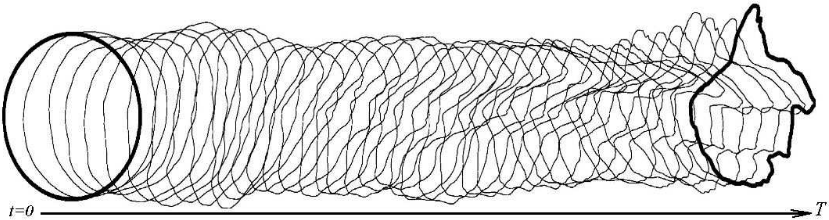
Hausdorff distance (1914)



- $d_H(X,Y) = \max\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\}$.
- First template and image are pre-processed via an edge detector (pattern recognition).
- Lack of structure morphing between shapes.

Motion in the space of shapes

Deterministic or stochastic (D. Mumford, ICM 2002 plenary lecture)



- How do we tune shapes (forgetting shift, rotation, and scaling)?
- How do we compare shapes? (Hausdorff distance is not satisfactory!)
- We will find sub-Riemannian geodesic equations for morphing one shape to another on the Virasoro-Bott group as a space of shapes with the distribution provided by the Teichmüller space and curve.

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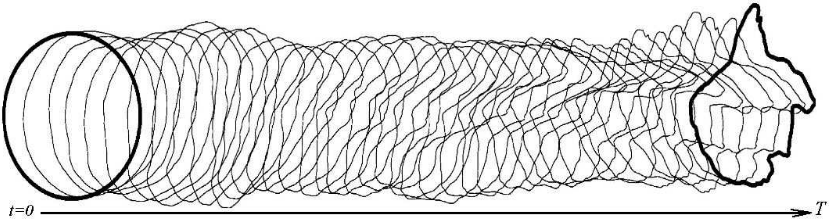

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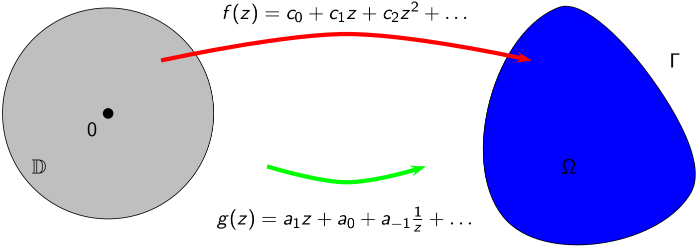
Conformal welding



Conformal welding

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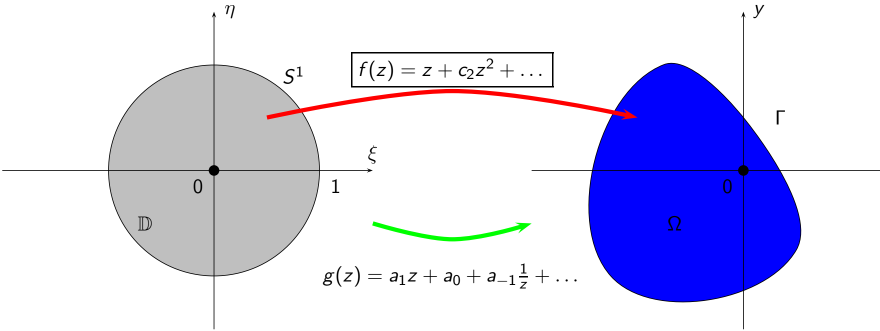

- By $\mathcal{F}_0$ we denote the class of all conformal embeddings of the unit disk $\mathbb{D}$ to $\mathbb{C}$ normalized as $f(z) = z + c_2 z^2 + \dots$, and C^∞-smooth on the boundary $S^1 = \partial \mathbb{D}$.

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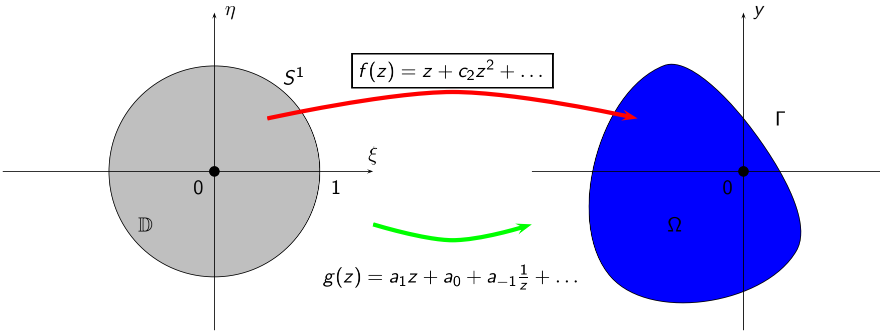

- $\gamma = f^{-1} \circ g|_{S^1} \in \mathrm{Diff}\ S^1/\mathrm{Rot}$— 'fingerprints';
- $f \in \mathcal{F}_0 \leftrightarrows \gamma \in \mathrm{Diff}\ S^1/\mathrm{Rot}$;
- Kirillov manifold= smooth Teichmüller curve.

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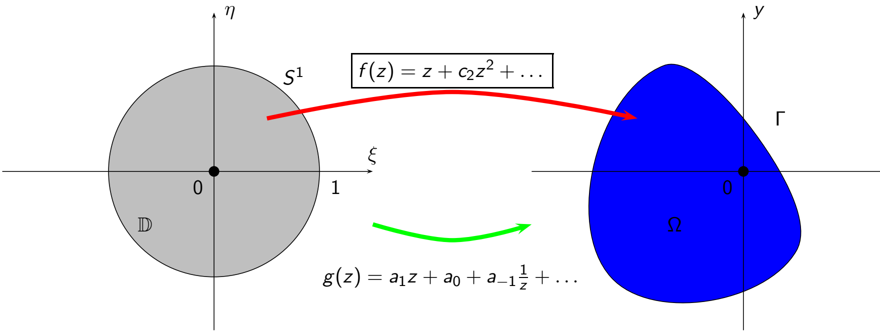

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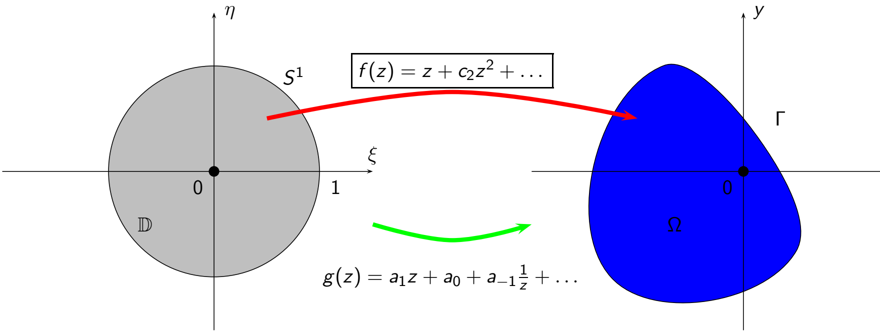

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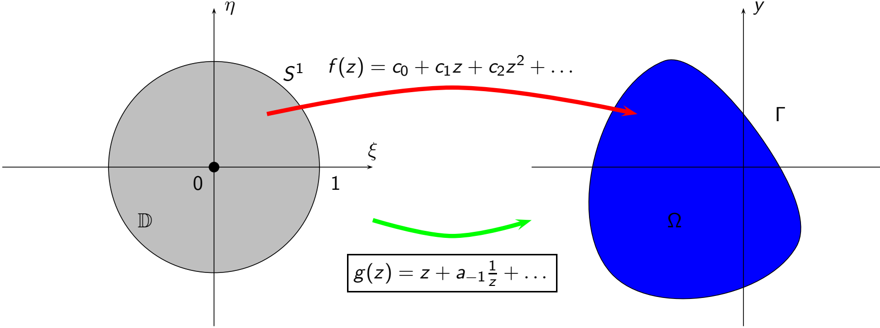

- By $\mathcal{G}_0$ we denote the class of all conformal embeddings of the exterior of the unit disk $\mathbb{D}^-$ to $\hat{\mathbb{C}}$ with the hydrodynamics normalisation normalized as $g(z) = z + \frac{a_{-1}}{z} + \dots$, and C^∞-smooth on the boundary $S^1 = \partial\mathbb{D}$.

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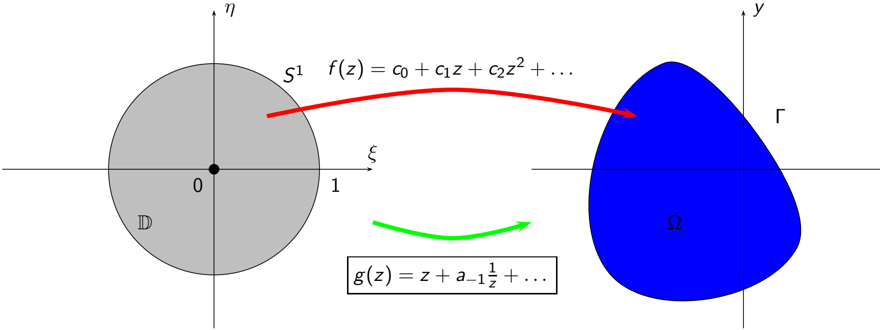

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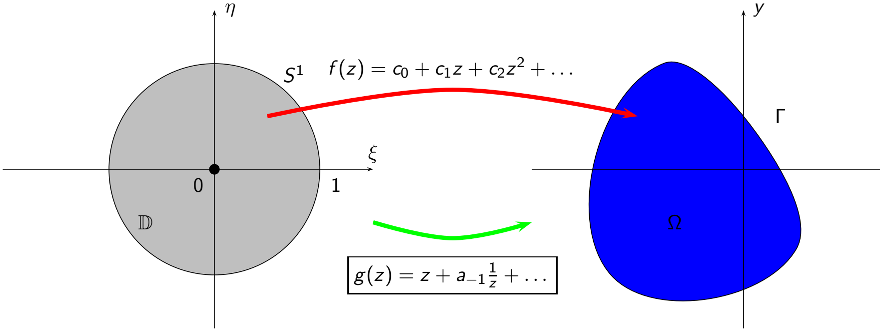

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Witt algebra

- $\mathrm{Vect}\ S^1$– Lie algebra of smooth vector fiels on S^1;
- It can be identified with the Lie algebra $\mathfrak{diff}\, S^1$;
- Lie bracket $[\varphi_1(\theta)\partial_\theta, \varphi_2(\theta)\partial_\theta] = (\varphi_1'\varphi_2 - \varphi_2'\varphi_1)\partial_\theta$;
- If $\{-ie^{ik\theta}\}_{k\in\mathbb{Z}}$ is a complex Fourier basis, then

$$[-ie^{im\theta}\partial_\theta, -ie^{in\theta}\partial_\theta] = -i(n-m)e^{i(n+m)\theta}\partial_\theta;$$

or

$L_k = -ie^{im\theta}\partial_\theta \Longrightarrow [L_m, L_n] = (n-m)L_{n+m}$– Witt relation.

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Virasoro algebra

- $\mathfrak{vir} = \mathrm{Vect}\ S^1 \oplus \mathbb{R}$:
- $(\varphi, c) \in \mathfrak{vir} \Rightarrow [(\varphi_1, c_1), (\varphi_2, c_2)]_{\mathfrak{vir}} = \big([\varphi_1, \varphi_2], \frac{c}{12}\omega_{\mu\nu}(\varphi_1, \varphi_2)\big)$,
- Gelfand-Fuchs cosycle is unique non-trivial:

$$\omega_{\mu\nu}(\varphi_1, \varphi_2) = \frac{1}{2\pi}\int_0^{2\pi} \left(\mu\varphi_1\varphi_2' + \nu\varphi_1'\varphi_2''\right) d\theta;$$

- $\omega_{\mu\nu}(L_{-n}, L_n) = n(\mu + \nu n^2)$.
- Virasoro commutation relations:

$$\boxed{[L_m, L_n]_{\mathfrak{vir}} = (n-m)L_{n+m} + \frac{c}{12} n(\mu + \nu n^2)\delta_{n,-m}.}$$

- $c-$ central charge.

Virasoro-Bott group

- Vir$=\mathrm{Diff}\ S^1 \times \mathbb{R}$:
- $(\phi, a) \in$Vir $\Rightarrow$
  $(\phi_1, a_1) \circ (\phi_2, a_2) = \big((\phi_1 \circ \phi_2), a_1 + a_2 + \frac{c}{12}\Omega_{\mu\nu}(\phi_1, \phi_2)\big)$;
  (here we use the covering space for $\mathrm{Diff}\ S^1$)
- Thurston-Bott cosicle:

$$\Omega_{\mu\nu}(\phi_1, \phi_2) = \mu A(\phi_1, \phi_2) + \nu B(\phi_1, \phi_2);$$

- Extensions are non-trivial if and only if $\nu \neq 0$.

Coadjoint representation

- We will work with $\mathrm{Diff}\ S^1$ and Virasoro-Bott group $\mathrm{Vir}{=}\mathrm{Diff}\ S^1 \oplus \mathbb{R}$;
- Classification of orbits in the coadjoint representation of Vir;
- $\mathrm{Diff}\ S^1/\mathrm{Möb}$ and $\mathrm{Diff}\ S^1/\mathrm{Rot}$ carry the structure of infinite-dimensional, homogeneous, complex analytic Kählerian manifolds (*after complexification*).
- $\mathrm{Diff}\ S^1/\mathrm{Möb}$ is a smooth approximation of the universal Teichmüller space T;
- $\mathrm{Diff}\ S^1/\mathrm{Rot}$ is a smooth approximation of the universal Teichmüller curve $\hat{T}$ = complex line bundle over T;
- Fibration $\pi\colon \hat{T} \to T$ with typical fiber $\mathrm{Möb}/S^1 \simeq \mathbb{D}^- = \{|z| > 1\}$.

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Fréchet manifolds

- $\mathrm{Diff}\ S^1$ is a Lie-Fréchet group;
- $\hat{T} = \mathrm{Diff}\ S^1/\mathrm{Rot}$ and $T = \mathrm{Diff}\ S^1/\mathrm{Möb}$ are Fréchet homogeneous manifolds.
- $\mathrm{Diff}\ S^1$ acts on $\hat{T}$ and T;
- $\hat{T}$ is a base space for the principal bundle $\mathrm{Rot} \longrightarrow \mathrm{Diff}\ S^1 \xrightarrow{\pi_0} \hat{T}$;
- T is a base space for the principal bundle $\mathrm{Möb} \longrightarrow \mathrm{Diff}\ S^1 \xrightarrow{\pi_1} T$.

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Lie-Fréchet algebras

- The Lie algebra of $\text{Diff}\ S^1$ is the Witt algebra $\mathfrak{diff} \simeq \text{Vect}\ S^1$;
- Real-valued form

$$\eta_0(v) = \frac{1}{2\pi}\int_0^{2\pi} v(\theta)d\theta;$$

- Complex-valued form

$$\eta_1(v) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta} v(\theta)d\theta.$$

- $\mathfrak{v}_0 = \ker \eta_0$ and $\mathfrak{v}_1 = \ker \eta_0 \cap \ker \eta_1$;
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Lie-Fréchet algebras

- The Lie algebra of Vir is the Virasoro algebra $\mathfrak{vir}^c_{\mu\nu}$;
- Real-valued form is the same $\eta_0(v)$
- $\mathfrak{e} = (\ker \eta_0, 0)$ and $\mathfrak{k} = (a_0\partial_\theta, b) = (\mathfrak{rot}, b)$;
- Then $\mathfrak{vir}_{\mu\nu} = \mathfrak{e} \oplus \mathfrak{k}$.

Sub-bundles for $T\mathrm{Diff}\,S^1$

- $\mathcal{V}_0 =$ left translations of $\mathfrak{v}_0$ by $\mathrm{Diff}\,S^1$;
- $\mathcal{V}_1 =$ left translations of $\mathfrak{v}_1$ by $\mathrm{Diff}\,S^1$;
- $\mathcal{R} = \ker d\pi_0$, $\mathcal{M} = \ker d\pi_1$;
- $T\mathrm{Diff}\,S^1 = \mathcal{V}_0 \oplus \mathcal{R} = \mathcal{V}_1 \oplus \mathcal{M}$;
- Sub-bundles $\mathcal{V}_0$ and $\mathcal{V}_1$ are Ehresmann connections.

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SUB-BUNDLE FOR TVIR

- $\mathcal{E} =$ left translations of $\mathfrak{e}$ by Vir;
- $\mathcal{R} =$ ker $d\pi_0$;
- TVir$= \mathcal{E} \oplus (\mathcal{R} \times \mathbb{R})$;

Sub-Riemannian manifolds

- $\langle \cdot, \cdot \rangle_{\mathfrak{diff}}$ is an inner product on $\mathfrak{diff}$;
- $\langle \cdot, \cdot \rangle_{\mathfrak{vir}}$ its extension to $\mathfrak{vir}$.
- $\mathbf{g}$ is a Riemannian metric on $T\mathrm{Diff}\, S^1$ obtained by left actions of $\mathrm{Diff}\, S^1$ of $\langle \cdot, \cdot \rangle_{\mathfrak{diff}}$;
- $\hat{\mathbf{g}}$ is a Riemannian metric on $T\mathrm{Vir}$ obtained by left actions of Vir of $\langle \cdot, \cdot \rangle_{\mathfrak{vir}}$;
- $\mathbf{h_0}$= restriction of $\mathbf{g}$ to $\mathcal{V}_0$;
- $\mathbf{h_1}$= restriction of $\mathbf{g}$ to $\mathcal{V}_1$;
- $\hat{\mathbf{h}}$= restriction of $\hat{\mathbf{g}}$ to $\mathcal{E}$
- Manifolds $(\mathrm{Diff}\, S^1, \mathcal{V}_0, \mathbf{h_0})$, $(\mathrm{Diff}\, S^1, \mathcal{V}_1, \mathbf{h_1})$, and $(\mathrm{Vir}, \mathcal{E}, \hat{\mathbf{h}})$.

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- $\mathcal{V}$-horizontal curves $\gamma(t)$ connecting two points a_0 and a_1, $\gamma(0) = a_0$, $\gamma(1) = a_1$ on $\mathrm{Diff}\, S^1$;
- Energy functional
$$E(\gamma) = \frac{1}{2} \int_0^1 \mathbf{h}(\dot{\gamma}, \dot{\gamma}) dt;$$
- Critical values in the space of horizontal curves= geodesics;

Problem: ∞-dimensional manifolds.
Treatment: prof. Markina's talk:

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- Convenient calculus (most general notion of smoothness, smooth curves): A. Kriegl and P. W. Michor;
- Regular Lie groups: Omori, Maeda, Yoshioka (1980), Milnor (1984):
  - Lie group G modeled on a convienient vector space with the Lie algebra $\mathfrak{g}$;
  - ℓ_a is left multiplication by an element $a \in G$;
  - Left Maurer-Cartan form κ^ℓ is a $\mathfrak{g}$-valued one-form on G given by

$$\kappa^\ell(v) = d\ell_{a^{-1}} v, \qquad v \in T_a G;$$

  - The *left logarithmic derivative* of γ is a smooth curve $u(t) = \kappa^\ell(\dot\gamma(t))$;
  - G is regular if $\gamma \leftrightarrow u$ is one-to-one.

Our groups $\mathrm{Diff}\, S^1$ and $\mathrm{Vir}{=}\mathrm{Diff}\, S^1 \oplus \mathbb{R}$ are regular.

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Left logarithmic derivative

- For $\gamma(t) \subset \mathrm{Diff}\, S^1$ the left Maurer-Cartan form is

$$u(t) = \kappa^{\ell}(\dot{\gamma}(t)) = \frac{\dot{\gamma}}{\gamma'};$$

- For $(\gamma(t), b(t)) \subset$Vir the left Maurer-Cartan form is

$$\hat{u}(t) = \kappa^{\ell}(\dot{\gamma}(t), \dot{b}(t)) = \left(\frac{\dot{\gamma}}{\gamma'}, C(t)\right),$$

where

$$C(t) = \dot{b}(t) - \frac{\mu}{4\pi}\int_0^{2\pi} u(t) d\theta + \frac{\mu}{4\pi}\int_0^{2\pi} u(t) d\gamma(t) +$$

$$+\frac{\nu}{4\pi}\int_0^{2\pi} u'(t) d\log\gamma'(t).$$

Euler-Arnold equations

- Assume G is a Lie group with the Lie algebra $\mathfrak{g}$, and $\langle \cdot, \cdot \rangle$ is defined on $\mathfrak{g}$;
- $\mathrm{ad}_x : y \mapsto [x, y]$ exists for any $x \in \mathfrak{g}$;
- $\mathrm{ad}_x^\top$ to ad_x.

Theorem (Grong, Markina, A. V.)

Assume $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, where $\mathfrak{h}^\perp$ is the orthogonal complement to $\mathfrak{h}$ with respect to $\langle \cdot, \cdot \rangle$. If a horizontal curve γ is a geodesic, then it is either a semi-rigid curve or it is a normal geodesic. In the latter case it is a solution to the equations

$$u = \kappa^\ell(\dot\gamma), \qquad \dot u = \mathrm{pr}_{\mathfrak{h}} \, \mathrm{ad}_u^\top (u + \lambda), \quad \dot\lambda = \mathrm{pr}_{\mathfrak{h}^\perp} \, \mathrm{ad}_u^\top (u + \lambda),$$

for some curve λ in $\mathfrak{h}^\perp$.

Here $\mathrm{pr}_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ is the orthogonal projection with respect to $\langle \cdot, \cdot \rangle$.

Subgroup-invariant metric

- Special case G has a subgroup K, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, and $\mathfrak{h} = \mathfrak{k}^{\perp}$;
- Metric $\mathbf{g}$ is invariant under K;
- (for finite-dimensional K, equivalently, $\langle \cdot, \cdot \rangle$ is $\mathrm{ad}(\mathfrak{k})$-invariant).

Theorem (GMV)

$\dot{u} = \mathrm{pr}_{\mathfrak{h}}\, \mathrm{ad}_u^{\top}(u + \lambda)$, *and* $\lambda =$ *constant.*

We work with $\mathrm{Diff}\ S^1$**:**

- On $\mathrm{Vect}_0 = \ker \eta$, $\eta_x = \frac{1}{2\pi}\int_0^{2\pi} x(\theta)d\theta$, $x \in \mathrm{Vect}\ S^1 = \mathfrak{diff}$ define

$$\langle x, y\rangle_0^{\alpha\beta} := \frac{1}{2\pi}\int_0^{2\pi}(\alpha x(\theta)y(\theta) + \beta x'(\theta)y'(\theta))d\theta = \langle x, L_{\alpha\beta}y\rangle^{1,0},$$

$L_{\alpha\beta}x = \beta\partial_\theta^2 x - \alpha x.$

Extend $\langle x, y\rangle_0^{\alpha\beta}$ to $\langle x, y\rangle^{\alpha\beta}$ on $\mathrm{Vect}\ S^1$:

$$\langle x, y\rangle^{\alpha\beta} = \left\langle x - \eta_x, y - \eta_y\right\rangle_0^{\alpha\beta} + \eta_x\eta_y.$$

Principal bundle $\mathrm{Rot} \longrightarrow \mathrm{Diff}\, S^1 \xrightarrow{\pi_0} \hat{T} = \mathrm{Diff}\, S^1/\mathrm{Rot}$.

- The sub-Riemannian geodesic equation for $\gamma\colon [0,1] \to \mathrm{Diff}\, S^1$ with the base space $\mathrm{Diff}\, S^1/\mathrm{Rot}$ is

$$\boxed{\beta \dot{u}'' - \alpha \dot{u} = \beta(uu''' + 2u'u'') - 3\alpha uu' - 2\lambda u', \quad u(t) = \frac{\dot{\gamma}}{\gamma'}.}$$

Particular cases

- $\alpha = 1,\ \beta = 0 \Longrightarrow$ Riemann equation $\dot{u} = 3uu' + 2\lambda u'$;
- $\alpha = 0,\ \beta = 1 \Longrightarrow$ Hunter-Saxton equation $\dot{u}'' = uu''' + 2u'u'' - 2\lambda u'$;
- $\alpha = 1,\ \beta = 1 \Longrightarrow$ Camassa-Holm equation

$$\dot{u}'' - \dot{u} = uu''' + 2u'u'' - 3uu' - 2\lambda u'.$$

We work with Vir:

Principal bundle $\mathrm{Rot} \times \mathbb{R} \longrightarrow \mathrm{Vir} \xrightarrow{\hat{\pi}_0} \hat{T} = \mathrm{Diff}\, S^1/\mathrm{Rot}.$

- Extention of the Sobolev metric:

$$\langle (x, a_1), (y, a_2) \rangle^{\alpha\beta}_{\mu\nu} = \langle x, y \rangle^{\alpha\beta} + a_1 a_2.$$

- Particular case $\alpha = 1,\ \beta = 0 \Longrightarrow$ KdV:

$$\boxed{\dot{u} = 3uu' + (2\lambda_1 - \lambda_2\mu)u' + \lambda_2\nu u'''.}$$

($\lambda_1 = 0,\ \lambda_2 = 1,\ \mu = 0,\ \nu = 1-$ classical KdV)

Diff S^1 AND $\mathcal{F}_0$

- $\mathcal{A}_0$– the space of analytic functions in $\mathbb{D}$, $F(0)=0$, which is C^∞-smooth on $\hat{\mathbb{D}} = \mathbb{D} \cup S^1$;
- $\mathcal{A}_0$– is a complex Fréchet vector space with the topology defined by seminorms
$$\|F\|_m = \sup\{|F^{(m)}(z)| \mid z \in \hat{\mathbb{D}}\};$$
- $\mathcal{F}_0 \subset \mathcal{A}_0$ open subset;
- $\mathrm{Diff}\, S^1/S^1 \simeq \mathcal{F}_0$.

- For any smooth curve f_t in $\mathcal{F}_0$ with $f_0 = id_{\mathbb{D}}$ we can write

$$f_t(z) = z + tzF(z) + o(t), \qquad F \in \mathcal{A}_0;$$

- Velling-Kirillov metric $\mathcal{F}_0 \subset \mathcal{A}_0$, $T_{id_{\mathbb{D}}}\mathcal{F}_0 \simeq \mathcal{A}_0$;

$$\begin{aligned}\mathbf{b}_{\alpha\beta}\big|_{id_{\mathbb{D}}}(F_1, F_2) &= \frac{1}{\pi}\iint_{\mathbb{D}} \left(\alpha F_1'\overline{F}_2' + \beta(zF_1')'\overline{(zF_2')'}\right) d\sigma(z), \\ &= \sum_{n=1}^{\infty}(\alpha n + \beta n^3)a_n\overline{b}_n;\end{aligned}$$

$F_1(z) = \sum_{n=1}^{\infty} a_n z^n$, $F_2(z) = \sum_{n=1}^{\infty} b_n z^n$.

- $\alpha = 1$ and $\beta = 0 \Rightarrow$ Weil-Petersson metric;

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Teichmüller curve (Kirillov manifold)

- Principal bundle $\mathrm{Rot}\longrightarrow\mathrm{Diff}\, S^1 \xrightarrow{\pi_0} \hat{T}$;
- J is the Hilbert transform;
- $L_{\alpha\beta} = \beta\partial_\theta^2 - \alpha$ (Hill operator);

Theorem (GMV)

If γ is a normal geodesic in the Velling-Kirillov metric with the logarithmic derivative $u(t) = \kappa^\ell(\dot\gamma(t))$, then

$$L_{\alpha\beta} J\dot{u}' = uL_{\alpha\beta}Ju'' + 2u'L_{\alpha\beta}Ju' + 2\lambda u', \quad \lambda \in \mathbb{R}.$$

For $(\alpha, \beta) = (1, 0)$, this is a special case of the modified Constantin-Lax-Majda (CLM) equation.

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- Principal bundle $\text{Möb} \longrightarrow \text{Diff}\, S^1 \xrightarrow{\pi_1} T$;

THEOREM (GMV)

If γ is a normal geodesic in the Weil-Petersson metric with the logarithmic derivative $u(t) = \kappa^{\ell}(\dot{\gamma}(t))$, then

$$L_{-1,1} J \dot{u}' + \dot{\lambda} = \mathrm{ad}_u^{\top}(L_{-1,1} u' + \lambda).$$

λ in no longer constant and we must solve an additional equation.

Discussion

The equation

$$L_{\alpha\beta}J\dot{u}' = uL_{\alpha\beta}Ju'' + 2u'L_{\alpha\beta}Ju' + 2\lambda u', \quad \lambda \in \mathbb{R}$$

is equivalent to

$$\boxed{\dot{v}''' + \dot{v}' = uv'''' + uv'' + 2u'v''' + 2u'v' + 2\lambda u' + u\lambda' - 3i(\bar{w}c_2 - w\bar{c}_2 e^{-i\theta}),}$$

where $v = Ju$, $\lambda = \lambda_0 + we^{i\theta} + \bar{w}e^{-i\theta}$, $\dot{w} = 3i\bar{w}c_2$,
$c_2(t)$ is an arbitrary function $|c_2| < 2$.
For example $w = 0$, $\lambda \equiv \lambda_0$

$$\dot{v}''' + \dot{v}' = uv'''' + uv'' + 2u'v''' + 2u'v' + 2\lambda u'$$

CONTROLLABILITY

- No Chow-Rashevskiĭ theorem in general;

THEOREM (YURIĬ LEDYAEV, 2004)

If M is an infinite-dimensional manifold modelled on a Hilbert space, if $\mathcal{V}$ is a bracket generating distribution, then for any pair of points $a_0, a_1 \in M$ there is a sequence of horizontal curves $\gamma_n \in C_{\mathcal{V}}(a_0.a_1)$, such that $\gamma_n(0) = a_0$, $\gamma_n(1) \to a_1$.

Controllability

- Generalization of Ledyev's theorem to Fréchet manifolds (Irina Markina and Mahdi Salehani);
- For both

$$\mathrm{M\ddot{o}b} \longrightarrow \mathrm{Diff}\, S^1 \xrightarrow{\pi_1} T$$

  and

$$\mathrm{Rot} \longrightarrow \mathrm{Diff}\, S^1 \xrightarrow{\pi_0} \hat{T}$$

  we proved complete controllability (Erlend Grong, Irina Markina, and A.V.).

Děkuju mnohokrát

Na zdraví!