Symmetries and conservation laws for a generalization of Kawahara equation

Jakub Vašíček

Mathematical institute in Opava
Silesian University in Opava

Local and Nonlocal Geometry of PDEs and Integrability
Conference dedicated to the 70th birthday of Joseph Krasil'shchik

Trieste 2018

The Kawahara equation

The Kawahara equation, see Kawahara '72, Gandarias et. al. '17 and references therein, reads

$$u_t = \mu u_{5x} + \gamma u_{xxx} + \beta u^2 u_x + \alpha u u_x, \tag{1}$$

where $\alpha, \beta, \gamma, \mu$ are constants, $\mu \neq 0$ and $u\alpha + \beta \neq 0$.

The Kawahara equation

The Kawahara equation, see Kawahara '72, Gandarias et. al. '17 and references therein, reads

$$u_t = \mu u_{5x} + \gamma u_{xxx} + \beta u^2 u_x + \alpha u u_x, \tag{1}$$

where $\alpha, \beta, \gamma, \mu$ are constants, $\mu \neq 0$ and $u\alpha + \beta \neq 0$. By rescaling t we can WLOG set $\mu = 1$.

Equation (1) has a number of applications in physics, in particular in the study of plasma waves and water waves, cf. Kawahara '72, Gandarias et. al. '17 and references therein.

The Kawahara equation

The Kawahara equation, see Kawahara '72, Gandarias et. al. '17 and references therein, reads

$$u_t = \mu u_{5x} + \gamma u_{xxx} + \beta u^2 u_x + \alpha u u_x, \tag{1}$$

where $\alpha, \beta, \gamma, \mu$ are constants, $\mu \neq 0$ and $u\alpha + \beta \neq 0$. By rescaling t we can WLOG set $\mu = 1$.

Equation (1) has a number of applications in physics, in particular in the study of plasma waves and water waves, cf. Kawahara '72, Gandarias et. al. '17 and references therein. We consider the following generalization of (1) with $\mu = 1$:

$$u_t = u_{5x} + b u_{xxx} + f(u) u_x, \tag{2}$$

The Kawahara equation

The Kawahara equation, see Kawahara '72, Gandarias et. al. '17 and references therein, reads

$$u_t = \mu u_{5x} + \gamma u_{xxx} + \beta u^2 u_x + \alpha u u_x, \tag{1}$$

where $\alpha, \beta, \gamma, \mu$ are constants, $\mu \neq 0$ and $u\alpha + \beta \neq 0$. By rescaling t we can WLOG set $\mu = 1$.

Equation (1) has a number of applications in physics, in particular in the study of plasma waves and water waves, cf. Kawahara '72, Gandarias et. al. '17 and references therein.
We consider the following generalization of (1) with $\mu = 1$:

$$u_t = u_{5x} + b u_{xxx} + f(u) u_x, \tag{2}$$

where b is a constant and f is a function of u.
We shall refer to (2) as to **GKE**.

The Kawahara equation

The Kawahara equation, see Kawahara '72, Gandarias et. al. '17 and references therein, reads

$$u_t = \mu u_{5x} + \gamma u_{xxx} + \beta u^2 u_x + \alpha u u_x, \tag{1}$$

where $\alpha, \beta, \gamma, \mu$ are constants, $\mu \neq 0$ and $u\alpha + \beta \neq 0$. By rescaling t we can WLOG set $\mu = 1$.

Equation (1) has a number of applications in physics, in particular in the study of plasma waves and water waves, cf. Kawahara '72, Gandarias et. al. '17 and references therein. We consider the following generalization of (1) with $\mu = 1$:

$$u_t = u_{5x} + b u_{xxx} + f(u) u_x, \tag{2}$$

where b is a constant and f is a function of u.
We shall refer to (2) as to **GKE**.
Blanket assumption: the function f is nonconstant (so GKE is necessarily nonlinear).

More on the Kawahara equation

The authors Gandarias et. al. '17 considered a slightly broader generalization of (1) than GKE (2), namely,

$$u_t = a(t)u_{5x} + b(t)u_{xxx} + c(t)f(u)u_x, \tag{3}$$

but they study only its *Lie point* symmetries and *low-order* conservation laws.

More on the Kawahara equation

The authors Gandarias et. al. '17 considered a slightly broader generalization of (1) than GKE (2), namely,

$$u_t = a(t)u_{5x} + b(t)u_{xxx} + c(t)f(u)u_x, \qquad (3)$$

but they study only its *Lie point* symmetries and *low-order* conservation laws.

On the other hand, we obtain below a *complete description* of *generalized* symmetries and *local* conservation laws of all orders for GKE (2).

Preliminaries

Consider an evolution equation in two independent and one dependent variable of the form

$$u_t = K(x, u, u_x, \ldots, u_{nx}), \qquad n \geq 2, \tag{4}$$

where $u_{jx} = \partial^j u / \partial x^j$.

Preliminaries

Consider an evolution equation in two independent and one dependent variable of the form

$$u_t = K(x, u, u_x, \dots, u_{nx}), \qquad n \geq 2, \tag{4}$$

where $u_{jx} = \partial^j u / \partial x^j$.
Following mostly Krasil'shchik and Verbovetsky '11, Mikhailov et. al. '87 and '09 and Olver '93 we recall the basic notions we need.

Preliminaries

Consider an evolution equation in two independent and one dependent variable of the form

$$u_t = K(x, u, u_x, \ldots, u_{nx}), \qquad n \geq 2, \tag{4}$$

where $u_{jx} = \partial^j u / \partial x^j$.
Following mostly Krasil'shchik and Verbovetsky '11, Mikhailov et. al. '87 and '09 and Olver '93 we recall the basic notions we need.
Let D_x and D_t be total derivatives in x and t restricted to (4), that is:

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{(i+1)x} \frac{\partial}{\partial u_{ix}}, \qquad D_t = \frac{\partial}{\partial t} + \sum_{i=0}^{\infty} D_x^i(K) \frac{\partial}{\partial u_{ix}}$$

Local functions

A *local* function Q in our context is a function that may depend on $x, t, u, u_x, \ldots, u_{kx}$ for an arbitrary but finite order k.

Local functions

A *local* function Q in our context is a function that may depend on $x, t, u, u_x, \ldots, u_{kx}$ for an arbitrary but finite order k.

Definition 1

A function $f(x, t, u, \ldots, u_s)$ is a *rational local function* if and only if it can be written as $f = g/h$ where g, h are local functions polynomial in all their arguments.

Local functions

A *local* function Q in our context is a function that may depend on $x, t, u, u_x, \ldots, u_{kx}$ for an arbitrary but finite order k.

Definition 1

A function $f(x, t, u, \ldots, u_s)$ is a *rational local function* if and only if it can be written as $f = g/h$ where g, h are local functions polynomial in all their arguments.

We shall denote the field of rational local functions by $\mathscr{A}_0$. Let $\mathscr{A}$ be an extension of $\mathscr{A}_0$ such that $K \in \mathscr{A}$ and $\mathscr{A}$ is closed under D_x and D_t, cf. Mikhailov '09.

Local functions

A *local* function Q in our context is a function that may depend on $x, t, u, u_x, \ldots, u_{kx}$ for an arbitrary but finite order k.

Definition 1

A function $f(x, t, u, \ldots, u_s)$ is a *rational local function* if and only if it can be written as $f = g/h$ where g, h are local functions polynomial in all their arguments.

We shall denote the field of rational local functions by $\mathscr{A}_0$. Let $\mathscr{A}$ be an extension of $\mathscr{A}_0$ such that $K \in \mathscr{A}$ and $\mathscr{A}$ is closed under D_x and D_t, cf. Mikhailov '09.
We shall further refer to the elements of $\mathscr{A}$ as to *differential functions*. Unless explicitly stated otherwise all functions below are assumed to belong to $\mathscr{A}$.

Symmetries and formal series

An *evolutionary vector field* $\mathbf{v}_Q = Q\partial/\partial u$ with the characteristic $Q \in \mathscr{A}$ is a *generalized symmetry* of (4) iff Q satisfies

$$D_t(Q) = \mathbf{D}_K(Q). \tag{5}$$

Symmetries and formal series

An *evolutionary vector field* $\mathbf{v}_Q = Q\partial/\partial u$ with the characteristic $Q \in \mathscr{A}$ is a *generalized symmetry* of (4) iff Q satisfies

$$D_t(Q) = \mathbf{D}_K(Q). \tag{5}$$

Here, for $F = F(x, t, u, u_x, \ldots, u_{kx}) \in \mathscr{A}$

$$\mathbf{D}_F = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i} D_x^i. \tag{6}$$

Symmetries and formal series

An *evolutionary vector field* $\mathbf{v}_Q = Q\partial/\partial u$ with the characteristic $Q \in \mathscr{A}$ is a *generalized symmetry* of (4) iff Q satisfies

$$D_t(Q) = \mathbf{D}_K(Q). \tag{5}$$

Here, for $F = F(x, t, u, u_x, \dots, u_{kx}) \in \mathscr{A}$

$$\mathbf{D}_F = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i} D_x^i. \tag{6}$$

Consider an algebra $\mathscr{L}$ of formal series (Kupershmidt '00):

$$L = \sum_{i=-\infty}^{k} a_i \xi^i, \qquad a_i \in \mathscr{A}. \tag{7}$$

Symmetries and formal series

An *evolutionary vector field* $\mathbf{v}_Q = Q\partial/\partial u$ with the characteristic $Q \in \mathscr{A}$ is a *generalized symmetry* of (4) iff Q satisfies

$$D_t(Q) = \mathbf{D}_K(Q). \tag{5}$$

Here, for $F = F(x, t, u, u_x, \ldots, u_{kx}) \in \mathscr{A}$

$$\mathbf{D}_F = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i} D_x^i. \tag{6}$$

Consider an algebra $\mathscr{L}$ of formal series (Kupershmidt '00):

$$L = \sum_{i=-\infty}^{k} a_i \xi^i, \qquad a_i \in \mathscr{A}. \tag{7}$$

For L (7) define its degree $\deg L = k$ assuming $a_k \neq 0$ with the convention that $\deg 0 = -\infty$.

More on formal series

The multiplication of two monomials is defined by the formula

$$a\xi^i \circ b\xi^j = a \sum_{k=0}^{\infty} \frac{i(i-1)\cdots(i-k+1)}{k!} D_x^k(b)\xi^{i+j-k}.$$

More on formal series

The multiplication of two monomials is defined by the formula

$$a\xi^i \circ b\xi^j = a \sum_{k=0}^{\infty} \frac{i(i-1)\cdots(i-k+1)}{k!} D_x^k(b)\xi^{i+j-k}.$$

For $L = \sum_{i=-\infty}^{q} a_i \xi^i$ its *formal adjoint* is

$$L^* = \sum_{i=-\infty}^{q} (-\xi)^i \circ a_i.$$

More on formal series

The multiplication of two monomials is defined by the formula

$$a\xi^i \circ b\xi^j = a \sum_{k=0}^{\infty} \frac{i(i-1)\cdots(i-k+1)}{k!} D_x^k(b)\xi^{i+j-k}.$$

For $L = \sum_{i=-\infty}^{q} a_i \xi^i$ its *formal adjoint* is

$$L^* = \sum_{i=-\infty}^{q} (-\xi)^i \circ a_i.$$

Let $L \in \mathscr{L}$, $L = \sum_{i=-\infty}^{k} a_i \xi^i$, have $\deg L = k > 0$. The k-th root $L^{1/k}$ of L is a formal series of degree one of the form

$$L^{1/k} = \sum_{i=-\infty}^{1} \bar{a}_i \xi^i, \quad \tilde{a}_i \in \mathscr{A}$$

such that $\underbrace{L^{1/k} \circ L^{1/k} \circ \cdots \circ L^{1/k}}_{k \text{ times}} = L.$

Formal symmetries

Definition 2

Let $u_t = K$, where $K \in \mathscr{A}$ be an n-th order differential equation. A *formal symmetry of rank* k for this equation is a formal series $L \in \mathscr{L}$ of degree m which satisfies

$$\deg(D_t(L) - [\widehat{\mathbf{D}}_K, L]) \leq m + n - k, \tag{8}$$

where $\widehat{\mathbf{D}}_K$ is defined as $\widehat{\mathbf{D}}_K = \sum_{i=0}^{n} \frac{\partial K}{\partial u_i} \xi^i$.

Formal symmetries

Definition 2

Let $u_t = K$, where $K \in \mathscr{A}$ be an n-th order differential equation. A *formal symmetry of rank* k for this equation is a formal series $L \in \mathscr{L}$ of degree m which satisfies

$$\deg(D_t(L) - [\widehat{\mathbf{D}}_K, L]) \leq m + n - k, \tag{8}$$

where $\widehat{\mathbf{D}}_K$ is defined as $\widehat{\mathbf{D}}_K = \sum_{i=0}^{n} \frac{\partial K}{\partial u_i} \xi^i$.

For a local function f we define its order ord f=deg $\widehat{\mathbf{D}}_f$.

Formal symmetries

Definition 2

Let $u_t = K$, where $K \in \mathscr{A}$ be an n-th order differential equation. A *formal symmetry of rank* k for this equation is a formal series $L \in \mathscr{L}$ of degree m which satisfies

$$\deg(D_t(L) - [\widehat{\mathbf{D}}_K, L]) \leq m + n - k, \tag{8}$$

where $\widehat{\mathbf{D}}_K$ is defined as $\widehat{\mathbf{D}}_K = \sum_{i=0}^{n} \frac{\partial K}{\partial u_i} \xi^i$.

For a local function f we define its order ord f=deg $\widehat{\mathbf{D}}_f$.

Lemma 3

If G is a characteristic of generalized symmetry of order q for $u_t = K$ then $\widehat{\mathbf{D}}_G$ is a formal symmetry of degree q and rank at least q for this equation.

Conservation laws

Any *local conservation law* for (4) can WLOG be assumed (Olver '93) to read

$$D_t(\rho) = D_x(\sigma), \qquad \rho, \sigma \in \mathscr{A}, \tag{9}$$

where ρ is called the *density* and σ the *flux*.

Conservation laws

Any *local conservation law* for (4) can WLOG be assumed (Olver '93) to read

$$D_t(\rho) = D_x(\sigma), \qquad \rho, \sigma \in \mathscr{A}, \tag{9}$$

where ρ is called the *density* and σ the *flux*.

Definition 4

The *characteristic of a conservation law* (9) for (4) is a function $P \in \mathscr{A}$, which satisfies

$$D_t(\rho) - D_x(\sigma) = P \cdot (u_t - K).$$

Conservation laws

Any *local conservation law* for (4) can WLOG be assumed (Olver '93) to read

$$D_t(\rho) = D_x(\sigma), \qquad \rho, \sigma \in \mathscr{A}, \tag{9}$$

where ρ is called the *density* and σ the *flux*.

Definition 4

The *characteristic of a conservation law* (9) for (4) is a function $P \in \mathscr{A}$, which satisfies

$$D_t(\rho) - D_x(\sigma) = P \cdot (u_t - K).$$

The conservation law is called *trivial* if it has zero characteristic. For any trivial conservation law we have $\rho = D_x(\zeta)$ and $\sigma = D_t(\zeta)$ for some $\zeta \in \mathscr{A}$.
In what follows we **tacitly assume** that the conservation laws are considered modulo trivial ones.

More on conservation laws

Proposition

The characteristic $P \in \mathscr{A}$ *of a conservation law satisfies*

$$D_t(P) + \mathbf{D}_K^*(P) = 0, \qquad (10)$$

where for any differential operator in total derivatives $\mathscr{P} = \sum_{i=0}^{q} p_i D_x^i$ *of order* q *and* $p_i \in \mathscr{A}$ *we define its formal adjoint as*

$$\mathscr{P}^* = \sum_{i=0}^{q} (-D_x)^i \circ p_i.$$

More on conservation laws

Proposition

The characteristic $P \in \mathscr{A}$ of a conservation law satisfies

$$D_t(P) + \mathbf{D}_K^*(P) = 0, \tag{10}$$

where for any differential operator in total derivatives $\mathscr{P} = \sum_{i=0}^{q} p_i D_x^i$ of order q and $p_i \in \mathscr{A}$ we define its formal adjoint as

$$\mathscr{P}^* = \sum_{i=0}^{q} (-D_x)^i \circ p_i.$$

Solutions of (10) are called cosymmetries. Cosymmetry defines a characteristic of a conservation law iff it lies in the image of variational derivative $\delta/\delta u$ where $\dfrac{\delta}{\delta u} H = \sum_{i=1}^{\infty} (-D_x)^i \left(\dfrac{\partial H}{\partial u_i} \right)$.

More on conservation laws II

Definition 5

A system of evolution partial differential equations is said to be *Hamiltonian* if it can be rewritten in the following form:

$$\frac{\partial u}{\partial t} = \mathscr{D}\delta\mathscr{H}. \tag{11}$$

where $\mathscr{D}$ is a Hamiltonian differential operator, δ is the operator of variational derivative and $\mathscr{H} = \int H\,dx$, where $H \in \mathscr{A}$ and $\delta\mathscr{H} = \frac{\delta}{\delta u}H$, is usually referred to as the *Hamiltonian functional*, or just the *Hamiltonian*.

More on conservation laws II

Definition 5

A system of evolution partial differential equations is said to be *Hamiltonian* if it can be rewritten in the following form:

$$\frac{\partial u}{\partial t} = \mathscr{D}\delta\mathscr{H}. \tag{11}$$

where $\mathscr{D}$ is a Hamiltonian differential operator, δ is the operator of variational derivative and $\mathscr{H} = \int H\,dx$, where $H \in \mathscr{A}$ and $\delta\mathscr{H} = \frac{\delta}{\delta u}H$, is usually referred to as the *Hamiltonian functional*, or just the *Hamiltonian*.

Proposition

Consider a Hamiltonian equation in the form (11) and let (9) define a conservation law. Then $\mathscr{D}\left(\delta\rho/\delta u\right)$ is a characteristic of a symmetry of this equation.

Main result

The following result has important implications for existence of symmetries and cosymmetries.

Main result

The following result has important implications for existence of symmetries and cosymmetries.

Theorem 6

Generalized Kawahara equation (2), *that is,*

$$u_t = u_{5x} + bu_{xxx} + f(u)u_x,$$

where b *is a constant and* f *is a function of* u, *has no nontrivial formal symmetry of rank 13 or greater.*

Main result

The following result has important implications for existence of symmetries and cosymmetries.

Theorem 6

Generalized Kawahara equation (2), *that is,*

$$u_t = u_{5x} + bu_{xxx} + f(u)u_x,$$

where b *is a constant and* f *is a function of* u, *has no nontrivial formal symmetry of rank 13 or greater.*

By Lemma 3 this theorem implies that GKE has no generalized symmetries of order greater than 12, so it cannot have an infinite hierarchy of generalized symmetries of increasing orders and therefore is not symmetry integrable.

Formal symmetries – outline of proof I

Seeking a contradiction suppose $\exists L \in \mathscr{L}$ with deg $L \neq 0$, which is a formal symmetry of rank 13. That is:

$$\deg\left(D_t(L) - [\widehat{\mathbf{D}}_K, L]\right) \leq \deg L + \deg \widehat{\mathbf{D}}_K - 13. \qquad (12)$$

Formal symmetries – outline of proof I

Seeking a contradiction suppose $\exists L \in \mathscr{L}$ with deg $L \neq 0$, which is a formal symmetry of rank 13. That is:

$$\deg\left(D_t(L) - [\widehat{\mathsf{D}}_K, L]\right) \leq \deg L + \deg \widehat{\mathsf{D}}_K - 13. \qquad (12)$$

WLOG we set deg $L = 1$, so $L = g\xi + \sum_{i=0}^{\infty} l_i \xi^{-i}$, and $g, l_i \in \mathscr{A}$. Then equation (12) will boil down to

$$\deg(D_t(L) - [\widehat{\mathsf{D}}_K, L]) \leq -7. \qquad (13)$$

Formal symmetries – outline of proof I

Seeking a contradiction suppose $\exists L \in \mathscr{L}$ with deg $L \neq 0$, which is a formal symmetry of rank 13. That is:

$$\deg\left(D_t(L) - [\widehat{\mathbf{D}}_K, L]\right) \leq \deg L + \deg \widehat{\mathbf{D}}_K - 13. \qquad (12)$$

WLOG we set $\deg L = 1$, so $L = g\xi + \sum_{i=0}^{\infty} l_i \xi^{-i}$, and $g, l_i \in \mathscr{A}$. Then equation (12) will boil down to

$$\deg(D_t(L) - [\widehat{\mathbf{D}}_K, L]) \leq -7. \qquad (13)$$

We need to equate to zero the coefficients at ξ^i. The first nontrivial equation occurs for $i = 5$ we get

$$-5D_x(g) = 0.$$

$\Rightarrow g$ is an arbitrary function of t only. Likewise we get $l_j = l_j(t), j = 0, -1, -2$.

Formal symmetries – outline of proof II

For $i = 1, \ldots, -2$ we get slightly more complicated equations, namely

$$-5D_x(l_{i-4}) = F_i \tag{14}$$

Formal symmetries – outline of proof II

For $i = 1, \ldots, -2$ we get slightly more complicated equations, namely

$$-5D_x(l_{i-4}) = F_i \tag{14}$$

Recall that a necessary condition for this kind of equations to be solvable in the class of local functions is that the equality $\delta F_i/\delta u = 0$ holds.

Formal symmetries – outline of proof II

For $i = 1, \dots, -2$ we get slightly more complicated equations, namely

$$-5D_x(l_{i-4}) = F_i \tag{14}$$

Recall that a necessary condition for this kind of equations to be solvable in the class of local functions is that the equality $\delta F_i/\delta u = 0$ holds.
The first case when the condition $\delta F_i/\delta u = 0$ is nontrivial appears for $i = -3$. We have to solve the following system

$$-\frac{1}{5}g\frac{\partial^3 f}{\partial u^3} = 0, \qquad \frac{3}{25}\frac{\partial f}{\partial u}\frac{\partial g}{\partial t} = 0. \tag{15}$$

Formal symmetries – outline of proof II

For $i = 1, \ldots, -2$ we get slightly more complicated equations, namely

$$-5D_x(l_{i-4}) = F_i \tag{14}$$

Recall that a necessary condition for this kind of equations to be solvable in the class of local functions is that the equality $\delta F_i/\delta u = 0$ holds.
The first case when the condition $\delta F_i/\delta u = 0$ is nontrivial appears for $i = -3$. We have to solve the following system

$$-\frac{1}{5}g\frac{\partial^3 f}{\partial u^3} = 0, \qquad \frac{3}{25}\frac{\partial f}{\partial u}\frac{\partial g}{\partial t} = 0. \tag{15}$$

The first equation tells us that if $\partial^3 f/\partial u^3 \neq 0$ we arrive at a contradiction with our initial assumption, because in this case g would have to be zero.

Formal symmetries – outline of proof III

Now turn to a case when $\partial^3 f/\partial u^3 = 0$. By assumption $\partial f/\partial u \neq 0$ so

$$\frac{\partial g}{\partial t} = 0 \quad \Rightarrow g = C_1.$$

Formal symmetries – outline of proof III

Now turn to a case when $\partial^3 f/\partial u^3 = 0$. By assumption $\partial f/\partial u \neq 0$ so

$$\frac{\partial g}{\partial t} = 0 \quad \Rightarrow g = C_1.$$

For $i = -4$ the condition $\delta F_{-4}/\delta u = 0$ yields $\dfrac{\partial l_0}{\partial t} = 0$, so l_0 is a constant. Constants are however trivial formal symmetries to any evolution equation so we put $l_0 = 0$.

Formal symmetries – outline of proof III

Now turn to a case when $\partial^3 f/\partial u^3 = 0$. By assumption $\partial f/\partial u \neq 0$ so

$$\frac{\partial g}{\partial t} = 0 \quad \Rightarrow g = C_1.$$

For $i = -4$ the condition $\delta F_{-4}/\delta u = 0$ yields $\dfrac{\partial l_0}{\partial t} = 0$, so l_0 is a constant. Constants are however trivial formal symmetries to any evolution equation so we put $l_0 = 0$.

For $i = -5, -6$ we have $\partial l_j/\partial t = 0$, for $j = 1, 2$ so we obtain that l_1 and l_2 are constants.

Finally for $i = -7$ we obtain the system which contains, among other, the equation $g = C_1 = 0$.

Formal symmetries – outline of proof III

Now turn to a case when $\partial^3 f/\partial u^3 = 0$. By assumption $\partial f/\partial u \neq 0$ so

$$\frac{\partial g}{\partial t} = 0 \quad \Rightarrow g = C_1.$$

For $i = -4$ the condition $\delta F_{-4}/\delta u = 0$ yields $\dfrac{\partial l_0}{\partial t} = 0$, so l_0 is a constant. Constants are however trivial formal symmetries to any evolution equation so we put $l_0 = 0$.

For $i = -5, -6$ we have $\partial l_j/\partial t = 0$, for $j = 1, 2$ so we obtain that l_1 and l_2 are constants.

Finally for $i = -7$ we obtain the system which contains, among other, the equation $g = C_1 = 0$.

This expresses the vanishing of the leading term of L and therefore contradicts the initial assumption that $\deg L = 1$ and hence the proof is completed.

Symmetries I

By Theorem 6 which we just proved GKE, that is,

$$u_t = u_{5x} + bu_{xxx} + f(u)u_x,$$

has no formal symmetries of rank 13 or greater. Now by Lemma 3 this implies that GKE has no generalized symmetries of order greater than 9. This result can be further strengthened as follows:

Symmetries I

By Theorem 6 which we just proved GKE, that is,

$$u_t = u_{5x} + bu_{xxx} + f(u)u_x,$$

has no formal symmetries of rank 13 or greater. Now by Lemma 3 this implies that GKE has no generalized symmetries of order greater than 9. This result can be further strengthened as follows:

Theorem 7

GKE admits only generalized symmetries which are equivalent to Lie point ones, i.e., it has no genuinely generalized symmetries.

Symmetries I

By Theorem 6 which we just proved GKE, that is,

$$u_t = u_{5x} + bu_{xxx} + f(u)u_x,$$

has no formal symmetries of rank 13 or greater. Now by Lemma 3 this implies that GKE has no generalized symmetries of order greater than 9. This result can be further strengthened as follows:

Theorem 7

GKE admits only generalized symmetries which are equivalent to Lie point ones, i.e., it has no genuinely generalized symmetries.

With this in mind we can readily obtain a complete description of generalized symmetries of GKE.

Symmetries II

Theorem 8

1) *If* f *is an arbitrary function of* u *such that* $u\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u}$ *and 1 are not linearly dependent, then GKE has just two linearly independent symmetries with the characteristics* $Q_1 = u_{5x} + bu_{xxx} + fu_x$ *and* $Q_2 = u_x$.

Symmetries II

Theorem 8

1) *If f is an arbitrary function of u such that $u\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u}$ and 1 are not linearly dependent, then GKE has just two linearly independent symmetries with the characteristics* $Q_1 = u_{5x} + bu_{xxx} + fu_x$ *and* $Q_2 = u_x$.
2) *If $u\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u}$ and 1 are linearly dependent we have two cases:*

Symmetries II

Theorem 8

1) *If f is an arbitrary function of u such that $u\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u}$ and 1 are not linearly dependent, then GKE has just two linearly independent symmetries with the characteristics* $Q_1 = u_{5x} + bu_{xxx} + fu_x$ *and* $Q_2 = u_x$.
2) *If $u\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u}$ and 1 are linearly dependent we have two cases:*
   i) *if* $\frac{\partial^2 f}{\partial u^2} = 0$,
   that is, $f = \alpha u + \beta$, *where* α, β *are constants,* $\alpha \neq 0$, *then GKE admits, in addition to two symmetries listed in 1), a symmetry with the characteristic* $Q_3 = tu_x + 1/\alpha$;

Symmetries II

Theorem 8

1) *If f is an arbitrary function of u such that $u\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u}$ and 1 are not linearly dependent, then GKE has just two linearly independent symmetries with the characteristics* $Q_1 = u_{5x} + bu_{xxx} + fu_x$ *and* $Q_2 = u_x$.
2) *If $u\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u}$ and 1 are linearly dependent we have two cases:*
   i) *if* $\frac{\partial^2 f}{\partial u^2} = 0$,
   that is, $f = \alpha u + \beta$, *where* α, β *are constants,* $\alpha \neq 0$, *then GKE admits, in addition to two symmetries listed in 1), a symmetry with the characteristic* $Q_3 = tu_x + 1/\alpha$;
   ii) *if* $\frac{\partial^2 f}{\partial u^2} \neq 0$, *so* $f = \gamma \ln(u + c) + \delta$,
   where γ, δ *and* c *are constants,* $\gamma \neq 0$, *then GKE admits, in addition to the two symmetries listed in 1), a symmetry with the characteristic* $Q_4 = tu_x + (u + c)/\gamma$.

Conservation laws I

Theorem 9

GKE admits only local conservation laws with the characteristics of order not greater than four.

Sketch of the proof. Recall, see e.g. Gandarias et. al. '17, that GKE (2) admits a Hamiltonian operator $\mathscr{D} = D_x$. In particular this implies that applying the operator $\mathscr{D}$ to a characteristic P of a conservation law yields a characteristic of a symmetry of order higher by one than that of P, so we have to consider only conservation laws with order one less than the greatest order of previously found symmetries which was five (cf. Vodová '16 for a similar argument).

Conservation laws I

Theorem 9

GKE admits only local conservation laws with the characteristics of order not greater than four.

Sketch of the proof. Recall, see e.g. Gandarias et. al. '17, that GKE (2) admits a Hamiltonian operator $\mathscr{D} = D_x$. In particular this implies that applying the operator $\mathscr{D}$ to a characteristic P of a conservation law yields a characteristic of a symmetry of order higher by one than that of P, so we have to consider only conservation laws with order one less than the greatest order of previously found symmetries which was five (cf. Vodová '16 for a similar argument). Thus, the most general conservation law for GKE has a characteristic of the form $P = P(x, t, u, u_x, u_{xx}, u_{xxx}, u_{4x})$.

Conservation laws II

Theorem 10

If $\frac{\partial^2 f}{\partial u^2} \neq 0$, *then GKE* $u_t = u_{5x} + bu_{xxx} + f(u)u_x$ *has just three linearly independent local conservation laws with conserved densities*

Conservation laws II

Theorem 10

If $\frac{\partial^2 f}{\partial u^2} \neq 0$*, then GKE* $u_t = u_{5x} + bu_{xxx} + f(u)u_x$ *has just three linearly independent local conservation laws with conserved densities*

$$\rho_1 = u, \quad \rho_2 = u^2 \quad \text{and} \quad \rho_3 = (1/2)u_{xx}^2 - (1/2)bu_x^2 + \widehat{r},$$

where $\hat{r}(u)$ *is defined by the formula* $\partial\widehat{r}/\partial u = r(u)$ *and* $\partial r/\partial u = f$*, with associated fluxes*

$\sigma_1 = (1/2)\frac{\partial f}{\partial u}u^2 + u_{xx}b + u_{4x}$,

$\sigma_2 = uu_{4x} - u_{xxx}u_x + (1/2)u_{xx}^2 + buu_{xx} - (1/2)bu_x^2 + (1/3)\frac{\partial f}{\partial u}u^3$

$\sigma_3 = -fbu_x^2 + \frac{\partial f}{\partial u}u_x^2u_{xx} - b^2u_xu_{xxx} + (1/2)u_{xx}^2b^2 + rbu_{xx} - fu_xu_{xxx} + fu_{xx}^2 + 2bu_{4x}u_{xx} - bu_{5x}u_x - u_{xxx}^2b + (1/2)r^2 + ru_{4x} + (1/2)u_{4x}^2 - u_{5x}u_{xxx} + u_{xx}u_{6x}$

Conservation laws III

Theorem 11

If $\frac{\partial^2 f}{\partial u^2} = 0$*, so* $f = \alpha u + \beta$ *with* $\alpha \neq 0$*, then GKE admits, in addition to the local conservation laws listed in the previous theorem, a local conservation law with the conserved density* $\rho_4 = xu + (1/2)\alpha t u^2$ *and associated flux* $\sigma_4 = (1/6)\alpha((-3bu_x^2 + 6buu_{xx} + 6u_{4x}u - 6u_{xxx}u_x + 3u_{xx}^2)t + 3xu^2) + (1/2)\alpha^2 tu^3 + 3b(xu_{xx} - u_x) + xu_{4x} - u_{xxx}$.

Conservation laws III

Theorem 11

If $\frac{\partial^2 f}{\partial u^2} = 0$*, so* $f = \alpha u + \beta$ *with* $\alpha \neq 0$*, then GKE admits, in addition to the local conservation laws listed in the previous theorem, a local conservation law with the conserved density* $\rho_4 = xu + (1/2)\alpha t u^2$ *and associated flux*
$\sigma_4 = (1/6)\alpha((-3bu_x^2 + 6buu_{xx} + 6u_{4x}u - 6u_{xxx}u_x + 3u_{xx}^2)t + 3xu^2) + (1/2)\alpha^2 t u^3 + 3b(xu_{xx} - u_x) + xu_{4x} - u_{xxx}.$

Theorem 9 reduces the proof of the above two theorems to the search of cosymmetries of order up to four which is very similar to the computation of symmetries in Theorem 8, so we omit the relevant details.

Conclusions

We considered the class of equations of the form

$$u_t = u_{5x} + bu_{xxx} + f(u)u_x$$

with nonconstant f, which generalizes the Kawahara equation, and obtained a complete description of generalized symmetries and local conservation laws for equations from this class.

Conclusions

We considered the class of equations of the form

$$u_t = u_{5x} + b u_{xxx} + f(u) u_x$$

with nonconstant f, which generalizes the Kawahara equation, and obtained a complete description of generalized symmetries and local conservation laws for equations from this class.

In particular, it was shown that the genuinely generalized symmetries in this case do not exist, so the equations from the above class are not symmetry integrable.

Conclusions

We considered the class of equations of the form

$$u_t = u_{5x} + bu_{xxx} + f(u)u_x$$

with nonconstant f, which generalizes the Kawahara equation, and obtained a complete description of generalized symmetries and local conservation laws for equations from this class.

In particular, it was shown that the genuinely generalized symmetries in this case do not exist, so the equations from the above class are not symmetry integrable.

As a byproduct of our research it became clear that there appear to be small imperfections in the classification results of Gandarias et. al. '17 on the Lie point symmetries of a slightly more general equation

$$u_t = a(t)u_{5x} + b(t)u_{xxx} + c(t)f(u)u_x.$$

This research was supported by the Specific Research grant SGS/6/2017 of the Silesian University in Opava.

Further details can be found at arXiv:1810.02863.

This research was supported by the Specific Research grant SGS/6/2017 of the Silesian University in Opava.

Further details can be found at arXiv:1810.02863.

Warmest congratulations to prof. Krasil'shchik on the occasion of his 70th birthday!

This research was supported by the Specific Research grant SGS/6/2017 of the Silesian University in Opava.

Further details can be found at arXiv:1810.02863.

Warmest congratulations to prof. Krasil'shchik on the occasion of his 70th birthday!

Thank you for your attention!

[1] Gandarias, M.L., Rosa, M., Recio, E., Anco, S. *Conservation laws and symmetries of a generalized Kawahara equation*, AIP Conference Proceedings 1836 (2017), no. 1, 020072.

[2] Kawahara, T., *Oscillatory Solitary Waves in Dispersive Media*, J. Phys. Soc. Jpn. 33 (1972), 260–264.

[3] Krasil'shchik, J., Verbovetsky, A., *Geometry of jet spaces and integrable systems*, Journal of Geometry and Physics 61 (2011), no. 9, 1633–1674.

[4] Krasil'shchik, J., Verbovetsky, A., Vitolo, R., *The symbolic computation of integrability structures for partial differential equations*, Springer, Cham, 2017.

[5] Kupershmidt, B.A. *KP or mKP. Noncommutative mathematics of Lagrangian, Hamiltonian, and integrable systems.* AMS, Providence, RI, 2000.

[6] Mikhailov, A.V., Shabat, A.B., Yamilov, R.I., *The symmetry approach to the classification of non-linear equations. Complete lists of integrable systems*, Russian Mathematical Surveys 42 (1987), no. 4, 1–63.

[7] Mikhailov, A.V.; Sokolov, V.V., *Symmetries of Differential Equations and the Problem of Integrability*, in *Integrability*, ed. Mikhailov A.V., Springer, Berlin Heidelberg, 2009, 19–88.

[8] Olver, P.J., *Applications of Lie Groups to Differential Equations*, Springer, New York, 2nd edition, 1993.

[9] Sergyeyev A., Vitolo R., *Symmetries and conservation laws for the Karczewska–Rozmej–Rutkowski–Infeld equation*, Nonlinear Anal. Real World Appl. 32 (2016), 1–9.

[10] Vodová, J., *A complete list of conservation laws for non-integrable compacton equations of $K(m, m)$ type*, Nonlinearity 26 (2013), no. 3, 757–762.