Hydrodynamic type systems and beyond: a long way towards integrability with Maxim Pavlov

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Our joint papers

- Pavlov M.V., Tsarev S.P. "On conservation laws of Benney equations." Russian Mathematical Surveys 46.4 (1991): 196.
- Pavlov M.V., Tsarev S.P. "Tri-Hamiltonian structures of Egorov systems of hydrodynamic type." Functional Analysis and Its Applications 37.1 (2003): 32-45.
- Pavlov M.V., Tsarev S.P. "Classical mechanical systems with one-and-a-half degrees of freedom and Vlasov kinetic equation." Topology, Geometry, Integrable Systems, and Mathematical Physics 234 (2014): 337-371.
- Pavlov M.V., Tsarev S.P. "On local description of two-dimensional geodesic flows with a polynomial first integral." Journal of Physics A: Mathematical and Theoretical 49.17 (2016): 175201.

The context: Hydrodynamic type systems

In 1983 B.A.Dubrovin and S.P.Novikov proposed a natural hamiltonian formalism for one physically important class of homogeneous systems of PDE

$$\begin{pmatrix} u_t^1 \\ \vdots \\ u_t^n \end{pmatrix} = \begin{pmatrix} v_1^1(u) & \cdots & v_n^1(u) \\ \cdot & \cdots & \cdot \\ v_1^n(u) & \cdots & v_n^n(u) \end{pmatrix} \begin{pmatrix} u_x^1 \\ \vdots \\ u_x^n \end{pmatrix}, \qquad (1)$$

$u^i = u^i(x,t), \quad i = 1, \cdots, n$

The main (and unexpected at that moment) was a deep and simple connection to (pseudo-)Riemannian geometry via

$$u_t^i(x) = \{u^i(x), H\} = (g^{ij}\partial_k\partial_j h + b_k^{ij}\partial_j h)u_x^k = v_k^i(u)u_x^k \qquad (2)$$

with $g^{ij}(x)$ being a *flat* metric and $b_k^{ij} = -g^{is}\Gamma_{sk}^j$.

Diagonalizable HTS and classical heritage

Suppose a hamiltonian (DN-type) HTS has the complete set of Riemann invariants
(i.e. is diagonalisable with a point transformation $u_i \to \bar{u}_k$).
Important examples of such systems are Whitham (averaged KdV) system and Benney(-Zakharov) system.

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A forgotten world of classical differential geometry was “reinvested” into modern theory of integrable nonlinear PDEs!

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Not that much effective ...