# Beyond Recursion Operators: HAANTJES ALGEBRAS

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**Motivation**: We wish to generalize the theory of **Nijenhuis operators**, introduced in 1980's by F. Magri in the study of infinite-dimensional integrable systems (KdV hierarchy). In particular, we are interested in the case of *Hamiltonian finite-dimensional systems*.

**Idea**: To construct a coherent theory of integrability based on **Haantjes operators**.

**Results**: I) New geometric structures:

- **Haantjes algebras**
- $\omega\mathscr{H}$ **manifolds**
- $P\mathscr{H}$ **manifolds**

II) A new theoretical perspective concerning integrability, superintegrability and separation of variables.

**Inspiration**: *Haantjes manifolds* proposed by F. Magri for infinite dimensional integrable systems (2012–. . . ). However, the geometric constructions proposed in this talk are different from Magri's ones (we answer to different questions).

# Outline

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# A classical problem in differential geometry

Let $\boldsymbol{K} : TM \to TM$ a smooth field of operators with **real eigenvalues** and with eigen–distributions of constant rank. When do local charts exist in which $\boldsymbol{K}$ takes a *simple* (for instance a diagonal or block-diagonal) form? Precisely, when does $\boldsymbol{K}$ admit an **integrable eigen-frame**?

A **reference frame** is a set of $n$ vector fields $\{Y_1, \ldots, Y_n\}$ such that they form a basis of the tangent space $T_{\boldsymbol{x}}U$ at each point $\boldsymbol{x}$ of an open set $U \subseteq M$. Two frames $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$ are said to be **equivalent** if $n$ nowhere vanishing smooth functions $f_i$ exist such that

$$X_i = f_i(\boldsymbol{x}) Y_i \ , \qquad\qquad i = 1, \ldots, n \ .$$

A **natural frame** is a frame of the form $\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$ associated with the local coordinate chart $\{U, (x^1, \ldots, x^n)\}$.

**Definition**

An *integrable* frame is a reference frame equivalent to a natural frame.

Previous results:

i) **A. Tonolo** (1949) and **J. A. Schouten** (1951) for Riemannian manifolds;

ii) **A. Nijenhuis** (1951) on generic manifolds (without Riemannian structure), in the case of distinct eigenvalues (simple operators);

iii) **J. Haantjes** (1955), in the more general case of semisimple (diagonalizable) operators and even in the most general context of non-semisimple operators (considering generalized eigen–distributions).

**Remark** From now on, for simplicity we will use the term **operator** instead of **field of operators**.

# Nijenhuis results

Let $M$ be a differentiable manifold and $\boldsymbol{K}: TM \to TM$ be a (1,1) tensor field. The **Nijenhuis torsion** of $\boldsymbol{K}$ is the skew-symmetric $(1,2)$ tensor field defined by

$$\mathcal{T}_{\boldsymbol{K}}(X,Y) := \boldsymbol{K}^2[X,Y] + [\boldsymbol{K}X,\boldsymbol{K}Y] - \boldsymbol{K}\Big([X,\boldsymbol{K}Y] + [\boldsymbol{K}X,Y]\Big),$$

where $X, Y \in TM$ and $[\ ,\ ]$ denotes the commutator of two vector fields.

**Theorem (Nijenhuis, 1951)**

*Let $\boldsymbol{K}: TM \to TM$ a smooth operator and suppose that*

i) *it has distinct real eigenvalues in each point of $M$;*
ii) *its Nijenhuis torsion $\mathcal{T}_{\boldsymbol{K}}$ vanishes.*

*Then, each eigen-frame of $\boldsymbol{K}$ is* **integrable**.

$$\mathcal{D}_i(\boldsymbol{x}) \bigoplus \mathcal{D}_j(\boldsymbol{x}) \ , \quad \mathcal{D}_i(\boldsymbol{x}) \bigoplus \mathcal{D}_j(\boldsymbol{x}) \bigoplus \mathcal{D}_k(\boldsymbol{x}) \ , \dots$$

The **Nijenhuis geometry** has been largely investigated in relation with the theory of integrable systems. Indeed,

- Nijenhuis operators play the role of **recursion operators** in the theory of infinite-dimensional integrable systems.
- They play a crucial role in the geometric theory of **separation of variables** of finite-dimensional integrable systems.
- They represent basic structures in the *PN* **manifolds** (F. Magri & Y. Kosmann-Schwarzback, 1990) and in the $\omega N$ **manifolds** (G. Falqui & M. Pedroni, 2003).

# Recursion operator of the KdV hierarchy: Magri chain

$$
\begin{array}{ll}
u_t = & u_x \\
 & \Big\downarrow N \\
u_t = & u_{xxx} + 6uu_x \\
 & \Big\downarrow N \\
u_t = & u_{xxxxx} + 10u_{xxx}u_x + 20u_{xx}u_x + 30u_x u^2 \\
 & \Big\downarrow N \\
u_t = & X_k \\
 & \Big\downarrow N \\
u_t = & \dots
\end{array}
$$

$$N = \partial_x^2 + 3u + 3\partial^{-1}u\partial \ , \qquad \mathcal{T}_N = 0 \ , \quad [X_j, X_k] = 0 \ , \qquad \mathcal{L}_{X_k}(N) = 0.$$

# A modern problem in the theory of finite-dimensional integrable systems

Most of the classical integrable Hamiltonian systems do not admit a **semiglobal Bi-Hamiltonian** formulation (Brouzet, 1993): The existence of a **Nijenhuis recursion operator** (compatible with $\omega$ and such that $\mathcal{L}_{X_H}(\boldsymbol{N}) = 0$) restricts the form of the Hamiltonians considered.
Then, extensions of the BH theory are in order:

- A BH formulation w.r.t. **alternative** (i.e. not including $P_0$) Poisson structures, still compatible each other (Marmo, Vilasi et al., 1984);
- a BH formulation w.r.t. a pair of **incompatible** Poisson tensors (Bogoyavlenskij, 1995).
- a BH formulation w.r.t. a **degenerate** bi–Hamiltonian structure in an *extended* phase space (BH manifold) (Ibort, Magri & Marmo, 2000);
- Generalized Magri-chains (**quasi-bi-Hamiltonian formulation**) in the $\omega N$ **manifolds** instead of Lenard-Magri chains. (Falqui, Magri & T., 2000, Falqui & Pedroni, 2003)

# Haantjes geometry

The **Haantjes geometry** offers a new and natural answer to the two, seemingly unrelated problems we discussed:

- The construction of **local coordinate charts** $\{(x_1, \ldots, x_n)\}$ where given operators can be put in a simple (e.g. diagonal, block-diagonal, or triangular) form.
- The formulation of a coherent and sufficiently general geometry where the **integrability** and **separability** properties of Hamiltonian systems can be discussed.

# Haantjes results

The **Haantjes torsion** of $\boldsymbol{K}$ is the $(1,2)$ tensor field defined by

$$\mathcal{H}_{\boldsymbol{K}}(X,Y) := \boldsymbol{K}^2\mathcal{T}_{\boldsymbol{K}}(X,Y)+\mathcal{T}_{\boldsymbol{K}}(\boldsymbol{K}X,\boldsymbol{K}Y)-\boldsymbol{K}\Big(\mathcal{T}_{\boldsymbol{K}}(X,\boldsymbol{K}Y)+\mathcal{T}_{\boldsymbol{K}}(\boldsymbol{K}X,Y)\Big).$$

**Theorem (Haantjes, 1955)**

*Let $\boldsymbol{K} : TM \rightarrow TM$ a smooth operator and suppose that*

i) *it is semisimple (diagonalizable) in each point of $M$;*

ii) *its eigen-distributions have constant rank in each point of $M$.*

*Then, an integrable eigen–frame of $\boldsymbol{K}$ exists if and only if the Haantjes tensor of $\boldsymbol{K}$ vanishes.*

If $\boldsymbol{K}$ is not semisimple, Haantjes claimed that the vanishing of the Haantjes tensor is only sufficient to the existence of a (generalized) integrable eigen-frame of $\boldsymbol{K}$.

# Semisimple Haantjes operators

## Definition

A Haantjes operator is an operator whose Haantjes tensor identically vanishes.

## Theorem

*Let $\boldsymbol{K}$ be a smooth operator on $M$. If there exists a local coordinate chart $\{(x_1, \ldots, x_n)\}$, where $\boldsymbol{K}$ takes a diagonal form, i.e.*

$$\boldsymbol{K} = \sum_{i=1}^{n} l_i(\boldsymbol{x}) \frac{\partial}{\partial x_i} \otimes \mathrm{d}x_i,$$

*then the Haantjes torsion of $\boldsymbol{K}$ vanishes. In particular, if $l_i(x) = \lambda_i(x_i)$, $i = 1, \ldots, n$, the Nijenhuis torsion of $\boldsymbol{K}$ also vanishes.*

## Example 1: Hydrodynamic systems of PDEs

Integrable hydrodynamic systems

$$\boldsymbol{u}_t = \boldsymbol{K}(\boldsymbol{u})\boldsymbol{u}_x \qquad \boldsymbol{u}(x,t) = (u^1, u^2, \dots, u^n)^T$$

If $\boldsymbol{K}$ has distinct real eigenvalues, it is diagonalizable if and only if its Haantjes torsion vanishes. The eigenvalues of $\boldsymbol{K}$ are the characteristic speeds (**Riemann invariants**) of the systems.

## Example 2: Inertia tensors in classical mechanics

The **planar** inertia tensor (or Euler tensor) of a massive body

$$\boldsymbol{E}_P(\vec{v}) = \sum_\gamma m_\gamma \big((P_\gamma - P)\cdot\vec{v}\big)\,(P_\gamma - P) \qquad \vec{v} \in T_P\mathcal{E}_n \equiv \mathbb{E}_n$$

is a Nijenhuis operator (Benenti, 1992). The **inertia** tensor

$$\mathbb{I}_P = tr(\boldsymbol{E}_P)\boldsymbol{I}_n - \boldsymbol{E}_P$$

is a Haantjes operator (P.Tempesta & G. T., 2015).

# A remarkable property of Haantjes operators

**Theorem (Bogoyavlenskij, 2004)**

*Let $\boldsymbol{K}$ be an operator with vanishing Haantjes tensor in $M$. Then for any polynomial in $\boldsymbol{K}$, with arbitrary coefficients $a_j \in C^\infty(M)$, the associated Haantjes tensor also vanishes, i.e.*

$$\mathcal{H}_{\boldsymbol{K}}(X, Y) = 0 \implies \mathcal{H}_{(\sum_{j=0}^{n-1} a_j(\boldsymbol{x})\boldsymbol{K}^j)}(X, Y) = 0.$$

This means that a single Haantjes operator generates an associative algebra with identity over the ring of the smooth functions of $M$.
In the light of this result, it is natural to introduce the notion of **Haantjes algebra**, by requiring that linear combination and the product of two distinct Haantjes operators be still a Haantjes operator.

# Haantjes algebras: a novel algebraic structure

(P. Tempesta and G. T., (2017))

A **Haantjes algebra** of rank $m$ is a pair $(M, \mathscr{H})$ that satisfies the following conditions

- $M$ is a differentiable manifold of dimension $n$;
- $\mathscr{H}$ is a set of Haantjes operators $\boldsymbol{K} : TM \rightarrow TM$, that generate
  i) a **free module** of rank $m$ over the ring of smooth functions on $M$

$$\mathcal{H}_{\left(f(\boldsymbol{x})\boldsymbol{K}_1+g(\boldsymbol{x})\boldsymbol{K}_2\right)}(X,Y)=0\ , \quad \forall \boldsymbol{K}_1, \boldsymbol{K}_2 \in \mathscr{H} \quad , \forall X, Y \in TM\ ,$$

  being $f(\boldsymbol{x}), g(\boldsymbol{x})$ arbitrary smooth functions on M.
  ii) a **ring** w.r.t. the composition operation

$$\mathcal{H}_{\left(\boldsymbol{K}_1\,\boldsymbol{K}_2\right)}(X,Y)=\mathcal{H}_{\left(\boldsymbol{K}_2\,\boldsymbol{K}_1\right)}(X,Y)=0\,,\forall \boldsymbol{K}_1,\boldsymbol{K}_2\in\mathscr{H},\forall X,Y\in TM\,.$$

- In addition, if $\boldsymbol{K}_1\,\boldsymbol{K}_2=\boldsymbol{K}_2\,\boldsymbol{K}_1\ \ \forall \boldsymbol{K}_1,\boldsymbol{K}_2\in\mathscr{H}$ the algebra $\mathscr{H}$ will be said an **Abelian Haantjes algebra**.

Compatibility conditions for the module structure. We propose the following

## Definition

The *Haantjes binary tensor* of $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ is the skew-symmetric $(1,2)$ tensor field defined by

$$
\begin{aligned}
\mathcal{H}_{\boldsymbol{K}_1,\boldsymbol{K}_2}(X,Y): \;=\;& \Big(\boldsymbol{K}_1\boldsymbol{K}_2+\boldsymbol{K}_2\boldsymbol{K}_1\Big)[\boldsymbol{K}_1,\boldsymbol{K}_2](X,Y)\\
&+ [\boldsymbol{K}_1,\boldsymbol{K}_2](\boldsymbol{K}_1X,\boldsymbol{K}_2Y)+[\boldsymbol{K}_2,\boldsymbol{K}_1](\boldsymbol{K}_2X,\boldsymbol{K}_1Y)\\
&- \boldsymbol{K}_1\Big([\boldsymbol{K}_1,\boldsymbol{K}_2](X,\boldsymbol{K}_2Y)+[\boldsymbol{K}_1,\boldsymbol{K}_2](\boldsymbol{K}_2X,Y)\Big)\\
&- \boldsymbol{K}_2\Big([\boldsymbol{K}_2,\boldsymbol{K}_1](X,\boldsymbol{K}_1Y)+[\boldsymbol{K}_2,\boldsymbol{K}_1](\boldsymbol{K}_1X,Y)\Big).
\end{aligned}
$$

This tensor is symmetric in $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ thanks to the symmetry of the Frölicher–Nijenhuis bracket; for $\boldsymbol{K}_1=\boldsymbol{K}_2$ it reduces to (four times) the standard Haantjes torsion.

(P. Tempesta & G. T., 2018 ArXiv)

**Theorem**

*Let $M$ be a differentiable manifold, $f \in C^\infty(M)$ and $\boldsymbol{K}_1, \boldsymbol{K}_2 : TM \to TM$ be two $(1,1)$ tensor fields such that $\boldsymbol{K}_1\boldsymbol{K}_2 = \boldsymbol{K}_2\boldsymbol{K}_1$. Then*

$$\mathcal{H}_{f\boldsymbol{K}_1,\boldsymbol{K}_2}(X,Y) = f^2\,\mathcal{H}_{\boldsymbol{K}_1,\boldsymbol{K}_2}(X,Y) \ .$$

**Theorem**

*Let $\boldsymbol{K}_1$, $\boldsymbol{K}_2 : TM \to TM$ two $(1,1)$ tensor fields which can be simultaneously diagonalized in a local chart of a differentiable manifold $M$. Then for any $X, Y \in TM$, the Haantjes binary tensor*

$$\mathcal{H}_{\boldsymbol{K}_1,\boldsymbol{K}_2}(X,Y) = 0 \ .$$

Conjecture: if $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are two Haantjes operators that commute and $\mathcal{H}_{\boldsymbol{K}_1,\boldsymbol{K}_2}(X,Y) = 0$, then $\boldsymbol{K}_1 + \boldsymbol{K}_2$ is also a Haantjes operator.

# Cyclic Haantjes algebras

An especially relevant class of Haantjes algebras is represented by those generated by a *single* Haantjes operator $\boldsymbol{L}: TM \mapsto TM$. In fact, one can consider the Haantjes algebra $\mathcal{L}$ of any powers of $\boldsymbol{L}$

$$\mathcal{L}(\boldsymbol{L}) := \left\langle \boldsymbol{I}, \boldsymbol{L}, \boldsymbol{L}^2, \ldots, \boldsymbol{L}^{n-1}, \ldots \right\rangle \ ,$$

that we shall call a *cyclic* Haantjes algebra. Its rank is equal to the degree of the minimal polynomial of $\boldsymbol{L}$ that is not greater than $n$.

**Definition**

Let $(M, \mathscr{H})$ be a Haantjes algebra with identity, of rank $\mathrm{m}$. An operator $\boldsymbol{L}$ having its minimal polynomial of degree $\mathrm{m}$ will be called a cyclic generator of $\mathscr{H}$ if

$$\mathscr{H} \equiv \mathcal{L}(\boldsymbol{L}) \ .$$

The set $\mathcal{B}_{cyc} = \{\boldsymbol{I}, \boldsymbol{L}, \boldsymbol{L}^2, \ldots, \boldsymbol{L}^{m-1}\}$ will be called a cyclic basis of $\mathscr{H}$.

# Diagonalization of Haantjes algebras

**Theorem (P. Tempesta & G. T., 2017, ArXiv )**

*Let $(M, \mathscr{H})$ be an abelian Haantjes algebra.*
*i) If $\mathscr{H}$ is a* **semisimple Haantjes algebra***, then there exist* **local coordinate charts** $\{(x_1, \ldots, x_n)\}$ *in which every $\boldsymbol{K} \in \mathscr{H}$ can be simultaneously diagonalized.*

*Conversely, let $\{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_m\}$ be a commuting set of $\mathrm{m}$ $C^\infty(M)$–linearly independent operators. If they share a set of local coordinate charts in which they take a diagonal form, then they generate a semisimple abelian Haantjes algebra of rank not smaller than $\mathrm{m}$.*

*ii) If the abelian Haantjes algebra $(M, \mathscr{H})$ is* **non-semisimple***, then there exist local coordinate charts $\{(x_1, \ldots, x_n)\}$ in which every $\boldsymbol{K} \in \mathscr{H}$ takes a block-diagonal form.*

# Magri-Haantjes chains

The theory of *Lenard–Magri chains* is a fundamental piece of the geometric approach to soliton hierarchies. Inspired by these results, we introduce their extension à la Haantjes: the *Magri-Haantjes chains*:

## Definition

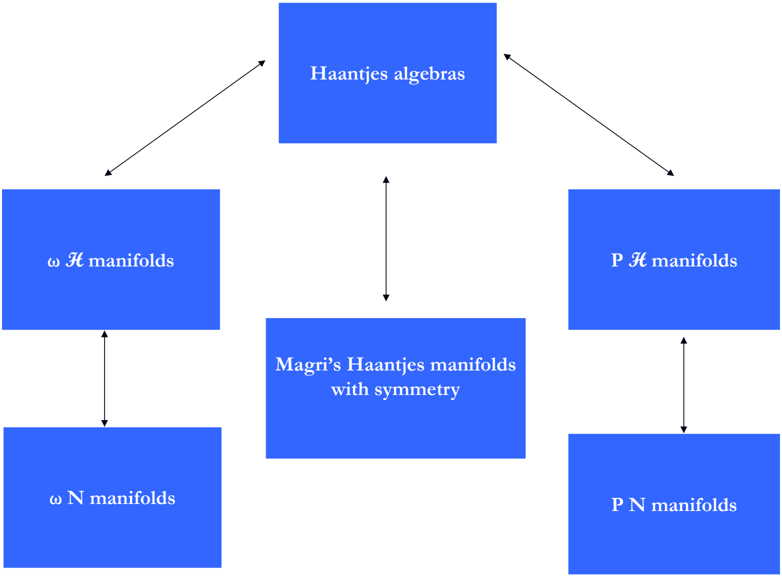
Let $(M, \mathscr{H})$ be a Haantjes algebra of rank $\mathrm{m}$. A smooth function $H$ is said to generate a **Magri–Haantjes chain** of 1-forms of length $\mathrm{m}$ if there exist a distinguished basis $\{\tilde{\boldsymbol{K}}_1, \ldots, \tilde{\boldsymbol{K}}_m\}$ of $\mathscr{H}$ such that

$$\mathrm{d}(\tilde{\boldsymbol{K}}_\alpha^T \,\mathrm{d}H) = 0 \qquad \alpha = 1, \ldots, m.$$

The (locally) exact linearly independent 1-forms $\mathrm{d}H_i$ such that

$$\mathrm{d}H_\alpha = \tilde{\boldsymbol{K}}_\alpha^T \,\mathrm{d}H \ ,$$

are called the elements of the Magri–Haantjes chain of length $\mathrm{m}$ generated by $H$, and the functions $H_\alpha$ its **potential functions**.


Haantjes algebras
ω ℋ manifolds
P ℋ manifolds
Magri's Haantjes manifolds with symmetry
ω N manifolds
P N manifolds

# *PN* manifolds

$(M, P, N)$

- $(M, P)$ is a Poisson manifold;
- $N : TM \to TM$, is a Nijhenuis (or hereditary) operator;
- $(P, N)$ are *compatible*, i.e. $(M, P_0 := P, P_1 := NP)$ must be a bi-Hamiltonian manifold with the two Poisson brackets

$$\{f, g\}_0 =< df, P_0 \mathrm{d}g > \qquad \{f, g\}_1 =< df, P_1 \mathrm{d}g >$$

# Lenard-Magri Chains in *PN* manifolds

*Lenard-Magri chain*

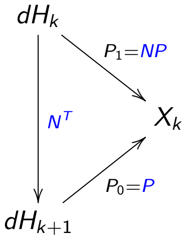
$dH_1 = dH$

$dH_2 = N^T dH$

$\vdots$

$dH_k = (N^T)^{k-1} dH$

BH formulation of $X_k$




Example: $I_k := \frac{1}{2k} tr(N^k), \quad k \in \mathbb{N}$

$\{H_i, H_j\}_{0,1} = 0$

$\mathcal{L}_{X_k}(N) = 0 \Rightarrow$ Recursion operator

# $P\mathscr{H}$ manifolds

(G. T., ArXiv (2018))

## Definition

A $P\mathscr{H}$ (or Poisson–Haantjes) manifold of class $\mathrm{m}$ is a triple $(M, \boldsymbol{P}, \mathscr{H})$ which satisfies the following properties:

i) $(M, P)$ is a Poisson manifold;

ii) $\mathscr{H}$ is an Abelian Haantjes algebra of rank $\mathrm{m}$;

iii) $(P, \mathscr{H})$ are algebraically compatible, that is

$$\boldsymbol{K} P = P \boldsymbol{K}^T \qquad \forall \boldsymbol{K} \in \mathscr{H} \ ,$$

A basis of the Haantjes algebra $\mathscr{H}$ will be denoted by

$$\{K_1, K_2, \ldots, K_m\}$$

In particular, if $P$ is invertible, $M$ is a $\omega\mathscr{H}$ manifold, with $\omega^\flat = P^{-1}$.

# Magri–Haantjes Chains in $P\mathscr{H}$ manifolds

| *Lenard-Magri chains* | *Magri-Haantjes chains* |
|---|---|
| $(M, P, N)$ | $(M, P, \mathscr{H})$ |
| $\mathrm{d}H_1 = \mathrm{d}H$ | $\mathrm{d}H_1 = \tilde{K}_1^T \mathrm{d}H$ |
| $\mathrm{d}H_2 = N^T \mathrm{d}H$ | $\mathrm{d}H_2 = \tilde{K}_2^T \mathrm{d}H$ |
| $\vdots$ | $\vdots$ |
| $\mathrm{d}H_m = (N^T)^{m-1} \mathrm{d}H$ | $\mathrm{d}H_m = \tilde{K}_m^T \mathrm{d}H$ |
| $\mathcal{L}_{X_H}(N) = 0$ | $\mathcal{L}_{X_H}(\tilde{K}_\alpha) \neq 0$ |
| $\{H_i, H_j\} = 0$ | $\{H_i, H_j\} = < dH_i, P^{-1} dH_j > = < \tilde{K}_i{}^T dH, P \tilde{K}_j{}^T dH > = < dH, \tilde{K}_i P \tilde{K}_j{}^T dH > = 0$ |

# A characterization of integrable systems

## Theorem

*Let $M$ be a $\omega\mathcal{H}$ manifold of class $n$ and $\{H_1, H_2, \ldots, H_n\}$ be smooth potentials functions of a Magri-Haantjes chain generated by a function $H$. Then, the foliation generated by these functions turns out to be Lagrangian. Consequently, each Hamiltonian system, with Hamiltonian functions $H_j$, $1 \leq j \leq n$ is* **integrable by quadratures**.

*Conversely, if a non degenerate Hamiltonian system with $n$ degrees of freedom is completely integrable in the Liouville-Arnold sense, then* **it admits an associated $\omega\mathcal{H}$ structure** *in any neighborhood of an Arnold torus, given by*

$$\tilde{K}_\alpha = \sum_{i=1}^{n} \frac{\nu_i^{(\alpha)}}{\nu_i}(\boldsymbol{J})\left(\frac{\partial}{\partial J_i} \otimes \mathrm{d}J_i + \frac{\partial}{\partial \phi_i} \otimes \mathrm{d}\phi_i\right) \quad \alpha = 1, \ldots, n \ ,$$

*where $\nu_i^\alpha := \frac{\partial H_\alpha}{\partial J_i}$, are the frequencies of the $\alpha - th$ linear flow on the torus, and $\nu_i$ the frequencies of the Hamiltonian flow of $H$.*

# Darboux–Haantjes charts in $\omega\mathcal{H}$ manifolds

## Definition

Let $(M, \omega, \mathscr{H})$ be a $\omega\mathcal{H}$ manifold of class $n$. A set of local coordinates $(\boldsymbol{q}, \boldsymbol{p})$ will be said to be a set of Darboux-Haantjes (DH) coordinates if the symplectic form in these coordinates assumes the *Darboux form*

$$\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i$$

and each Haantjes operator $\boldsymbol{K} \in \mathscr{H}$ diagonalizes

$$\boldsymbol{K} = \sum_{i=1}^{n} l_i(\boldsymbol{q}, \boldsymbol{p})\Big(\frac{\partial}{\partial q_i} \otimes \mathrm{d}q_i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i\Big).$$

## Theorem (Darboux-Haantjes coordinates and SoV)

*Let $(M, \omega, \mathscr{H})$ be a semisimple $\omega\mathcal{H}$ manifold of class n. Then*

*i) There exist Darboux-Haantjes local coordinates in $M$.*

*ii) Let $\{H_1, H_2, \ldots, H_n\}$ be a set of n potential functions of a Magri–Haantjes chain generated by a function $H$. Then, each set $(\boldsymbol{q}, \boldsymbol{p})$ of $\mathrm{DH}$ coordinates provides separation variables for the Hamilton–Jacobi equations associated with each function $H_j$.*

*Conversely, if $\{H_1, H_2, \ldots, H_n\}$ are a set of n independent functions separable in a set of Darboux coordinates $(\boldsymbol{q}, \boldsymbol{p})$, then they belong to a Magri–Haantjes chain w.r.t. the $\omega\mathcal{H}$ structure $(M, \omega, \mathscr{H})$ given by*

$$\tilde{K}_\alpha = \sum_{i=1}^{n} \frac{\frac{\partial H_\alpha}{\partial p_i}}{\frac{\partial H}{\partial p_i}} \left( \frac{\partial}{\partial q_i} \otimes \mathrm{d}q_i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i \right) \qquad \alpha = 1, \ldots, n\ .$$

# Applications of the theory of $P\mathscr{H}$ and $\omega\mathscr{H}$ manifolds

In $P\mathscr{H}$ manifolds:

- **the KdV stationary flows**;
- the Lagrange top.

In $\omega\mathscr{H}$ manifolds:

- Functionally separated systems;
- Telescopic systems;
- Gantmacher systems;
- Stäckel systems;
- the Jacobi-Calogero system;
- Benenti systems;
- the Goldfish system;
- Drach-Holt systems.

# First stationary flow of $7^{th}$ order of KdV hierarchy: $X_1$

$$M_7 := \{0 = u^{(7)} + 4u^{(5)}u + 42u^{(4)}u^{(1)} + 70u^{(3)}(u^{(2)} + u^2) + 280u^{(2)}u^{(1)}u + 70u^{(1)}\left((u^{(1)})^2 + 2u^3\right)\}$$

$$\begin{aligned}
\dot{u} &= u^{(1)} \\
\dot{u}^{(1)} &= u^{(2)} \\
\dot{u}^{(2)} &= u^{(3)} \\
\dot{u}^{(3)} &= u^{(4)} \\
\dot{u}^{(4)} &= u^{(5)} \\
\dot{u}^{(5)} &= u^{(6)} \\
\dot{u}^{(6)} &= -\Big(4u^{(5)}u + 42u^{(4)}u^{(1)} + 70u^{(3)}(u^{(2)} + u^2) + \\
&\quad + 280u^{(2)}u^{(1)}u + 70u^{(1)}\left((u^{(1)})^2 + 2u^3\right)\Big)
\end{aligned}$$

# Second stationary flow of $7^{th}$ order of KdV hierarchy: $X_2$

$$M_7 := \{(u_0 := u(x_0), u_1 := u^{(1)}(x_0), u_2 := u^{(2)}(x_0), \ldots, u_5 := u^{(5)}(x_0), u_6 := u^{(6)}(x_0))\}$$

$$\begin{aligned}
\dot{u}_0 =& u_3 + 6u_1 u_0 \\
\dot{u}_1 =& u_4 + 6u_1^2 + 6u_2 u_0 \\
\dot{u}_2 =& u_5 + 6u_3 u_0 + 18u_2 u_1 \\
\dot{u}_3 =& u_6 + 6u_4 u_0 + 24u_3 u_1 + 18u_2^2 \\
\dot{u}_4 =& -8u_5 u_0 - 12u_4 u_1 - 10u_3(u_2 + 7u_0^2) - 70u_1^3 - 140u_1 u_0^2 - 280u_2 u_1 u_0 \\
\dot{u}_5 =& -8u_6 u_0 + \ldots \\
\dot{u}_6 =& -28u_6 u_1 + \ldots
\end{aligned}$$

# Third stationary flow of $7^{th}$ order of KdV hierarchy: $X_3$

$$
\begin{aligned}
\dot{u}_0 &= u_5 + 20 u_2 u_1 + 10 u_3 u_0 + 30 u_1 u_0^2 \\
\dot{u}_1 &= u_6 + 10 u_4 u_0 + 30 u_3 u_1 + 10 u_2 (u_2 + 3 u_0^2) + 60 u_1^2 u_0 \\
\dot{u}_2 &= -4 u_5 u_0 + \dots \\
\dot{u}_3 &= -4 u_6 u_0 + \dots \\
\dot{u}_4 &= -10 u_6 u_1 + \dots \\
\dot{u}_5 &= 2 u_6 (-9 u_2 + 8 u_0^2) + \dots \\
\dot{u}_6 &= -28 u_6 u_3 + 112 u_6 u_1 u_0 + \dots
\end{aligned}
$$

# $M_7$ is a bi-Hamiltonian manifold

$$M_7 := \{(v_1 = \partial_x^{-1}\hat{X}_1, v_2 = \partial_x^{-1}\hat{X}_2, v_3 = \partial_x^{-1}\hat{X}_3, w_1 = \hat{X}_1, w_2 = \hat{X}_2, w_3 = \hat{X}_3, E)\}$$

$$\hat{P}_0 = \left[\begin{array}{cc|c} 0_3 & B & 0 \\ -B^T & 0_3 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \qquad B = 4\begin{bmatrix} 0 & 1 & 0 \\ 1 & v_1 & 0 \\ 0 & 0 & -v_3 \end{bmatrix},$$

$$\hat{P}_1 = \left[\begin{array}{cc|c} 0_3 & C & * \\ -C^T & 0_3 & * \\ \hline * & * & 0 \end{array}\right], \qquad C = 4\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & v_1 \\ 1 & v_1 & v_2 \end{bmatrix},$$

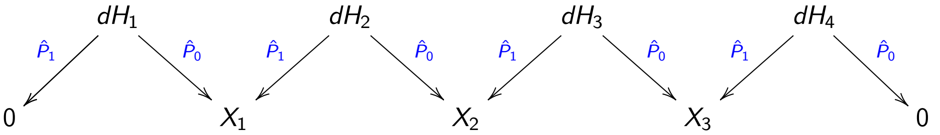
# Gelfand-Zakharevich systems of co-rank=1

$$H(\lambda) = H_1\lambda^3 + H_2\lambda^2 + H_3\lambda + H_4$$

$$H_1 = \frac{1}{4}w_1\left(w_2 - \frac{1}{2}w_1 v_1\right) - \frac{1}{8}\frac{w_3^2}{v_3} - \frac{1}{8}v_1^4 + \frac{3}{8}v_1^2 v_2 - \frac{1}{4}v_1 v_3 - \frac{1}{8}v_2^2 - \frac{1}{8}\frac{E}{v_3} \ ,$$

$$H_2 = \frac{1}{4}w_1\left(-\frac{1}{2}w_1 v_2 + w_3\right) + \frac{1}{8}w_2^2 - \frac{1}{8}\frac{w_3^2 v_1}{v_3} + \frac{1}{4}v_1 v_2\left(v_2 - \frac{1}{2}v_1^2\right) - \frac{1}{4}v_2 v_3 \ ,$$
$$+ \frac{1}{8}v_1^2 v_3 - \frac{1}{8}\frac{v_1}{v_3}E \ ,$$

$$H_3 = -\frac{1}{8}w_1^2 v_3 + \frac{1}{4}w_3\left(w_2 - \frac{1}{2}w_3\frac{v_2}{v_3}\right) - \frac{1}{8}v_1^3 v_3 - \frac{1}{8}v_3^2 + \frac{1}{4}v_1 v_2 v_3 - \frac{1}{8}\frac{v_2}{v_3}E \ , \quad H_4 = -\frac{1}{8}E \ .$$



# Nijenhuis operator for the GZ system

We have observed that in $M_7$ there exists a (some) Nijenhuis operator $\hat{N}$

$$\hat{N} =: \left[\begin{array}{cc|c} A & 0_3 & 0 \\ 0_3 & A & 0 \\ \hline 0 & 0 & 0 \end{array}\right] , \qquad A = \begin{bmatrix} 0 & 0 & -\frac{1}{v_3} \\ 1 & 0 & -\frac{v_1}{v_3} \\ 0 & 1 & -\frac{v_2}{v_3} \end{bmatrix} ,$$

which is the generator of the following cyclic Haantjes algebra

# KdV Haantjes algebra

$$\hat{K}_\alpha = e_\alpha I_7 + f_\alpha \hat{N} + g_\alpha \hat{N}^2 + h_\alpha \hat{N}^3 \qquad \alpha = 1, 2, 3$$

such that

$$\hat{K}_1{}^T \mathrm{d}H_3 = \mathrm{d}H_2 \quad \hat{K}_2{}^T \mathrm{d}H_3 = \mathrm{d}H_1 \quad \hat{K}_3{}^T \mathrm{d}H_3 = \mathrm{d}H_4$$

$$\hat{K}_1 =: \left[\begin{array}{cc|c} A_1 & 0_3 & 0 \\ 0_3 & A_1 & 0 \\ \hline 0 & 0 & v_1/v_2 \end{array}\right], \qquad A_1 = \begin{bmatrix} \frac{v_2}{v_3} & 0 & \frac{1}{v_3} \\ 1 & \frac{v_2}{v_3} & -\frac{v_1}{v_3} \\ 0 & 1 & 0 \end{bmatrix},$$

$$\hat{K}_2 =: \left[\begin{array}{cc|c} A_2 & 0_3 & 0 \\ 0_3 & A_2 & 0 \\ \hline 0 & 0 & 1/v_2 \end{array}\right], \qquad A_2 = \begin{bmatrix} \frac{v_1}{v_3} & -\frac{1}{v_3} & 0 \\ \frac{v_2}{v_3} & 0 & -\frac{1}{v_3} \\ 1 & 0 & 0 \end{bmatrix},$$

$$\hat{K}_3 =: \left[\begin{array}{cc|c} 0_3 & 0_3 & 0 \\ 0_3 & 0_3 & 0 \\ \hline 0 & 0 & v_3/v_2 \end{array}\right],$$

$$\begin{bmatrix} \hat{K}_0 \\ \hat{K}_1 \\ \hat{K}_2 \\ \hat{K}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{c_2}{c_1} & 1 + \frac{c_2^2}{c_1 c_3} + \frac{c_1 c_2}{c_3} & \frac{c_1^2}{c_3} + \frac{c_2}{c_3} & -\left(\frac{c_1}{c_3} + \frac{c_2}{c_1 c_3}\right) \\ \frac{c_3}{c_1} & -c_1 + \frac{c_2}{c_1} + \frac{c_2^2}{c_3} & 2 + \frac{c_1 c_2}{c_3} & -\left(\frac{c_2}{c_3} + \frac{1}{c_1}\right) \\ -\frac{1}{c_1} & -\frac{c_2}{c_1 c_3} & -\frac{1}{c_3} & \frac{1}{c_1 c_3} \end{bmatrix} = \begin{bmatrix} I \\ \hat{N} \\ \hat{N}^2 \\ \hat{N}^3 \end{bmatrix}$$

where

$$m_{\hat{N}}(\lambda) = \lambda^4 - c_1\lambda^3 - c_2\lambda^2 - c_3\lambda = \lambda\left(\lambda^3 - \frac{v_2}{v_3}\lambda^2 - \frac{v_1}{v_3}\lambda - \frac{1}{v_3}\right)$$

is the minimal polynomial of the Nijenhuis operator $\hat{N}$ in $M_7$.

# Non-Abelian Haantjes algebras and Superintegrability

Also superintegrable systems can be interpreted in the framework of Haantjes geometry.
Consider a $n$-dimensional **superintegrable system**, with $l$ integrals of motion $(n+1 \leq l \leq 2n-1)$. The following result holds.

**Theorem**

- *1) There exists a* **non-commutative Haantjes algebra** *of rank n associated with the system and its independent integrals.*
- *2) This noncommutative Haantjes algebra admits at least a* **commutative Haantjes subalgebra**, *corresponding to the commutative family of integrals of motion admitted by the system.*

# Example: the Post-Winternitz system

$$H = \frac{1}{2}(p_1^2 + p_2^2) + k_2 \frac{x_1}{x_2^{2/3}} + \frac{k_3}{x_2^{2/3}}$$

$$H_2 = 3p_1 p_2^2 + 2p_1^3 + k_2\left(\frac{6x_1}{x_2^{2/3}} p_1 + 9x_2^{1/3} p_2\right) + 6\frac{k_3}{x_2^{2/3}} p_1$$

$$K = 3 \left[\begin{array}{cc|cc} 2p_1 & p_2 & 0 & 3x_2 \\ 0 & 2p_1 & -3x_2 & 0 \\ \hline 0 & 0 & 2p_1 & 0 \\ 0 & 0 & p_2 & 2p_1 \end{array}\right].$$

Such a Haantjes operator possesses only 1 eigenvalue $l = 6p_1$, of algebraic multiplicity equal to 4, whereas the associated eigen-distribution is of dimension 2. It is not semisimple! Nevertheless, the Haantjes algebra whose basis is $\{I, K\}$ admits the Magri-Haantjes chain

$$I\mathrm{d}H = \mathrm{d}H_1 , \qquad K^T \mathrm{d}H = \mathrm{d}H_2 .$$

Moreover, the Post-Winternitz system admits an additional (independent) integral of motion

$$H_3 = p_2^4 - 12ax_2^{\frac{1}{3}}p_1p_2 + 4a\frac{x_1}{x_2^{\frac{2}{3}}}p_2^2 - 2a^2\left(9x_2^{2/3} - \frac{2x_1^2}{x_2^{\frac{4}{3}}}\right) .$$

and another Haantjes operator

$$\boldsymbol{L} = 4\left[\begin{array}{cc|cc} p_2^2 + 2a\frac{x_1}{x_2^{2/3}} & -(p_1p_2 + 3ax_2^{1/3}) & 0 & -3x_2p_1 \\ 0 & p_2^2 + 2a\frac{x_1}{x_2^{2/3}} & 3x_2p_1 & 0 \\ \hline 0 & 0 & p_2^2 + 2a\frac{x_1}{x_2^{2/3}} & 0 \\ 0 & 0 & -(p_1p_2 + 3ax_2^{1/3}) & p_2^2 + 2a\frac{x_1}{x_2^{2/3}} \end{array}\right] .$$

The Haantjes algebra whose basis is $\{I, L\}$ provides the Magri-Haantjes chain

$$I\,\mathrm{d}H = \mathrm{d}H_1 \ , \qquad L^T\mathrm{d}H = \mathrm{d}H_3 \ .$$

The set $\{I, K, L\}$ is also a Haantjes algebra but it is a non Abelian one!

# Non semisimple Haantjes algebras

Y. Kodama & B. Konopelchenko (2016)

$$\boldsymbol{K} = \begin{bmatrix} x_3 + x_1 x_2 + \frac{1}{3}x_1^2 & x_2 + \frac{1}{2}x_1^2 & x_1 & 0 \\ 0 & x_3 + x_1 x_2 + \frac{1}{3}x_1^2 & x_2 + \frac{1}{2}x_1^2 & x_1 \\ 0 & 0 & x_3 + x_1 x_2 + \frac{1}{3}x_1^2 & x_2 + \frac{1}{2}x_1^2 \\ 0 & 0 & 0 & x_3 + x_1 x_2 + \frac{1}{3}x_1^2 \end{bmatrix}$$

$$\mathscr{H} = Span\{\boldsymbol{I}, \boldsymbol{K}, \boldsymbol{K}^2, \boldsymbol{K}^3\}$$

A local chart in which $\boldsymbol{K}$ takes a Jordan form does not exist! Nor, a local chart in which $\boldsymbol{K}$ takes a generalized Jordan form does exist!

$$\boldsymbol{K} = \begin{bmatrix} 6x_4 & 0 & 0 & -9x_1 \\ 3x_3 & 6x_4 & -9x_1 & 0 \\ 0 & 0 & 6x_4 & 3x_3 \\ 0 & 0 & 0 & 6x_4 \end{bmatrix}$$

$$\mathscr{H} = Span\{\boldsymbol{I}, \boldsymbol{K}\}$$

A local chart in which $\boldsymbol{K}$ takes a Jordan form does not exist! However, in the local chart $(q_1 = x_2, q_2 = x_1^{1/3} x_3, q_3 = x_3, q_4 = x_4)$ $\boldsymbol{K}$ takes the following generalized Jordan form

$$\boldsymbol{K} = \begin{bmatrix} 6q_4 & 9\frac{q_2^2}{q_3^2} & 0 & 0 \\ 0 & 6q_4 & 0 & 0 \\ 0 & 0 & 6q_4 & 3q_3 \\ 0 & 0 & 0 & 6q_4 \end{bmatrix}$$

# Few results known about non semisimple Haantjes operators

- (Nijenhuis) E. T. Kobayashi (1962): studied the case of a field of operators $\boldsymbol{K}$ supposed to have a Jordan canonical form equal to a fixed matrix $\boldsymbol{J}$ in each point of $M$. He found that if $\boldsymbol{K}$ is cyclic, the vanishing of the Nijenhuis torsion and (Errata Corrige) the assumption that the direct sum of the generalized eigen–distribution be integrable, assure the existence of a local chart in which the local form of $\boldsymbol{K}$ is $\boldsymbol{J}$.
- (Nijenhuis) M. De Filippo, G. Vilasi & M. Salerno (1985): case of fields of operators $\boldsymbol{K}$ with real eigenvalues and generalized eigenvectors with Riesz index 2. The vanishing of the Nijenhuis torsion of $\boldsymbol{K}$ assures the existence of local charts in which its local form is the Jordan canonical form.
- (Haantjes) P. Tempesta & G. T. (2017): normal forms of operators with real eigenvalues (block-diagonal).

  The vanishing of the Haantjes torsion implies the existence of local charts in which the local form of $\boldsymbol{K}$ is quasi-diagonal, that is block-diagonal.

- In 2017 Y. Kosmann-Schwarzbach, and (independently) us, have defined the **generalized Nijenhuis torsion of level $n$** as the

$$\tau_{\boldsymbol{L}}^{(n)}(X,Y) := \boldsymbol{L}^2\tau_{\boldsymbol{L}}^{(n-1)}(X,Y) + \tau_{\boldsymbol{L}}^{(n-1)}(\boldsymbol{L}X,\boldsymbol{L}Y) - \boldsymbol{L}\Big(\tau_{\boldsymbol{L}}^{(n-1)}(X,\boldsymbol{L}Y) + \tau_{\boldsymbol{L}}^{(n-1)}(\boldsymbol{L}X,Y)\Big), \qquad n \geq 1$$

where $\tau_{\boldsymbol{L}}^{(0)}(X,Y) = [X,Y]$, $X, Y \in TM$. Here $\tau_{\boldsymbol{L}}^{(1)} = \tau_{\boldsymbol{L}}$ and $\tau_{\boldsymbol{L}}^{(2)} = \mathcal{H}_{\boldsymbol{L}}$.

We wish to show that such generalized Nijenhuis torsions of level $n$ possess an interesting geometric meaning, when we consider the case of triangular nilpotent operators. The relevance of these operators is well known: according to the classical Jordan-Chevalley decomposition, given a vector space $V$, any linear endomorphism $L : V \rightarrow V$ can be decomposed (in a suitable basis) as the sum $\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{T}$, where $\boldsymbol{D}$ is a diagonal operator and $\boldsymbol{T}$ is a triangular nilpotent operator.

Let $M$ be a differentiable manifold, and $\boldsymbol{A}_n : TM \rightarrow TM$ be a nilpotent $(1,1)$ tensor field; we shall assume that there exists a local coordinate chart on $M$ where $\boldsymbol{A}_n$ takes the form

$$\boldsymbol{A}_n := \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & \dots & a_{1n} \\ 0 & 0 & a_{23} & a_{24} & \dots & a_{2n} \\ 0 & 0 & 0 & a_{34} & \dots & a_{3n} \\ 0 & 0 & 0 & 0 & \dots & a_{(n-1)n} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} , \qquad (1)$$

where $a_{ij} = a_{ij}(x_1, \dots, x_n)$ are arbitrary functions depending on the local coordinates on $M$. Observe that the (standard) Haantjes torsion of $\boldsymbol{A}_3$ vanishes.

However, a direct calculation shows that this is not the case for the triangular nilpotent operators $\boldsymbol{A}_4$ and $\boldsymbol{A}_5$, namely $\mathcal{H}_{\boldsymbol{A}_4}(X,Y) \neq 0$, $\mathcal{H}_{\boldsymbol{A}_5}(X,Y) \neq 0$. Nevertheless, their generalized Nijenhuis torsions of level three and four vanishes: $\tau^{(3)}_{\boldsymbol{A}_4}(X,Y) = 0$, $\tau^{(4)}_{\boldsymbol{A}_5}(X,Y) = 0$. Inspired by these observations, we conjecture the following result, which has been tested in many examples.

**Conjecture**. *Let $M$ be an $n$-dimensional differentiable manifold, and $\boldsymbol{A}_n : TM \rightarrow TM$ be a (1,1) tensor fields on $M$. The vanishing of its generalized Nijenhuis torsion of level $(n-1)$*

$$\tau^{(n-1)}_{\boldsymbol{A}_n}(X,Y) = 0$$

*is a necessary condition for the existence of a local chart where the operator field $\boldsymbol{A}_n$ takes the nilpotent triangular form* (1).

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# Happy 70's

# Professor Krasil'shchik!