Classical *r*-matrix like approach to Frobenius manifolds

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Motivation

- Frobenius manifolds are equivalent to pencils of hydrodynamic type (local) Poisson tensors, with some 'potentiality' properties. (Dubrovin)
- Classical r-matrix formalism (Semenov-Tian-Shansky, ...) is very efficient in construction of hydrodynamic Poisson tensors and related integrable hierarchies. (Luan-Chau Li, Błaszak, Sz., ...)

The goal is to develop the above scheme in order of construction of Frobenius manifolds and solutions of WDVV equations directly from principal hierarchy.

Standard approach

- The topological Landau-Ginzburg models (in 2d TFT):

$$\eta\left(\partial', \partial''\right) = -\sum_{|\lambda|<\infty} \mathrm{res}_{d\lambda=0} \frac{\partial'\lambda\partial''\lambda}{\lambda_z} dz$$

$$c\left(\partial', \partial'', \partial'''\right) = -\sum_{|\lambda|<\infty} \mathrm{res}_{d\lambda=0} \frac{\partial'\lambda\partial''\lambda\partial'''\lambda}{\lambda_z} dz,$$

  where $\lambda = \lambda(z)$ is a superpotential, as well as, the singularity theory and Saito's formalism. Simple singularities are labeled A_n, D_n, E_6, E_7, E_8.

- Construction from the point of view τ-functions, string equations, etc. (Krichever, Aoyama and Kodama, Takasaki, ...)

- Dubrovin (remarkable) formula ...

Frobenius manifold

Frobenius manifold

Smooth (or holomorphic) manifold M with (nondegenerate) metric η and Frobenius algebra structure on tangent bundle $(TM, \circ)$, i.e. $\circ$ is associative commutative unital and invariant:
$\eta(X \circ Y, Z) = \eta(X, Y \circ Z)$.

Further requirements:

1. η is flat;
2. ∇c is symmetric in all its four arguments, where $c(X, Y, Z) := \eta(X \circ Y, Z)$;
3. the unit e is flat, i.e. $\nabla e = 0$;
4. exists Euler field E (i.e. $\nabla \nabla E = 0$) s.t.

$$\mathcal{L}_E \circ = \circ \qquad \text{and} \qquad \mathcal{L}_E \eta = d\,\eta,$$

where d is a number. Normalisation: $\mathcal{L}_E e = -e$.

WDVV associativity equations

Prepotential

Let $\{t^\alpha\}$ be (local) flat coordinates of η s.t. $e = \partial_{t^1}$. Then, there exists (smooth) function $F(t)$ such that

$$c_{\alpha,\beta,\gamma} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \qquad \text{and} \qquad \eta_{\alpha,\beta} = \frac{\partial^3 F(t)}{\partial t^1 \partial t^\alpha \partial t^\beta}.$$

WDVV equations

The structure constants: $\partial_\alpha \circ \partial_\beta = c_{\alpha\beta}^\gamma(t)\partial_\gamma$, where $c_{\alpha\beta}^\gamma = c_{\alpha\beta\epsilon}\eta^{\epsilon\gamma}$. Then, the associativity equations on $F(t)$ are

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 F(t)}{\partial t^\delta \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\alpha}.$$

Quasi-homogeneity condition

$$EF = (3 - d)F + quadratic\ terms,$$

where

$$E = ((1 - q_\alpha)t^\alpha + r_\alpha)\,\partial_\alpha.$$

Intersection form

Second flat metric

$$\gamma^{-1}(\alpha, \beta) := \langle \alpha \circ \beta, E \rangle \qquad \alpha, \beta \in T^*M$$

Dubrovin-Novikov bracket

Loop manifold: $\mathcal{L}M = \left\{ \mathbb{S}^1 \to M \right\}$

Hydrodynamic Poisson bracket

has the form

$$\{h, f\} = \int_{\mathbb{S}^1} \frac{\partial f}{\partial u^\mu} \pi^{\mu\nu} \frac{\partial h}{\partial u^\nu} \, dx := \int_{\mathbb{S}^1} \left\langle df, \eta^{-1} \nabla_{u_x} dh \right\rangle dx, \qquad (1)$$

where

$$\pi^{\mu\nu} = \eta^{\mu\nu} \partial_x - \eta^{\mu\varepsilon} \Gamma^\nu_{\varepsilon\sigma} u^\sigma_x$$

and $u^\sigma : \mathbb{S}^1 \to M$ are (dynamical) coordinate fields.

Recall that (1) is a Poisson bracket wrt nondegenerate metric η iff
- (i) η is flat
- (ii) and $\Gamma^\nu_{\varepsilon\sigma}$ is the Levi-Civita connection of η.

Further properties

Deformed flat connection

$$\widetilde{\nabla}_X Y := \nabla_X Y + \xi\, X \circ Y,$$

which is symmetric and flat.

Thus

$$\widetilde{\nabla}_i dh_k(\xi) = 0 \qquad \Longleftrightarrow \qquad \partial_i \partial_j h_k = \xi\, c_{ij}^l\, \partial_l h_k$$

Expanding, $h_k(\xi) = \sum_{p=0}^{\infty} h_{k,p} \xi^p$, the coefficients can be determined recursively from

$$\partial_i \partial_j h_{k,p} = c_{ij}^l\, \partial_l h_{k,p-1} \qquad p = 1, 2, \ldots .$$

Principal hierarchy

Taking $h_{k,p}$ as Hamiltonian densities and applying D-N Poisson tensor wrt η, in flat coordinates the hierarchy takes the form

$$(t^\mu)_{\tau^{k,p}} = \eta^{\mu\nu}\partial_x\partial_{t^\nu}h_{k,p}(t).$$

Claim

Let $h_{\alpha,0} \equiv \eta_{\alpha\mu}t^\mu$, then the prepotential F is determined by $h_{\alpha,1}$:

$$F = \frac{1}{3-d}\sum_\alpha E^\alpha h_{\alpha,1} + q.t. \qquad d \neq 3$$

$$= \frac{1}{3-d}\sum_\alpha \left[(1-q_\alpha)t^\alpha + r_\alpha\right]h_{\alpha,1} + q.t. \qquad d \neq 3.$$

Proposition

Let $\mathfrak{g}$ be an associative algebra and second multiplication be

$$a \circ_\ell b := \ell(a)b + a\ell(b) \qquad a, b \in \mathfrak{g},$$

generated by a linear map $\ell : \mathfrak{g} \to \mathfrak{g}$. A sufficient condition for its associativity is the so-called (modified) Poincare-Bertrand formula

$$\ell(a \circ_\ell b) - \ell(a)\ell(b) = \delta\, ab, \qquad (2)$$

where $\delta \in Center(\mathfrak{g})$.

Besides, $\ell'(\cdot) = \ell(d\cdot)$ satisfies (2) for arbitrary $d \in Center(\mathfrak{g})$ iff ℓ satisfies (2), then $\delta' = \delta d^2$.

Construction of Frobenius algebras II

Let $\mathcal{A}$ be a commutative associative unital algebra, with a trace form $\mathrm{tr} : \mathcal{A} \to \mathbb{C}$ s.t. the pairing $(a, b)_{\mathcal{A}} := \mathrm{tr}(ab)$ is non-degenerate. Let $\circ_\ell$ be a second commutative associative multiplication on $\mathcal{A}$.

Invariant metric

Then, the metric

$$\eta(a, b) := \mathrm{tr}\left(a \circ_\ell b\right) \qquad a, b \in \mathcal{A}$$

is (naturally) invariant

$$\eta\left(a \circ_\ell b, c\right) = \mathrm{tr}\left(a \circ_\ell b \circ_\ell c\right) = \eta\left(a, b \circ_\ell c\right).$$

Classical *r*-matrices

Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra.

Classical *r*-matrix

is a linear map $r : \mathfrak{g} \to \mathfrak{g}$ such that

$$[a, b]_r := [r(a), b] + [a, r(b)] \qquad a, b \in \mathfrak{g}$$

defines second Lie bracket on $\mathfrak{g}$.

Modified Yang-Baxter equation

A sufficient condition for r to be a classical r-matrix is to satisfy:

$$[r(a), r(b)] - r([a, b]_r) + \alpha[a, b] = 0,$$

where α is a number.

Simplest solutions

Assume that $(\mathcal{A}, \cdot)$ can be decomposed into subalgebras, i.e.

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- \qquad \mathcal{A}_\pm \mathcal{A}_\pm \subset \mathcal{A}_\pm \qquad \mathcal{A}_+ \cap \mathcal{A}_- = \emptyset.$$

Then $\ell = \frac{1}{2}(P_+ - P_-)$ satisfies the Poincare-Bertrand equation for $\delta = \frac{1}{4}$.

Assume that $(\mathcal{A}, [\cdot, \cdot])$ can be decomposed into Lie subalgebras, i.e.

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Then $r = \frac{1}{2}(P_+ - P_-)$ satisfies the Yang-Baxter equation for $\alpha = \frac{1}{4}$.

Theorem by L.-C. Lie

Let $(\mathcal{A}, \{\cdot, \cdot\})$ be a Poisson algebra with a non-degenerate ad-invariant scalar product $(a, b)_{\mathcal{A}} = \mathrm{Tr}(ab)$. Assume that r is a classical r-matrix, then for each $n \geqslant 0$ the formula

$$\{h, f\}_n (\lambda) = (\lambda, \{r(\lambda^n df), dh\} + \{df, r(\lambda^n dh)\})_{\mathcal{A}},$$

where $h, f \in \mathcal{C}^\infty(\mathcal{A})$, defines a Poisson structure on $\mathcal{A}$. Moreover, all brackets are compatible.

Hydrodynamic Poisson structure

Let the Poisson bracket on $\mathcal{A}$ be given in the form $\{\cdot, \cdot\} = \partial \wedge \partial_x$, where $\partial, \partial_x \in Der(\mathcal{A})$ and ∂_x is such that $\partial_x r = r \partial_x$.

Linear metric

Then, the linear Poisson bracket takes the hydrodynamic form

$$\pi_0(\omega) = r^*(\lambda' \omega_x) - \lambda' r(\omega_x) - r^*(\lambda_x \omega') + \lambda_x \partial r(\omega)$$

for which the corresponding metric is $\eta^{-1}(\omega) = r^*(\lambda' \omega) - \lambda' r(\omega)$.

Proposition

If $\ell = r^*$ satisfies the Poincare-Bertrand formula. then the multiplications

$$a \circ_r b = r^*(\lambda' a)b + a r^*(\lambda' b)$$

is associative and on $\mathcal{A}$ there is a structure of Frobenius algebra.

Useful formulae

Proposition

Rewriting the recurrence formula on the cotangent bundle of Frobenius manifold one finds that

$$\widetilde{\nabla}_X dh = 0 \quad \Longleftrightarrow \quad \eta^{-1}\nabla_X dh(\lambda) = \xi dh(\lambda) \circ^\dagger X,$$

where $\langle \beta, \gamma \circ^\dagger X \rangle := \langle \gamma \circ \beta, X \rangle$.

Proposition

On $(\mathcal{A}, \cdot, \circ_\ell)$ with $\ell = r^*$ we have

$$\gamma \circ_r^\dagger \lambda_t = \lambda' r(\gamma \lambda_t) + r^*(\lambda' \gamma)\lambda_t.$$

Poisson algebras of Laurent series near ∞ I

Algebra

Let $\mathcal{A}^\infty = \left\{ \sum_i^N u_i z^i | u_i \in \mathbb{C} \right\} = \mathcal{A}_{\geqslant k} \oplus \mathcal{A}_{<k}$. Then $\ell = P_{\geqslant k} - \frac{1}{2}$ only for $k = 0$ or 1.

Multiplication

Let $\lambda \in \mathcal{A}$, consider multiplications given by

$$a \circ b := \ell(z^r \lambda_z a)b + a\ell(z^r \lambda_z b)$$
$$= (z^r \lambda_z a)_{\geqslant k} b + a(z^r \lambda_z b)_{\geqslant k} - z^r \lambda_z ab.$$

Metric

Trace form is $\mathrm{tr}(\cdot) := -\mathrm{res}_{z=\infty}(z^{-r}\cdot)$, then corresponding metric has the form

$$\eta^{-1}(\omega) = z^r \left[(\lambda_z \omega)_{\geqslant k-r} - \lambda_z(\omega)_{\geqslant r-k} \right].$$

Poisson algebras of Laurent series near ∞ II

Proposition

Then, we have the Frobenius algebra structure on $\mathcal{A}^\infty$ for:

- $r = k = 0$, corresponds to dispersionless KdV;
- $r = 1$ and $k = 1$: dToda;
- $r = 1$ and $k = 0$: modified dToda;
- $r = 2$ and $k = 1$: isomorphic to dKdV.

Proposition

The recurrence formula takes the form:

$$\frac{\partial h_p(\lambda)}{\partial \lambda} = -h_{p-1}(\lambda) \qquad p = 0, 1, 2, \ldots.$$

Euler vector field:

$$E(\lambda) = \lambda - \frac{1}{N} z \lambda_z.$$

Theorem

Respective Frobenius manifolds are:

(i) for $r = 0$ when

$$M = \left\{ \lambda = z^N + u_{N-2}z^{N-2} + \ldots + u_0 \right\} \subset \mathcal{A},$$

corresponding to dKdV (equivalent with A_N model);

(ii) for $r = 1$ when

$$M = \left\{ \lambda = z^N + u_{N-1}z^{N-1} + \ldots + u_{-m}z^{-m} \right\} \subset \mathcal{A} \qquad N, m \geqslant 0,$$

corresponding to dToda.

Now, λ is appropriate meromorphic function with 'finite' nonzero poles γ.

Trace forms: $\mathrm{tr}_\infty(\cdot) = -\mathrm{res}_{z=\infty}(\cdot)$ or $\mathrm{tr}_\gamma(\cdot) = -\mathrm{res}_{z=\gamma}(\cdot)$.

Then, $\mathrm{tr}_\infty(\beta_\infty \lambda_t) = \mathrm{tr}_\gamma(\beta_\gamma \lambda_t)$ and

$$\eta_\infty^{-1}(\omega) = (\lambda_z \omega)_{\geqslant 0}^\infty - \lambda_z(\omega)_{\geqslant 0}^\infty$$
$$\eta_\gamma^{-1}(\omega) = -(\lambda_z \omega)_{<0}^\gamma + \lambda_z(\omega)_{<0}^\gamma.$$

Proposition

All the above operators define the same metric, i.e.

$$\eta^{-1}(\alpha, \beta) \equiv \mathrm{tr}_\infty\left(\beta\, \eta_\infty^{-1}(\alpha)\right) = \mathrm{tr}_\gamma\left(\beta\, \eta_\gamma^{-1}(\alpha)\right),$$

and the same Frobenius algebras.

Superpotential

$\lambda = z^4 + uz^2 + vz + w = z^4 + t^3 z^2 + t^2 z + t^1 + \frac{1}{8}(t^3)^2$.

Linear Poisson tensor $\pi_0(\omega) = \{(\omega)_{\geqslant 0}, \lambda\} - (\{\omega, \lambda\})_{\geqslant 0}$. Then, for $i = 1, 2, 3$ Casimirs are $c_i = \int_{\mathbb{S}^1} t^i \, dx$, where

$$t^i = \frac{4}{i-4} \mathrm{res}_{z=\infty} \lambda^{-\frac{i}{4}+1} \qquad dt^i = \lambda^{-\frac{i}{4}}$$

s.t. $(dt^i)_{\geqslant 0}^{\infty} = 0$.

Flat coordinates

$t^1 = -\frac{1}{8} u^2 + w$, $t^2 = v$, $t^1 = u$.

$$E(\lambda) = \lambda - \frac{1}{4} z \lambda_z = \frac{1}{2} t^3 z^2 + \frac{3}{4} t^2 z + t^1 + \frac{1}{8} (t^3)^2.$$

Hence,

$$E = \frac{1}{2} t^3 \partial_{t^3} + \frac{3}{4} t^2 \partial_{t^2} + t^1 \partial_{t^1}.$$

One finds: $h_{i,1} = \frac{16}{i(i+4)} \operatorname{res}_{z=\infty} \lambda^{\frac{i}{4}+1}$ for $i = 1, 2, 3$.

Prepotential

$$F = -\frac{5}{4} t^1 (t^2)^2 - \frac{5}{4} (t_1)^2 t^3 + \frac{5}{32} (t^2)^2 (t^3)^2 - \frac{1}{384} (t^3)^5$$

Example: two-pole case, $r = k = 0$. I

Superpotential: $\lambda = z + t_1 (z - t_3)^{-1} + t_2 (z - t_4)^{-1}$

Flat coordinates:

$$t_1 = h_{3,0} = \mathrm{res}_{t_3} \lambda \qquad t_2 = h_{4,0} = \mathrm{res}_{t_4} \lambda$$
$$t_3 = h_{1,0} = -\mathrm{res}_\infty Log \lambda + \mathrm{res}_{t_3} Log \lambda$$
$$t_4 = h_{2,0} = -\mathrm{res}_\infty Log \lambda + \mathrm{res}_{t_4} Log \lambda.$$

Hamiltonian densities:

$$h_{3,1} = -\frac{1}{2}\mathrm{res}_{t_3}\lambda^2 \qquad h_{4,1} = -\frac{1}{2}\mathrm{res}_{t_4}\lambda^2$$
$$h_{1,1} = \mathrm{res}_\infty (\lambda Log \lambda - \lambda) - \mathrm{res}_{t_3} (\lambda Log \lambda - \lambda)$$
$$h_{2,1} = \mathrm{res}_\infty (\lambda Log \lambda - \lambda) - \mathrm{res}_{t_4} (\lambda Log \lambda - \lambda)$$

Euler field: $E(\lambda) = \lambda - z\lambda_z = 2t_1\partial_{t_1}\lambda + 2t_2\partial_{t_2}\lambda + t_3\partial_{t_3}\lambda + t_4\partial_{t_4}\lambda$.

Prepotential:

$$F = -t_1 t_3^4 - t_2^3 - t_1^2 Log t_1 - t_2^2 Log t_2 - 2t_1 t_2 Log(t_3 - t_4)$$

Euler field: $E(\lambda) = \lambda - z\lambda_z = 2t_1\partial_{t_1}\lambda + 2t_2\partial_{t_2}\lambda + t_3\partial_{t_3}\lambda + t_4\partial_{t_4}\lambda$.

Prepotential:

$$F = -t_1 t_3^4 - t_2^3 - t_1^2 Log t_1 - t_2^2 Log t_2 - 2t_1 t_2 Log(t_3 - t_4)$$

Thank you!