# Novikov algebras and a classification of multicomponent Camassa-Holm equations


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# Motivation

The Camassa-Holm equation:

$$v_t - v_{xxt} = -3vv_x + 2v_x v_{xx} + vv_{xxx} + cv_{xxx}$$

has intriguing properties like:

(i) the existence of multi 'peakon' solutions,

(ii) the non-existence of a $\tau$-function or functions,

(iii) that it can be found by exploiting the tri-Hamiltonian structure of the KdV hierarchy:

$$\frac{d}{dx}, \qquad \frac{d^3}{dx^3}, \qquad u\frac{d}{dx} + \frac{1}{2}u_x,$$

which may be recombined to form the bi-Hamiltonian structure

$$\mathcal{P}_1 = \frac{d}{dx} + \frac{d^3}{dx^3}, \qquad \mathcal{P}_2 = u\frac{d}{dx} + \frac{1}{2}u_x.$$

Then, the Lenard-Magri recursion scheme results in the Camassa-Holm equation, where $u = v - v_{xx}$ .

**Goal**

Construct and classify multicomponent versions of Camassa-Holm equations.

Homogeneous first order operator:

$$\mathcal{P}^{ij} = g^{ij}(u)\frac{d}{dx} + b^{ij}_k u^k_x\,, \qquad x \in \mathbb{S}^1,$$

where $g^{ij}(u) = c^{ij}_k u^k$ is symmetric and $b^{ij}_k$, $c^{ij}_k$ are constants.

General case by Dubrovin-Novikov (1984) – $\mathcal{P}$ is Hamiltonian iff $(g, \Gamma)$ is flat.

## Balinskii-Novikov (1985)

The operator is a Poisson operator if and only if

- $c^{ij}_k = b^{ij}_k + b^{ji}_k$;
- $b^{ij}_k$ is the set of structure constants of an algebra $\mathbb{A}$, that is $e^i \cdot e^j = b^{ij}_k e^k$ where $e^1, \ldots, e^n$ are basis vectors, such that

$$(a \cdot b) \cdot c = (a \cdot c) \cdot b\,,$$
$$(a \cdot b) \cdot c - a \cdot (b \cdot c) = (b \cdot a) \cdot c - b \cdot (a \cdot c)\,.$$

This structure $(\mathbb{A}, \cdot)$ is called a **Novikov algebra**.

Rewrite in terms of left and right multiplications $L_a b = R_b a = a \cdot b$ :

$$[R_a, R_b] = 0 \,, \qquad [L_a, L_b] = L_{[a,b]} \,,$$

where $[a, b] = a \cdot b - b \cdot a$.

- Novikov algebras are Lie admissible;
- commutative $\Rightarrow$ associative;
- left unity $\Rightarrow$ commutative (and hence associative).

**Classification of Novikov algebras**

- Classification: Bai & Meng (2001) – $dim \leq 3$, transitive case for $dim = 4$
- Burde & de Graaf (2013) – $dim = 4$ with abelian and nilpotent Lie algebras

Classification for $dim \geqslant 4$ is far from being complete.

# Lie-Poisson structure

## Associated translationally invariant Lie algebra $\mathscr{L}_{\mathbb{A}}$

The space $\mathscr{L}_{\mathbb{A}}$ of $\mathbb{A}$-valued functions of $x \in \mathbb{S}^1$, with a bracket of the form

$$[\![a, b]\!] := a_x \cdot b - b_x \cdot a \,, \qquad a_x \equiv \frac{da}{dx},$$

which defines a Lie bracket if and only if the algebra $\mathbb{A}$ with the multiplication $\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is a Novikov algebra.

The Lie-Poisson bracket associated to the Lie algebra $\mathscr{L}_{\mathbb{A}}$ is

$$\{\mathcal{H}, \mathcal{F}\}[u] := \int_{\mathbb{S}^1} \frac{\delta \mathcal{F}}{\delta u^i} \mathcal{P}^{ij} \frac{\delta \mathcal{H}}{\delta u^j}\, dx \equiv \left\langle u, [\![\delta_u \mathcal{F}, \delta_u \mathcal{H}]\!] \right\rangle, \qquad u \in \mathscr{L}_{\mathbb{A}}^*,$$

where $\mathcal{H}, \mathcal{F} \in \mathscr{F}(\mathscr{L}_{\mathbb{A}}^*)$ are functionals.

# Deformations and cocycles

Hamitonian condition $[\mathcal{H},\mathcal{H}]_S = 0$ (Schouten bracket). Deform

$$\mathcal{H} \mapsto \mathcal{H} + \lambda\,\mathcal{K}, \qquad \lambda = const,$$

while preserving Hamiltonian property implies:

$$[\mathcal{K},\mathcal{K}]_S = 0\,, \qquad\qquad [\mathcal{H},\mathcal{K}]_S = 0\,.$$

We will restrict to constant deformations.

First order deformation:

$$\mathcal{K} = g^{ij}\frac{d}{dx} \qquad g - \text{symmetric}$$

Extra condition: $g(a \cdot b, c) = g(a, c \cdot b)$ (quasi-Frobenius condition).

Second order deformation:

$$\mathcal{K} = f^{ij}\frac{d^2}{dx^2} \qquad f - \text{anti-symmetric}$$

Extra conditions: $f(a \cdot b, c) = f(a, c \cdot b)$, $f(a \cdot b, c) + f(b \cdot c, a) + f(c \cdot a, b) = 0$.

Third order deformation:

$$\mathcal{K} = h^{ij}\frac{d^3}{dx^3} \qquad h - \text{symmetric}$$

Extra condition: $h(a \cdot b, c)$ – totally symmetric (Frobenius condition).

# Outline

- Assemble Hamitonian bits to form bi-Hamiltonian operators and derive:
  - $(i)$ KdV-type equations;
  - $(ii)$ Camassa-Holm type equations.
- Complete classification of cocycles (partially done by Bei and Meng).
- Classify systems in terms of algebraic structures on Novikov algebras.

**Aim**

Understand properties of integrable systems in terms of underlying Novikov algebras and related cocycles.

The characteristic matrix of a Novikov algebra $\mathbb{A}$ is $\mathcal{B} = (b_{ij})$ defined by $b_{ij} := e_i \cdot e_j = b_{ij}^k e_k$.

Table : Classification of bilinear forms associated with one and two-dimensional Novikov algebras.

| type | charact. matrix | $g$ | $f$ | $h$ | comments |
|---|---|---|---|---|---|
| $\mathbb{C}$ | $e_1$ | $g_{11}$ | $0$ | $h_{11}$ | |
| $(T2)$ | $\begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & 0 \end{pmatrix}$ | transitive |
| $(T3)$ | $\begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & h_{22} \end{pmatrix}$ | transitive |
| $(N3)$ | $\begin{pmatrix} e_1 & e_2 \\ e_2 & 0 \end{pmatrix}$ | $\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & 0 \end{pmatrix}$ | |
| $(N4)$ | $\begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}$ | $\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$ | $\begin{pmatrix} 0 & f_{12} \\ -f_{12} & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & h_{22} \end{pmatrix}$ | $\det f \neq 0$ |
| $(N5)$ | $\begin{pmatrix} 0 & e_1 \\ 0 & e_1 + e_2 \end{pmatrix}$ | $\begin{pmatrix} 0 & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & h_{22} \end{pmatrix}$ | |
| $(N6)$ | $\begin{pmatrix} 0 & e_1 \\ \kappa e_1 & e_2 \end{pmatrix}$ | $\begin{pmatrix} 0 & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 \\ 0 & h_{22} \end{pmatrix}$ | $\kappa \neq 0, 1$ |

Consider pair of compatible Poisson operators $\mathcal{P}_0$ and $\mathcal{P}_1$ on $\mathscr{L}_{\mathbb{A}}$, associated with Novikov algebra $\mathbb{A}$ :

$$\mathcal{P}_1\gamma = (R^*_\gamma u)_x + L^*_{\gamma_x} u + g_1\gamma_x + f_1\gamma_{xx} + h_1\gamma_{xxx}$$

and

$$\mathcal{P}_0\gamma = g_0\gamma_x + f_0\gamma_{xx} + h_0\gamma_{xxx},$$

where $u \in \mathscr{L}^*_{\mathbb{A}}$ and $\gamma \in \mathscr{L}_{\mathbb{A}}$.

Here $g_0$ and $g_1$ generate first order deformations, $f_0$ and $f_1$ deormations of order two, while $h_0$ and $h_1$ third order deformations.

Bi-Hamiltonian chain:

$$\begin{aligned} u_{t_0} &= \mathcal{P}_0\delta_u\mathcal{H}_0 \equiv 0\,, \\ u_{t_1} &= \mathcal{P}_1\delta_u\mathcal{H}_0 = \mathcal{P}_0\delta_u\mathcal{H}_1 \\ u_{t_2} &= \mathcal{P}_1\delta_u\mathcal{H}_1 = \mathcal{P}_0\delta_u\mathcal{H}_2 \\ &\vdots \end{aligned}$$

An inertia operator $\Lambda : \mathscr{L}_{\mathbb{A}} \to \mathscr{L}_{\mathbb{A}}^*$ defined by

$$\Lambda\gamma := g_0\gamma + f_0\gamma_x + h_0\gamma_{xx},$$

such that $\mathcal{P}_0\gamma \equiv \Lambda\gamma_x$. $\Lambda$ is self-adjoint, i.e. $\Lambda^\dagger = \Lambda$, and it is assumed that it is invertible. Thus we impose that $g_0$ is nondegenerate in the generic case.

Convenient change of coordinates:

$$v := \Lambda^{-1}u,$$

which is of (linear) Miura-type.

Thus:

$$\begin{aligned} v_{t_0} &= \tilde{\mathcal{P}}_0\delta_v\mathcal{H}_0 \equiv 0 \\ v_{t_1} &= \tilde{\mathcal{P}}_1\delta_v\mathcal{H}_0 = \tilde{\mathcal{P}}_0\delta_v\mathcal{H}_1 \\ v_{t_2} &= \tilde{\mathcal{P}}_1\delta_v\mathcal{H}_1 = \tilde{\mathcal{P}}_0\delta_v\mathcal{H}_2 \\ &\vdots \end{aligned}$$

where $\tilde{\mathcal{P}}_i = \Lambda^{-1}\mathcal{P}_i(\Lambda)^{-1}$ and $\delta_v\mathcal{H}_j = \Lambda\delta_u\mathcal{H}_j$.

## Theorem

The first two evolution equations from the hierarchy are

$$
\begin{aligned}
v_{t_1} &= v_x \cdot c\,, \\
g_0(v_{t_2}) + f_0(v_{xt_2}) + h_0(v_{xxt_2}) &= g_0(v_x \cdot (v \cdot c)) + g_0(v \cdot (v_x \cdot c)) + L^*_{v \cdot c} g_0(v_x) \\
&\quad + f_0(v_x \cdot (v_x \cdot c)) + f_0(v_{xx} \cdot (v \cdot c)) \\
&\quad + 2h_0((v_x \cdot c) \cdot v_{xx}) + h_0((v \cdot c) \cdot v_{xxx}) \\
&\quad + g_1(v_x \cdot c) + f_1(v_{xx} \cdot c) + h_1(v_{xxx} \cdot c)\,,
\end{aligned}
$$

where $c = \mathrm{const} \in \mathscr{L}_{\mathbb{A}}$ and $\delta_u \mathcal{H}_0 = c$.

The densities of the first three Hamiltonian functionals are

$$
\begin{aligned}
H_0 &= g_0(c, v)\,, \\
H_1 &= \frac{1}{2}\, g_0(v, v \cdot c) + \frac{1}{2}\, f_0(v_x, v \cdot c) + \frac{1}{2}\, h_0(v_{xx}, v \cdot c)\,, \\
H_2 &= \frac{1}{3}\, g_0(v, v \cdot (v \cdot c)) + \frac{1}{3}\, f_0(v_x, v \cdot (v \cdot c)) + \frac{1}{3}\, h_0(v \cdot c, v \cdot v_{xx}) \\
&\quad + \frac{1}{6}\, g_0(v \cdot c, v \cdot v) + \frac{1}{6}\, h_0(v_x \cdot c, v_x \cdot v) \\
&\quad + \frac{1}{2}\, g_1(v, v \cdot c) + \frac{1}{2}\, f_1(v_x, v \cdot c) + \frac{1}{2}\, h_1(v_{xx}, v \cdot c)\,.
\end{aligned}
$$

## Remark

The Hamiltonian flow on the dual space $\mathscr{L}_{\mathbb{A}}^*$ can be interpreted as the Euler equation corresponding to the centrally extended Lie algebra $\mathscr{L}_{\mathbb{A}}$ with the quadratic Hamiltonian

$$\mathcal{H}_1 = \frac{1}{2}\langle u, \Lambda^{-1} R_c^* u\rangle.$$

This Euler equation transformed to $\mathscr{L}_{\mathbb{A}}$ through $v := \Lambda^{-1}u$ is exactly the second flow from the above theorem.

- The only relevant Novikov algebras are in dimension one: the field of complex numbers $\mathbb{C}$; in dimension two: $(N3)$–$(N6)$; in dimension three: $(C6)$, $(C8)$, $(C9)$, $(C16)$, $(C19)$, $(D2)$–$(D5)$; in dimension four (within the considered sub-class of the Novikov algebras): $\widetilde{A}_{3,3}$, $\widetilde{A}_{3,4}$, $N_{22}^{\mathfrak{h}_1}$, $N_{23}^{\mathfrak{h}_1}$, $N_{24}^{\mathfrak{h}_1}$, $N_{27}^{\mathfrak{h}_2}$ and $\mathbb{A}_4$.
- Most of the relevant Novikov algebras lead to the construction of evolution equations in a triangular form.
- The only non-triangular systems are associated to the algebras $(N4)$, $(C8)$ and $\mathbb{A}_4$.
- Many Novikov algebras with nontrivial algebraic properties result in systems of evolution equations which are degenerate, for example, not fully nonlinear in all of the variables.

- The only relevant one-dimensional Novikov algebra $\mathbb{A}$ is $\mathbb{C}$.
- Let $g_0 = g$, $g_1 = \alpha$, $h_0 = h$ and $h_1 = \beta$ (and $f_1 = f_2 = 0$).
  For $c = 1$ we have
  $$g v_t + h v_{xxt} = \alpha v_x + 3 g v v_x + 2 h v_x v_{xx} + h v v_{xxx} + \beta v_{xxx} \,,$$
  here $v \in \mathscr{L}_{\mathbb{C}}$.
- It was obtained before by Khesin and Misiołek (2003).
- Particular cases:
  - Korteweg–de Vries equation: $v_t = 3 v v_x + v_{xxx}$
  - Camassa-Holm equation: $v_t - v_{xxt} = \alpha v_x + 3 v v_x - 2 v_x v_{xx} - v v_{xxx} + \beta v_{xxx}$
  - Hunter-Saxton equation: $v_{xxt} = 2 v_x v_{xx} + v v_{xxx}$.

**Proposition**

For any dimension $n$ the algebra $\mathbb{A}_n$ defined by

$$(a \cdot b)^i := a^i b^n \qquad \Longleftrightarrow \qquad b^k_{ij} = \delta^k_i \delta^n_j .$$

is an associative Novikov algebra. If $n \geq 2$ it is non-abelian. The associated Lie algebra structure on $\mathbb{A}_n$ is non-nilpotent. Moreover:

- an arbitrary symmetric bilinear form $g$ on $\mathbb{A}_n$ satisfies the quasi-Frobenius condition;
- an anti-symmetric bilinear form $f = (f_{ij})$ on $\mathbb{A}_n$ satisfies the second-order conditions iff $f_{ij} = 0$ for $i \neq n$ and $j \neq n$;
- a symmetric bilinear form $h = (h_{ij})$ on $\mathbb{A}_n$ satisfies the Frobenius conditions if and only if $h_{ij} = 0$ for $i \neq n$ or $j \neq n$.

**Remark**

The translationally invariant Lie algebra $\mathscr{L}_{\mathbb{A}_n}$ is isomorphic to $\mathrm{Vect}(\mathbb{S}^1) \ltimes C^\infty(\mathbb{S}^1)^{\oplus n-1}$.

- Consequently the most general bilinear forms are given by

$$(g_0)_{ij} = g_{ij}, \qquad (f_0)_{ij} = \delta_j^n f_i - \delta_i^n f_j, \qquad (h_0)_{ij} = \delta_i^n \delta_j^n h$$

and

$$(g_1)_{ij} = \alpha_{ij}, \qquad (f_1)_{ij} = \delta_j^n \gamma_i - \delta_i^n \gamma_j, \qquad (h_1)_{ij} = \delta_i^n \delta_j^n \beta.$$

- The element $c = (c^i) \in \mathbb{A}_n$ is the right unity iff $c^n = 0$.
- For $v = (v^i) \in \mathscr{L}_{\mathbb{A}_n}$ we find:

$$i \neq n: \qquad g_{ij} v_t^j + f_i v_{xt}^n = \left( g_{ij} v^j v^n + f_i v^n v_x^n + \alpha_{ij} v^j + \gamma_i v_x^n \right)_x,$$

$$i = n: \qquad g_{nj} v_t^j - f_j v_{xt}^j + h v_{xxt}^n = \Big( g_{nj} v^j v^n + \frac{1}{2} g_{jk} v^j v^k - f_j v_x^j v^n + \frac{1}{2} h (v_x^n)^2 + h v^n v_{xx}^n + \alpha_{nj} v^j - \gamma_j v^j + \beta v_{xx}^n \Big)_x.$$

- Various examples of 2-component Camassa-Holm equations that have appeared already in the literature fall into this scheme by identifying the underlying Novikov algebras and bilinear forms.

- Particularly for $(N4) = \mathbb{A}_2$:

  (i) Ito equation and its Camassa-holm type extension by Guha & Olver

  (ii) Dispersive water waves (DWW) by Kupershmidt and Kaup-Broer system

  (iii) 2-component Camassa-Holm equation derived by Chen & Liu & Zhang

- Taking the dispersionless limit we obtain in coordinates $u$ the following equations of hydrodynamic type:

$$u_{t_1} = R_c^* u_x \,,$$
$$u_{t_2} = (R^*_{R_c\Lambda^{-1}u} u)_x + L^*_{\Lambda^{-1}u_x} R_c^* u + g_1 R_c \Lambda^{-1} u_x \,, \qquad (1)$$

where $\Lambda \equiv g_0$.

- For all the explicit examples of considered Novikov algebras and bilinear forms the associated Haantjes tensor vanishes.
- But only those systems associated to $(N4)$, $(C8)$ and $\mathbb{A}_n$ are hyperbolic and thus are diagonalisable.

# Thank you!

More details at **arXiv:1309.3188 [nlin.SI]**