Higher holonomy

Representations up to homotopy

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Прежде всего, я хотел бы поблагодарить организаторов этой конференции, вие чест Александер Михайловича. Я рад встретиться с русскими и украинскими математиками и физиками. Я надеюсь, что в будущем многие из вас смогут посетить Филаделфиа.

Каждый любит язык своей страны.
Поэтому я буду говорить по-английски.

Introduction

Chen's work on the “de Rham” cohomology of loop spaces on a manifold was a breakthrough; he

delete some of the following

> *introduced a series of elegant new ideas. A ”differentiable space” generalizes manifolds (the loop space turns out to be a differentiable space). A calculus of exterior forms on differentiable spaces is invented, and it is shown that iterated integrals on a manifold X lead to forms on the loop space. This gives the required exterior calculus. A pairing is constructed of this iterated integral complex and the cobar construction of a smooth singular chain complex*

A few years later, Vinogradov in his introduction of the secondary calculus and diffeology provided a broader context for pursuing differential geometry in an infinite dimensional setting. In the context of diffeology (Vinogradov) or differentiable spaces (Chen), not only is there a de Rham theory, but also an analog of classical differential geometry ("a language the muse did not sing at my cradle"). In particular, classical *parallel transport* is defined in the context of a *connection* on a smooth vector bundle $p : E \to M$. *Connection* can refer to a differential 1-form or a set of horizontal subspaces in the tangent bundle $Tp : TE \to TM$[1]. Let $PM = M^I$, the space of (piecewise smooth) paths in M. The corresponding *parallel transport* $\tau : PM \times_M E \to E$ is constructed by lifting a path $\lambda : I \to M$ in M to a horizontal path in E, *unique!* if a starting point is specified.

[1] Also related to covariant derivatives, but which play no explicit role in our treatment.

Remark

There is great ambiguity in the literature concerning the terms

- *connection*
- *connection form*
- *parallel transport*
- *holonomy*
- *monodromy*
- *covariant derivative.*

There is a correspondence between these objects, so one of them is often referred to by one of the other names. We have tried to adopt a consistent nomenclature. For this talk, connection will usually mean connection form.

Remark

Similarly, there is a great variety of relevant exposition evolving over time. Of particular relevance today:

- *Adams*
- *Chen*
- *Igusa*
- *Block and Smith.*

According to n-lab:

> *Apparently one of the oldest occurrences of the idea that a principal bundle with connection over a connected base space may be reconstructed from its holonomies around all smooth loops (for any fixed base point)*

appeared in or was implied by Kobayashi in 1954.

Established for principal bundles, there is a corresponding result for associated vector bundles with typical fiber V.

For vector bundles with flat connection on M, there is an equivalence between such bundles and $Reps(\pi_1(M))$, where $Reps(\pi_1(M))$ denotes the space of representations $\pi_1(M) \to GL(V)$.

Regarding τ as homotopy lifting (rather than just path lifting), we have $\tau : I \times V \to E$ so that

$$\begin{array}{ccc} I \times V & \xrightarrow{\tau} & E \\ \downarrow & & \downarrow p \\ I & \xrightarrow{\lambda} & M \end{array}$$

is commutative.

The *holonomy* with respect to a curve is given by the evaluation of τ on the path in M.

The *holonomy group* is the image $\tau_* : \pi_1(M) \to GL(V)$ as a subgroup of the structure group of the bundle.

That the image is a group follows from the uniqueness of the lifting. It is well defined up to conjugation depending on the choice of base point.

The holonomy descends to a representation of $\pi_1(M)$ as a result of the flatness.

Going from a representation $\pi_1(M) \to GL(V)$ to a bundle with flat connection, is achieved by the associated $GL(V)$-bundle construction.

As classical holonomy is given by the evaluation of the lifting τ, consider *higher holonomy* by lifting $\sigma : \Delta^n \to M$ to $\tau^n : \Delta^n \times V \to E$, but now with $E \to M$ a graded vector bundle with fiber V, a graded vector space.

The higher holonomy corresponds to a *generalized connection* as a set of forms of total degree 0 on the path space $PM = M^I$ of M

Consider a graded vector bundle $p : E = \pi E^k \to M$.

Let $End^p(E)$ denote the degree p part of the endomorphism bundle of E:

$$End^p(E) = \prod Hom(E^k, E^{k+p}).$$

A $\mathbb{Z}$-graded connection *form* is just the analog of a classical connection form, but with careful attention to grading and signs.

As such, it corresponds to (a family of) differential forms with values in $End(E)$.

Let $\Omega^\bullet(M)$ be the graded algebra of smooth differential forms on M and let $\Omega^\bullet(M; E)$ be the graded $\Omega^\bullet(M)$-module of forms with values in $End(E)$. As usual, it is useful to describe graded connections locally, so the appropriate covariant derivative can be written: $d + A$, where $A = A_1 + A_2 + \cdots$ with $A_p \in \Omega^p(M; E^{1-p})$

Need to check that degree and literature more thoroughly

What then corresponds to 'higher holonomy'?

The idea of higher holonomy was introduced by Chen as *generalized holonomy* in conjunction with his technology of iterated integrals. It was developed further elaborated by Igusa [**?**] and later was related to a notion of *representation up to homotopy*.

To capture such 'higher structure', Chen used maps of simplices $\sigma : \Delta^k \to M$. Denote the standard ordered n-simplex Δ^n as $< 0, 1, \cdots, n >$ with vertices labelled $0, 1, \cdots, n$. Sub-simplices are denoted $< i_0, i_1, \cdots, i_j >$. The face and degeneracy maps for a simplicial set are:

$$\partial_q < 0, 1, \cdots, n > = < 0, 1, \cdots, q-1, q+1, \cdots, n >$$
$$s_q < 0, 1, \cdots, n > = < 0, 1, \cdots, q, q, \cdots, n >$$

Definition

Sing(M) *is the set of (smooth) maps of simplices* $\sigma : \Delta^k \to M$
For $\sigma : \Delta^k \to M$*, we denote by* V_i *the fibre over the image of the vertex* $i \in \Delta$*.*

Definition

A representation up to homotopy of Sing(M) on a graded vector space V is a collection of maps $\{\theta_k\}_{k\geq 0}$ which assign to any k-simplex $\sigma : \Delta^k \to M$ a map $\theta_k(\sigma) : I^{k-1} \times V_0 \to V_k$ satisfying, for any $e \in V_0$, the relations:

θ_0 is the identity on V_0

$\theta_k(\sigma)(t_1, \cdots, t_{k-1}, -) : V_0 \to V_k$ is an isomorphism for any $(t_1, \cdots, t_{k-1})$.

For any $1 \leq p \leq k-1$ and $e \in V_0$,

$$\theta_k(\sigma)(\cdots, t_p = 0, \cdots, e) = \theta_{k-1}(\partial_p\sigma)(\cdots, \hat{t}_p, \cdots, , e)$$

$$\theta_k(\sigma)(\cdots, t_p = 1, \cdots, e) =$$

$$\theta_p(< 0, \cdots, p >)(t_1, \cdots, t_{p-1}, \theta_q(< p, \cdots, k >)(t_{p+1}, \cdots, t_k, e)).$$

There is a similar but earlier use of that name [**?**] and there are also *homotopy coherent representations*.

Notice the 'cubical' nature of the condition, similar to Sugawara's *strong homotopy multiplicative* for A_∞-maps between associative H-spaces..

In this setting, *coherence* refers to the compatibility of the θ_n with respect to the boundary of the cubes and simplices.

Alternatively, in terms of forms, write for every $t \in I$, the evaluation map

$$ev_t : PM \to M$$

sending $\mathfrak{g}$ to $\mathfrak{g}(t)$. Let W_t be the pull back of E to PM along ev_t. That is,

$$(W_t)_{\mathfrak{g}} = V_{\mathfrak{g}(t)}.$$

For $0 \leq s \leq t \leq 1$, let

$$Hom^q(W_s, W_t)$$

be the space of degree q graded homomorphisms from W_s to W_t. Define a smooth bundle

$$\Omega^p(PM, Hom^q(W_s, W_t)) = \Omega^p(PM) \otimes Hom^q(W_s, W_t)$$

whose fiber over $\mathfrak{g}$ is the vector space of smooth p-forms with coefficients in

$$Hom^q(W_s, W_t)_{\mathfrak{g}} = Hom^q(V_{\mathfrak{g}(s)}, V_{\mathfrak{g}(t)}).$$

Definition

Given a graded vector bundle $p : E \to M$, a homotopy coherent representation on E of the smooth singular simplicial set $Sing(M)$ of M consists of a family of forms $\Psi_p(s,t)$ for all $0 \leq s \leq t \leq 1$

$$\Psi_p(s,t) \in \Omega^p(PM, Hom^{-p}(W_s, W_t))$$

satisfying the following at each $\mathfrak{g} \in PM$:

1. $\Psi_0(s,s)_{\mathfrak{g}}$ *is the identity map in*
$$Hom^0(V_{\mathfrak{g}(s)}, V_{\mathfrak{g}(s)})$$
2. $\Phi(s,t)_{\mathfrak{g}}$ *satisfies a certain first order linear differential equation.*

The differential equation determines $\Psi_p(s,t)$ uniquely as a p-form on PM. Starting with $\Psi_0(s,t)$, one can find $\Psi_p(s,t)$ by induction on p using a version of Chen's iterated integrals.

Definition

A $\mathbb{Z}$-connection form A on a graded vector bundle $p : E \to M$ is a form of total degree 1, i.e. in $\pi\Omega^p(M; End^{1-p}(E))$.

Theorem

Given a graded vector bundle $p : E \to M$ with typical fiber V and a flat $\mathbb{Z}$-connection form A,

there is a homotopy coherent representation on V of the smooth singular simplicial set $Sing(M)$ of M

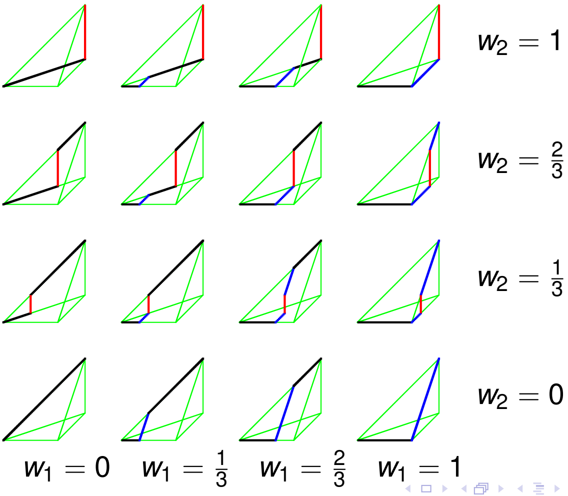
A crucial piece of structure is a set of *coherent* maps

$$\gamma_n : I^{n-1} \to P\Delta^n.$$

These involve realizing any simplex as a family of paths with fixed endpoints 0 and n.

Such maps were first produced by Adams [**?**] in the topological context by induction using the contractability of Δ^n. Later specific formulas were introduced by Chen [**?**] and later equivalently but more transparently, by Igusa [**?**].

Figure: After Igusa



In a variant of the classical Riemann–Hilbert equivalence, a map

$$Flat(M) \to Reps(\pi_1(M))$$

is developed by calculating the holonomy with respect to a flat connection. The holonomy descends to a representation of $\pi_1(M)$ as a result of the flatness.

The functor

$$Reps(\pi_1(M)) \to Flat(M),$$

can be achieved by the associated bundle construction for the structure group $GL(V)$.

more Block and Smith

Block and Smith remark:

> *It would be an interesting problem in its own right to define an inverse functor which makes use of a kind of associated bundle construction.*

This is exactly what we are working on now.