Metric Invariants of Spherical Harmonics

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September 22, 2021
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Lie group $\textbf{SO}(3)$ is obvious symmetry group of these equations and all $\mathcal{E}^{(i)}$ are affine algebraic manifolds equipped with the algebraic $\textbf{SO}(3)$ - action.
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We say that a rational $\text{SO}(3)$-invariant function on affine manifold $E_i$ is a differential metric invariant of spheric harmonics, having order $i$. The field of differential invariants we denote by $\mathbb{F}_d^k$. 

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2 The Lie algebra $\mathfrak{sl}(2) \subset A_3$, generated by the following operators

$$X_+ = \frac{r^2}{2}, \quad H = \delta + \frac{3}{2}, \quad X_- = \frac{\Delta}{2},$$

where

$$r^2 = x^2 + y^2 + z^2, \quad \delta = x\partial_x + y\partial_y + z\partial_z, \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2,$$

and operators $(X_+, H, X_-)$ form the Weyl basis in $\mathfrak{sl}(2)$ :

$$[H, X_+] = 2X_+, \quad [H, X_-] = -2X_-, \quad [X_-, X_+] = H.$$
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The Lie algebra \( \mathfrak{so}(3) \subset A_3 \) generated by the angular momentum operators

\[
L_z = x\partial_y - y\partial_x, \quad L_y = x\partial_z - z\partial_x, \quad L_x = y\partial_z - z\partial_y.
\]
These Lie algebras mutually commute and the *universal enveloping algebra* $U(\mathfrak{sl}(2)) \subset A_3$ is the subalgebra of $so(3)$-invariant operators in $A_3$. 

Casimir operator in Lie algebra $so(3)$ is the orbital angular momentum operator $\mathcal{M} = L_x^2 + L_y^2 + L_z^2$ and it coincides with Casimir operator in Lie algebra $\mathfrak{sl}(2)$: $\mathcal{M} = r^2 \Delta^2 \delta^2$. 

Operator $\mathcal{M}$ is also the spherical Laplace operator.
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Operator $M$ is also the spherical Laplace operator.
The following sequence

\[ 0 \rightarrow \mathbb{H}_k \rightarrow \mathbb{P}_k \xrightarrow{\Delta} \mathbb{P}_{k-2} \rightarrow 0 \]

is exact, and \( \dim \mathbb{H}_k = 2k + 1 \).
Harmonic polynomials

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2. **Splitting** \( \mathbb{P}_k \): for any homogeneous polynomial \( p_k \in \mathbb{P}_k \) there are (and unique) spheric harmonics \( h_{k-2i} \in \mathbb{H}_{k-2i}, 0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \), such that

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p = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} r^{2i} h_{k-2i}.
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4. The restriction of spheric harmonics on the unit sphere \( S^2 \subset \mathbb{R}^3 \) are eigenfunctions of the spherical laplacian \( \Delta_S \) with eigenvalues \(-k(k+1)\) and any continuous function on \( S^2 \) could be approximated (with any accuracy) by linear combination of spherical harmonics.
Harmonic projections $\eta_{k,2i} : \mathbb{P}_k \to \mathbb{H}_{k-2i}$ are the following

$$\eta_{k,2i} = r^{-2i} Q_{k,2i}(M),$$

where

$$Q_{k,2i}(\lambda) = \prod_{j \neq i} \frac{\lambda - \lambda_j}{\lambda_i - \lambda_j}, \quad \lambda_i = - (k - 2i)(k - 2i + 1).$$
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3. Define product of spheric harmonics \( h_k \in \mathbb{H}_k, h_l \in \mathbb{H}_l \) as follows

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h_k \ast h_l = \eta_{k+l,0}(h_k h_l) \in \mathbb{H}_{k+l}.
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Here

$$\eta_{k+l,0} = \prod_{j=1}^{[k+l/2]} \frac{M + (k + l - 2j)(k + l - 2j + 1)}{2j (2j - 2k - 2l - 1)}.$$
Spherical harmonics

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Algebra \((H_*, *)\) generated by linear functions \((x, y, z)\) satisfying the relation

\[ x * x + y * y + z * z = 0. \]
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Algebra $(\mathbb{H}_*, \ast)$ generated by linear functions $(x, y, z)$ satisfying the relation

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The complixification $\mathbb{H}_* = \mathbb{H} \otimes \mathbb{C}$ is the algebra of regular functions on the null cone $\{x^2 + y^2 + z^2 = 0\}$ in $\mathbb{C}^3$. 
Algebra of spherical harmonics

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4. Example.

\[ x \ast x = xx - \frac{r^2}{3}, \quad x \ast y = xy. \]
The space $\mathbb{H}_k$ of spherical harmonics is a vector space of dimension $2k + 1$. The Lie group $\text{SO}(3)$ acts in algebraic way on $\mathbb{H}_k$, and in $\mathbb{H}_k$ are realized all irreducible representations of $\text{SO}(3)$.
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Due to Hilbert theorem polynomial invariants of this action (i.e. polynomial invariants of spherical harmonics) form a finite generated commutative algebra.

Due to Rosenlicht theorem rational invariants of this action (i.e. rational invariants of spherical harmonics) form a field of transcendence degree equals the codimension of regular orbit.

Regular orbit has codimension $(2k - 2)$, when $k > 1$, and codimension 1, when $k = 1$. Therefore, in order to define a regular orbit we need $2k$ algebraically independent rational invariants, for $k > 1$, and only one invariant, for $k = 1$. 

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Equations $\mathcal{E}^{(i)}$ are affine manifolds of dimension $2i + 4$, if $2 \leq i < k$. The regular $\text{SO}(3)$-orbits (that correspond to smooth points of quotient $\mathcal{E}^{(i)}/\text{SO}(3)$) Thus, due to Hilbert theorem, the quotients are affine manifolds of dimension $2i + 1$. Rational differential invariants of order $\leq i$ are rational functions on $\mathcal{E}^{(i)}/\text{SO}(3)$ and therefore the transcendence degree of field $\mathcal{F}_i^d$ equals to $2i + 1$. 
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As we have seen, the transcendence degree of field $\mathcal{F}^{a}_k$ equals $2(k - 1)$.

Take a regular harmonic $h \in \mathbb{H}_k$. Then it is easy to check that the $\text{SO}(3)$--orbit of the 2-jet $j_2(h)$ into $\mathcal{E}^{(2)}$ is a 6-dimensional submanifold into 8-dimensional manifold $\mathcal{E}^{(2)}$ and therefore we need 2 differential invariants of order 2 to describe the orbit (compare with $2(k - 1)$ algebraic invariants).
Invariant coframe

Total differentials of the obvious invariants \( J_{-1} = \frac{r^2}{2} \) and \( J_0 = u \) give us two \( \text{SO}(3) \)-invariant horizontal 1–forms:

\[
\omega_1 = x dx + y dy + z dz,
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\omega_2 = u_x dx + u_y dy + u_z dz.
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3. Then coframe $(\omega_1, \omega_2, \omega_3)$ is $\textbf{SO}(3)$-invariant.
Invariant frame

\begin{align*}
D_1 &= x \frac{d}{dx} + y \frac{d}{dy} + z \frac{d}{dz}, \\
D_2 &= u_x \frac{d}{dx} + u_y \frac{d}{dy} + u_z \frac{d}{dz}, \\
D_3 &= (yu_z - zu_y) \frac{d}{dx} + (zu_x - xu_z) \frac{d}{dy} + (xu_y - yu_x) \frac{d}{dz}.
\end{align*}
First invariants

\[ J_{-1} = \frac{r^2}{2}, \quad J_0 = u, \]
\[ J_1 = D_2(J_0) = u_x^2 + u_y^2 + u_z^2, \]
\[ J_{21} = \frac{D_2(J_1)}{2} = u_x^2 u_{xx} + u_y^2 u_{yy} + u_z^2 u_{zz} + 
2 \left( u_x u_y u_{xy} + u_x u_z u_{xz} + u_y u_z u_{yz} \right). \]
Invariant symmetric forms and operators

1. Symmetric differential $i$-forms

\[ \Theta_i = \sum_{i_1 + i_2 + i_3 = i} u_{i_1, i_2, i_3} \frac{dx^{i_1}}{i_1!} \cdot \frac{dy^{i_2}}{i_2!} \cdot \frac{dz^{i_3}}{i_3!} \]

are invariants with respect to Lie group of affine transformations in $\mathbb{R}^3$. 
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are invariants with respect to Lie group of affine transformations in $\mathbb{R}^3$.

2. Differential operators

$$\hat{\Theta}_i = \sum_{i_1+i_2+i_3=i} u_{i_1,i_2,i_3} \frac{d^k}{i_1!i_2!i_3!} \frac{dx^{i_1}}{i_1!} \frac{dy^{i_2}}{i_2!} \frac{dz^{i_3}}{i_3!}$$

are $SO(3)$-invariant.
**Invariants**

Let

\[ dx = t_{11} \omega_1 + t_{12} \omega_2 + t_{13} \omega_3, \]
\[ dy = t_{21} \omega_1 + t_{22} \omega_2 + t_{23} \omega_3, \]
\[ dz = t_{31} \omega_1 + t_{32} \omega_2 + t_{33} \omega_3, \]

where \( t_{ij} \) are rational functions on \( J^1 (\mathbb{R}^3) \), and let

\[ \Theta_i = \sum_{i_1 + i_2 + i_3 = i} T_{i_1, i_2, i_3} \frac{\omega_1^{i_1}}{i_1!} \cdot \frac{\omega_2^{i_2}}{i_2!} \cdot \frac{\omega_3^{i_3}}{i_3!}. \]

**Theorem**

Functions \( T_{i_1, i_2, i_3} \) are rational differential \( \text{SO}(3) \)-invariants of order \( i = i_1 + i_2 + i_3 \) and any rational differential \( \text{SO}(3) \)-invariants of order \( i \) is a rational function of them.
Remark that invariants

\[ G_i = \Theta_i(u) = \sum_{i_1 + i_2 + i_3 = i} \frac{u_{i_1,i_2,i_3}^2}{i_1! i_2! i_3!} \]

are squares of lengths of symmetric forms \( \Theta_i \).

Thus,

\[ \Theta_1 = u_x \frac{d}{dx} + u_y \frac{d}{dy} + u_z \frac{d}{dz}, \]
\[ \Theta_2 = \frac{1}{2} \left( u_{xx} \frac{d^2}{dx^2} + u_{yy} \frac{d^2}{dy^2} + u_{zz} \frac{d^2}{dz^2} \right) + u_{xy} \frac{d^2}{dxdy} + u_{xz} \frac{d^2}{dxdz} + u_{yz} \frac{d^2}{dydz} \]

and

\[ \Theta_1(u) = u_x^2 + u_y^2 + u_z^2, \]
\[ \Theta_1(u) = J_{22} = \frac{u_{xx}^2 + u_{yy}^2 + u_{zz}^2}{2} + u_{xy}^2 + u_{xz}^2 + u_{yz}^2. \]
Theorem

The field of rational differential $\textbf{SO}(3)$-invariants of spherical harmonics is generated by invariants $(J_{-1} = \frac{r^2}{2}, J_0 = u, J_{22})$ and derivation $\nabla = \hat{\Theta}_1$. 
Monoid of invariants

\[ \text{SO}(3) \text{ -- Invariants} \iff \text{SO}(3) \text{ -- invariant differential operators:} \]

\[ \phi \in C^\infty \left( \mathbf{J}^k \left( \mathbb{R}^3 \right) \right) \iff \Delta_\phi : C^\infty \left( \mathbb{R}^3 \right) \to C^\infty \left( \mathbb{R}^3 \right) , \]

\[ \Delta_\phi \left( f \right) = j_k \left( f \right)^* \left( \phi \right) . \]
Monoid of invariants

1. $\text{SO}(3)$—Invariants $\iff \text{SO}(3)$—invariant differential operators:

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$$\Delta_\phi (f) = j_k (f)^* (\phi).$$

2. Monoid structure on $\text{SO}(3)$—invariants defines by the composition of invariant operators, and $\text{id} = u$. 

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2. Monoid structure on \( \operatorname{SO}(3) \) — invariants defines by the composition of invariant operators, and \( \text{id} = u \).

3. Thus, the field \( \mathcal{F}_k \) is the monoid.
Let

\[ W = x \partial_x + y \partial_y + z \partial_z + ku \partial_u, \]

and let \( W^* \) be its \( \infty \)-prolongation.
Let

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and let \( W^* \) be its ∞-prolongation.

We say that a polynomial differential invariant \( I \) has weight \( w(I) \) if

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In other words, if \( h \) is a homogeneous polynomial of degree \( k \) then \( I(h) \) has degree \( w(I) \).
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2. Let $I$ be a polynomial differential invariant of weight $w$, and $h \in \mathbb{H}_k$. Then $I(h) = P w$, $(\eta_w, 2 l \Delta I)(h) = P w$ and its length $\Delta G w^{2 l} \eta_w, 2 l \Delta I$ is a scalar, i.e. invariant $G w^{2 l} \eta_w, 2 l \Delta I$ is an algebraic invariant.
1. Algebraic invariants on $\mathbb{H}_k$ are differential invariants of order $k$.
2. Let $I$ be a polynomial differential invariant of weight $w$, and $h \in \mathbb{H}_k$.
3. Then $I(h) \in \mathbb{P}_w$, $(\eta_{w,2l} \circ \Delta_l)(h) \in \mathbb{H}_{w-2l}$ and its length $(\Delta_{G_{w-2l}} \circ \eta_{w,2l} \circ \Delta_l)(h)$ is a scalar, i.e invariant $G_{w-2l} \circ \eta_{w,2l} \circ I$ is an algebraic invariant.