# Algebraic quantum Hamiltonians on the plane

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# 1. Introduction

Consider integrable Hamiltonians

$$H = \Delta + U(x_1, ..., x_n), \qquad \text{where} \qquad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{1}$$

related to simple Lie algebras. For such Hamiltonians the potential $U$ is a rational, trigonometric or elliptic function. For instance, the elliptic Calogero-Moser Hamiltonian is given by

$$H = \Delta + g \sum_{i>j} \wp(x_i - x_j),$$

where $g$ is arbitrary constant.

**Observation 1** (A.Turbiner). For many of these Hamiltonians there exists a change of variables and a gauge transformation that bring the Hamiltonian to a differential operator with polynomial coefficients.

**Example.** Consider the Calogero model with $n = 3$:

$$H = \Delta + g \sum_{i>j}^{3} \frac{1}{(x_i - x_j)^2}.$$

Let $Y = \sum_{i=1}^{3} x_i$ and $y_i = x_i - \frac{Y}{3}$. Then

$$\Delta = -3\frac{\partial^2}{\partial Y^2} - \frac{2}{3}\Big(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2}\Big).$$

Thus we have reduced the Hamiltonian to the following two dimensional one:

$$\mathcal{H} = -\frac{1}{3}\Big(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2}\Big) + \nu(\nu - 1) \sum_{i>j}^{3} \frac{1}{(y_i - y_j)^2}. \quad (2)$$

Here $y_3 = -y_1 - y_2$.

The change of variables

$$x = -y_1^2 - y_2^2 - y_1 y_2, \qquad y = -y_1 y_2 (y_1 + y_2)$$

and the gauge transformation $\mathcal{H} \to h^{-1} \mathcal{H} h$, where

$$h = (x-y)^\nu (2x+y)^\nu (x+2y)^\nu,$$

bring $\mathcal{H}$ to the polynomial form

$$L = x \frac{\partial^2}{\partial x^2} + 3y \frac{\partial^2}{\partial x \partial y} - \frac{1}{3} x^2 \frac{\partial^2}{\partial y^2} + (1+3\nu) \frac{\partial}{\partial x}. \quad \square$$

In the trigonometric case the transformation to a polynomial form is given by

$$x = \cos y_1 + \cos y_2 + \cos(y_1 + y_2) - 3,$$

$$y = \sin y_1 + \sin y_2 - \sin(y_1 + y_2).$$

Recently (**A.Turbiner, VS**) the transformation

$$x = \frac{\wp'(y_1) - \wp'(y_2)}{\wp(y_1)\wp'(y_2) - \wp(y_2)\wp'(y_1)}, \quad y = \frac{\wp(y_1) - \wp(y_2)}{\wp(y_1)\wp'(y_2) - \wp(y_2)\wp'(y_1)}$$

that brings the elliptic Calogero-Moser Hamiltonian to a polynomial form has been found. The above rational and trigonometric transformations are its degenerations. $\square$

**Conjecture (M. Matushko)**. The analog of the above transformation for arbitrary $n$ is given by the solution of the linear system

$$M\mathbf{u} = \mathbf{e},$$

where $\mathbf{u} = (u_1, \ldots, u_n)^t$, $\mathbf{e} = (1, 1, \ldots, 1)^t$ with

$$M^i_j = \frac{d^{j-1}\wp(y_i)}{dy_i^{j-1}} \ .$$

This formala gives correct transformations in the cases $n = 1, 2, 3$.

Obviously, for any polynomial form $P$ of Hamiltonian (1)

**1:** the contravariant metric g defining by the symbol of P is flat

and

**2:** $P$ can be reduced to a self-adjoint operator by a gauge transformation $P \to fPf^{-1}$, where $f$ is a function.

Besides evident properties 1,2 we have in mind the following non-trivial

**Observation 2.** (A. Turbiner). For all known cases

**3:** $P$ preserves some nontrivial finite - dimensional vector space $V$ of polynomials.

In the most interesting case the vector space V coincides with the space $V_k$ of all polynomials of degrees $\leqslant k$ for some k.

# 1. ODE elliptic case.

Let

$$Q = \sum_{i=0}^{m} a_i(x) \frac{d^i}{dx^i}$$

be an ordinary differential operator of degree $m$ with polynomial coefficients.

**Conjecture**. The vector space $V_k$ of polynomials of degree $\leqslant k$, where $k \geqslant m$, is invariant with respect to $Q$ iff $Q$ is a polynomial in generators

$$J_1 = 1, \qquad J_2 = \frac{d}{dx}, \qquad J_3 = x\frac{d}{dx}, \qquad J_4 = x^2\frac{d}{dx} - kx.$$

The Lie algebra generated by $J_1, ..., J_4$ is isomorphic to $gl(2)$. □

**Remark**. Consider operators $D = T - 1$ and $X = xT^{-1}$, where $T(f(x)) = f(x+1)$. Then $[D, X] = 1$.

**Lemma.** If $Q$ preserves $V_k$, где $k \geqslant m$, then $\deg a_i \leqslant m + i$.

In particular, any such operator $P$ of second order has the following structure:

$$P = (a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0) \frac{d^2}{dx^2} +$$

$$(b_3 x^3 + b_2 x^2 + b_1 x + b_0) \frac{d}{dx} + c_2 x^2 + c_1 x + c_0,$$

where the coefficients are related by the following identities

$$b_3 = 2(1-k)\, a_4, \qquad c_2 = k(k-1)\, a_4, \qquad c_1 = k(a_3 - k a_3 - b_2).$$

The transformation group

$$x \to \frac{s_1 x + s_2}{s_3 x + s_4}, \qquad P \to (s_3 x + s_4)^{-k} P (s_3 x + s_4)^k, \tag{3}$$

acts on the nine-dimensional vector space of such operators. The coefficient $a(x)$ at the second derivative is a fourth order polynomial which transforms as follows

$$a(x) \to (s_3 x + s_4)^4 a\left(\frac{c_1 x + c_2}{c_3 x + c_4}\right).$$

If $a(x)$ has four distinct roots, we call the operator $P$ *elliptic*. In the elliptic case using transformations (3), we may reduce $a$ to

$$a(x) = 4\,x(x-1)(x-\kappa).$$

Define parameters $n_1, ..., n_5$ by identities

$$b_0 = 2(1+2n_1), \quad b_1 = -4\Big((\kappa+1)(n_1+1) + \kappa n_2 + n_3\Big),$$

$$b_2 = -2\,(3+2n_1+2n_2+2n_3),$$

$$k = -\frac{1}{2}(n_1+n_2+n_3+n_4),$$

$$n_5 = c_0 + n_2(1-n_2) + \kappa n_3(1-n_3) + (n_1+n_3)^2 + \kappa(n_1+n_2)^2.$$

Then the operator $H = hPh^{-1}$, where

$$h = x^{\frac{n_1}{2}}(x-1)^{\frac{n_2}{2}}(x-\kappa)^{\frac{n_3}{2}}$$

has the form

$$H = a(x)\frac{d^2}{dx^2} + \frac{a'(x)}{2}\frac{d}{dx} + n_5 + n_4(1-n_4)\,x + \frac{n_1(1-n_1)\kappa}{x} +$$

$$\frac{n_2(1-n_2)(1-\kappa)}{x-1} + \frac{n_3(1-n_3)\kappa(\kappa-1)}{x-\kappa}.$$

Now after the transformation $y = f(x)$, where

$$f'^2 = 4f(f-1)(f-\kappa)$$

we arrive at

$$H = \frac{d^2}{dy^2} + n_5 + n_4(1-n_4)\,f + \frac{n_1(1-n_1)\kappa}{f} +$$

$$\frac{n_2(1-n_2)(1-\kappa)}{f-1} + \frac{n_3(1-n_3)\kappa(\kappa-1)}{f-\kappa}.$$

In general here $n_i$ are arbitrary parameters.

When

$$k = -\frac{1}{2}(n_1 + n_2 + n_3 + n_4)$$

is a natural number, the operator $H$ preserves the finite-dimensional polynomial vector space $V_k$.

Another form of this Hamiltonian (up to a constant) is given by

$$H = \frac{d^2}{dy^2} + n_4(1-n_4)\,\wp(y) + n_1(1-n_1)\,\wp(y+\omega_1) +$$

$$n_2(1-n_2)\,\wp(y+\omega_2) + n_3(1-n_3)\,\wp(y+\omega_1+\omega_2),$$

where $\omega_i$ are half-periods of the Weierstrass function $\wp(x)$. If $n_1 = n_2 = n_3 = 0$ we get the Lame operator. In general, it is the Darboux-Treibich-Verdier operator.

## 2. Two-dimensional case.

Consider second order differential operators

$$L = a(x,y)\frac{\partial^2}{\partial x^2} + 2b(x,y)\frac{\partial^2}{\partial x \partial y} + c(x,y)\frac{\partial^2}{\partial y^2} + d(x,y)\frac{\partial}{\partial x} + e(x,y)\frac{\partial}{\partial y} + f(x,y) \tag{4}$$

with polynomial coefficients. Denote by $D(x,y)$ the determinant $a(x,y)c(x,y) - b(x,y)^2$. We assume that $D \neq 0$.

The operators we are interested in should possess three important properties:

**Property 1.** We assume that the associated contravariant metric

$$g^{1,1} = a\ ,\ g^{1,2} = g^{2,1} = b\ ,\ g^{2,2} = c\ ,$$

is flat or $R_{1,2,1,2} = 0$.

This is equivalent to

$$2\Big(b^2 a_{xx} - 2abb_{xx} + a^2 c_{xx} + 2bca_{xy} - 2(b^2 + ac)b_{xy} + 2abc_{xy} +$$

$$+c^2 a_{yy} - 2bcb_{yy} + b^2 c_{yy}\Big) \times D + \text{first order terms} = 0.$$

**Example 1.** For any constant $\kappa$ the metric $g$ with

$$\begin{aligned} a &= (x^2 - 1)(x^2 - \kappa) + (x^2 + \kappa)\, y^2, \\ b &= xy\,(x^2 + y^2 + 1 - 2\kappa), \\ c &= (\kappa - 1)(x^2 - 1) + (x^2 + 2 - \kappa)\, y^2 + y^4 \end{aligned} \tag{5}$$

is flat.

In this case we have

$$D = (y^2 + x^2 + 2x + 1)(y^2 + x^2 - 2x + 1)\Big(\kappa y^2 + (\kappa - 1)x^2 + \kappa(1 - \kappa)\Big).$$

**Property 2.** The operator should be potential. This means that

$$\frac{\partial}{\partial y}\Big(\frac{be - cd + c(a_x + b_y) - b(b_x + c_y)}{D}\Big) \tag{6}$$

$$= \frac{\partial}{\partial x}\Big(\frac{bd - ae + a(b_x + c_y) - b(a_x + b_y)}{D}\Big).$$

The properties 1 and 2 guaranty that $L$ can be reduced to the form

$$\bar{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + V(x, y)$$

by a proper change of variables and by a guage transform.

**Observation 2.** (A. Turbiner). Known polynomial forms for the Calogero-Moser type Hamiltonians preserve some finite - dimensional vector spaces of polynomials.

In this talk we consider operators (4) with polynomial coefficients that satisfy the following condition:

**Property 3.** The operator has to preserve the vector space $V_n$ of all polynomials $P(x, y)$ such that $\deg P \leqslant n$ for some $n > 2$.

**Proposition.** The operator $L$ satisfies Property 3 iff the coefficients of $L$ have the following structure

$$a = q_1x^4 + q_2x^3y + q_3x^2y^2 + k_1x^3 + k_2x^2y + k_3xy^2 +$$

$$a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6;$$

$$b = q_1x^3y + q_2x^2y^2 + q_3xy^3 + \frac{1}{2}\Big(k_4x^3 + (k_1 + k_5)x^2y + (k_2 + k_6)xy^2 + k_3y^3\Big)$$

$$+b_1x^2 + b_2xy + b_3y^2 + b_4x + b_5y + b_6;$$

$$c = q_1x^2y^2 + q_2xy^3 + q_3y^4 + k_4x^2y + k_5xy^2 + k_6y^3 +$$

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6;$$

$$d=(1-n)\Big(2(q_1x^3+q_2x^2y+q_3xy^2)+k_7x^2+(k_2+k_8-k_6)xy+k_3y^2\Big)+$$

$$d_1x+d_2y+d_3;$$

$$e=(1-n)\Big(2(q_1x^2y+q_2xy^2+q_3y^3)+k_4x^2+(k_5+k_7-k_1)xy+k_8y^2\Big)+$$

$$e_1x+e_2y+e_3;$$

$$f=n(n-1)\Big(q_1x^2+q_2xy+q_3y^2+(k_7-k_1)x+(k_8-k_6)y\Big)+f_1.$$

The dimension of the space of such operators equals 36. The group $GL_3$ acts on this vector space by the formula

$$\tilde{x}=\frac{P}{R},\qquad \tilde{y}=\frac{Q}{R},\qquad \tilde{L}=R^{-n}LR^{n},$$

where $P,Q,R$ are polynomials of degree one in $x$ and $y$.

This representation is a sum of irreducible representations $W_1$, $W_2$ and $W_3$ of dimensions 27, 8 and 1 correspondingly. A basis in $W_2$ is given by

$$x_1 = 5k_7 - k_5 - 7k_1, \qquad x_2 = 5k_8 - k_2 - 7k_6,$$

$$x_3 = 5d_1 + 2(n-1)(2a_1 + b_2), \qquad x_4 = 5e_1 + 2(n-1)(2b_1 + c_2),$$

$$x_5 = 5d_2 + 2(n-1)(2b_3 + a_2), \qquad x_6 = 5e_2 + 2(n-1)(2c_3 + b_2),$$

$$x_7 = 5d_3 + 2(n-1)(a_4 + b_5), \qquad x_8 = 5e_3 + 2(n-1)(b_4 + c_5).$$

The generic orbit of the action on $W_2$ has dimension 6. There are two polynomial invariants of the action:

$$I_1 = x_3^2 - x_3x_6 + x_6^2 + 3x_4x_5 + 3(n-1)(x_1x_7 + x_2x_8),$$

and

$$I_2 = 2x_3^3 - 3x_3^2x_6 - 3x_3x_6^2 + 2x_6^3 + 9x_4x_5(x_3 + x_6)+$$

$$9(n-1)(x_1x_3x_7+x_2x_6x_8-2x_1x_6x_7-2x_2x_3x_8+3x_2x_4x_7+3x_1x_5x_8).$$

## Flat potential operators with discrete symmetries

For almost all known examples the operator $L$ that satisfies Properties 1-3 its symbol admits additional finite group of discrete symmetries.

**Example 2.** The operator with coefficients

$$a = x^2(x^2+y^2)+\alpha x^2+\beta y^2, \qquad b = xy(x^2+y^2)+(\alpha-\beta)xy,$$

$$c = y^2(x^2+y^2)+\beta x^2+\alpha y^2, \qquad d = 2(n-1)x(\lambda-x^2-y^2),$$

$$e = 2(n-1)y(\lambda-x^2-y^2), \quad f = n(n-1)(x^2+y^2).$$

satisfies Properties 1-3, and possesses the discrete group of symmetries isomorphic $D_4$, generated by reflections

$$x\to -x,\ y\to y, \qquad x\to x,\ y\to -y, \qquad x\to y,\ y\to x. \quad (7)$$

Consider the case when $L$ is invariant with respect to a reflection. Using a transformation, we reduce the reflection to the form $\tilde{x} = x$, $\tilde{y} = -y$. Then the coefficients of the operator $L$ have the following symmetry properties:

$$a(x,-y) = a(x,y), \quad b(x,-y) = -b(x,y), \quad c(x,-y) = c(x,y),$$

$$d(x,-y) = d(x,y), \quad e(x,-y) = -e(x,y), \quad f(x,-y) = f(x,y).$$

The class of such operators admits the transformation group

$$\tilde{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \tilde{y} = \frac{y}{\gamma x + \delta}. \tag{8}$$

Transformations $\tilde{L} = c_1 L + c_2$ are also allowed.

Transformations (8) act on 15-dimensional vector space of coefficients of polynomials $a, b, c$. The irreducible components of this representation have dimensions $5, 3, 3, 3, 1$.

If we write the coefficients $a, b$ and $c$ in the form

$$a = P + Qy^2, \qquad b = \frac{1}{4}(P' - R)y + \frac{1}{2}Q'y^3,$$

$$c = S + \left(\frac{1}{12}P'' - \frac{1}{4}R' + \sigma\right)y^2 + \frac{1}{2}Q''y^4.$$

where $\deg P = 4, \quad \deg Q = \deg R = \deg S = 2$, then the coefficients of these polynomials and the constant $\sigma$ correspond to irreducible components.

In particular, the polynomial $P$ changes under transformations (8) as follows

$$\tilde{P} = (\gamma x + \delta)^4 P\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right). \tag{9}$$

**Definition.** A differential operator $L$ is called **elliptic** if the polynomial $P$ has four different roots on the Riemann sphere. It is called **trigonometric** if $P$ has one double root.

## Classification of the elliptic models

**Proposition 1.** If Property 1 holds and $P, S \neq 0$ then any root of the polynomial $S$ is a root of the polynomial $P$. $\square$

Consider elliptic models with symbols that admit the discrete group of symmetries (7) isomorphic $D_4$. Without loss of generality we set $P(x) = (x^2 - 1)(x^2 - \kappa)$. Taking into account Proposition 1 and the condition $S(-x) = S(x)$, we may put $S(x) = (x^2 - 1)$. Another possibility is $S(x) = 0$.

In both cases the system on algebraic equations for 6 unknown coefficients of polynomials $Q, R$ and $\sigma$ equivalent to zero-curvature condition $R_{1,2,1,2} = 0$ can be easily solved. As a result we obtain

**Proposition 2.** Any elliptic symbol that admit symmetries (7) coincides up to a scaling with the symbol from Example 1. $\square$

Consider now any elliptic symbols invariant with respect to $\tilde{x} = x$, $\tilde{y} = -y$. Without loss of generality we set

$$P(x) = x(x-1)(x-\kappa).$$

It follows from Proposition 1 that there are two alternatives: multiple roots **A:** $S = kx^2$ and distinct roots **B:** $S = kx(x-1)$.

**Theorem 1.** In Case **A** with $k \neq 0$ we obtain from $R_{1,2,1,2} = 0$ that

$$S(x) = x^2, \quad R(x) = -\frac{5}{3}(x^2 - 2x + 3\kappa - 2\kappa x),$$

$$Q(x) = \frac{1}{9}(x^2 - x + 1 + \kappa^2 - \kappa x - \kappa), \qquad \sigma = 0. \quad \square$$

**Theorem 2.** In Case **B** we have

$$S(x) = x(x-1), \quad R(x) = -3(x^2 - 2\kappa x + \kappa),$$
$$Q(x) = \frac{1}{2}(x^2 - 2\kappa x + 2\kappa^2 - \kappa), \qquad \sigma = \frac{1}{3}(2\kappa - 1). \quad \square$$

It turns out that in the case $S = 0$ we have no elliptic symbols.

It is easy to verify that the symbol from Theorem 2 is equivalent to the symbol (5) from Example 1.

Now for both elliptic symbols found in Theorems 1,2 we are going to find from (6) the coefficients $d(x, y), e(x, y)$ at the first derivatives in the Hamiltonian. It is trivial since given $a, b, c$ condition (6) is equivalent to a system of linear equations for the coefficients of polynomials $d$ and $e$.

However there is a non-trivial observation here. For symbol (5) polynomials $d$ and $e$ depend on three arbitrary parameters. But due to $D_4$-symmetry the symbol admits the transformation $\bar{x} = x^2$, $\bar{y} = y^2$. After this transformation and a scaling, we get

$$a = x(x-1)(x-\kappa) + (1-\kappa)\, x(x+\kappa)\, y,$$

$$b = x(x+1-2\kappa)\, y + (1-\kappa)\, x\, y^2,$$

$$c = (1-x)\, y + (x+2-\kappa)\, y^2 + (1-\kappa)\, y^3.$$

It follows from (6) that

$$d = \lambda_1 x(x+y-\kappa y) + \lambda_2(1+y-\kappa y) + p\, x,$$

$$e = \lambda_1 y(x+y-\kappa y) + \lambda_3(x-1) + q\, y,$$

$$f = \lambda_4(x+y-\kappa y) + \lambda_5,$$

where

$$\kappa p + (1-\kappa)q + \lambda_1(2\kappa-1) + \lambda_2(2-\kappa) + \lambda_3(1-\kappa^2) = 0.$$

Thus, now $d$ and $e$ depend on 5 arbitrary parameters!

Let $L$ be the corresponding second order operator with the above polynomial coefficients. If we bring the operator $\bar{L} = -4L$ to the canonical form by transformation

$$\bar{x} = f(x)f(y), \qquad \bar{y} = \frac{(f(x)-1)(f(y)-1)}{\kappa - 1},$$

where

$$f'^2 = 4f(f-1)(f-\kappa),$$

and by a gauge transformation, we get

$$H = \Delta + 2m(m-1)(\wp(x+y) + \wp(x-y)) +$$

$$\sum_{i=0}^{3} n_i(n_i - 1)(\wp(x+\omega_i) + \wp(y+\omega_i)),$$

where $\omega_0 = 0, \omega_3 = \omega_1 + \omega_2$ and $\omega_1, \omega_2$ are the half-periods of the Weierstrass function $\wp(x)$.

This is so called Inozemtsev $BC_2$ Hamiltonian. Its polynomial form preserves $V_k$ if

$$k = -\frac{1}{2}(2m + \sum n_i)$$

is a natural number.

## $A_2$ and $G_2$ elliptic models

For the symbol from Theorem 1 condition (6) leeds to

$$d = \frac{1}{9}(1-n)\left(3(5x^2-4x-4\kappa x+3\kappa)+(2x-1-\kappa)\,y^2\right),$$

$$e = \frac{2}{9}(1-n)\,y\left(9x+y^2-6\kappa-6\right), \qquad f = \frac{1}{9}n(n-1)\left(6x+y^2\right). \qquad \square$$

Therefore, in this case we have no arbitrary constants in $d$ and $e$ except for $n$. It turns out that this is a polynomial form of the $A_2$-elliptic model.

In contrast to the $BC_2$ Inozemtsev case, the finding of transformations that bring the operator to the Schrodinger form (1) is a rather difficult problem.

To receive arbitrary parameters in $d$ and $e$ we apply the same trick as in Example 1. Namely we apply transformation $\bar{x} = x, \bar{y} = y^2$. Due to the symmetry properties of the symbol the new symbol is also polynomial:

$$a = x(x-1)(x-\kappa) + \frac{1}{9}(1 - x + x^2 - \kappa - \kappa x + \kappa^2)\, y,$$

$$b = \frac{1}{3}(7x^2 - 8x - 8\kappa x + 9\kappa)\, y + \frac{1}{9}(2x - 1 - \kappa)\, y^2,$$

$$c = 4x^2 y + \frac{4}{3}(4x - 3 - 3\kappa)\, y^2 + \frac{4}{9}\, y^3.$$

The determinant $D$ is given by $D = -\frac{y}{27}\, K(x,y)$, where $K = (k_3 y^3 + 6k_2 y^2 + 9k_1 y + 108k_0)$,

$$k_3 = (\kappa - 1)^2, \quad k_0 = x^3(x-1)(x-\kappa),$$

$$k_2 = (\kappa+1)x^2 + 2(\kappa^2 - 4\kappa + 1)x - (\kappa - 2)(\kappa + 1)(2\kappa - 1),$$

$$k_1 = x^4 + 8(\kappa+1)x^3 - 2(4\kappa^2 + 23\kappa + 4)x^2 + 36\kappa(\kappa+1)x - 27\kappa^2.$$

Condition (6) implies

$$d = \frac{1}{9}(1-n)(6x+y)(2x-1-\kappa) + \frac{s}{3}(x^2 - 2x - 2\kappa x + 3\kappa),$$

$$e = \frac{2}{9}(9x^2 + 12xy + y^2 - 9y - 9\kappa y) + \frac{2s}{3}(3x^2 + xy - y - \kappa y) +$$
$$\frac{2(n-1)}{9}(9x^2 - 15xy - 2y^2 + 9y + 9\kappa y),$$
$$f = \frac{n(n-1)}{9}(3x+y) - \frac{s}{3}nx.$$

We see an extra parameter $s$ this formulas.

Introduce parameters $m_i$ by identities

$$n = -3m_1 - m_2, \qquad s = 1 + 3m_1 + 3m_2.$$

Then

$$hLh^{-1} = \Delta_g + m_2(1-m_2)\frac{x^2}{y} + 3m_1(1-m_1)\frac{P^2}{K} + \lambda. \qquad (10)$$

Here $\Delta_g$ is the Laplace-Beltrami operator, $\quad h = K^{\frac{m_1}{2}} y^{\frac{m_2}{2}}$,

$$P = 3x^3 - 6(\kappa+1)\,x^2 + (y + \kappa y + 9\kappa)\,x - 2(\kappa^2 - \kappa + 1)y,$$

$$\lambda = \frac{\kappa+1}{3}(3m_1 + m_2)(1 + 3m_1 + 3m_2).$$

Applying the transformation

$$x = \frac{f(y_1)^2 f'(y_2) - f(y_1)^2 f'(y_2)}{f(y_1)f'(y_2) - f(y_1)f'(y_2)}$$

$$y = -12\left(\frac{f(y_1)f(y_2)(f(y_1) - f(y_2))}{f(y_1)f'(y_2) - f(y_1)f'(y_2)}\right)^2,$$

where $f'^2 = 4f(f-1)(\kappa - f)$, to operator (10), we get

$$\mathcal{H} = -\frac{1}{3}\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2}\right) + V(y_1, y_2),$$

where

$$V = (m_1 - 1)m_1\Big(\wp(y_1 - y_2) + \wp(2y_1 + y_2) + \wp(y_1 + 2y_2)\Big) +$$

$$+\frac{(m_2 - 1)m_2}{3}\Big(\wp(y_1) + \wp(y_2) + \wp(y_1 + y_2)\Big).$$

This is just the elliptic $G_2$-model. The elliptic $A_2$-model corresponds to the special case $m_2 = 0$. The invariants of the $\wp$-function are related to the parameter $\kappa$ as follows

$$g_2 = \frac{4}{3}(\kappa^2 - \kappa + 1), \qquad g_3 = -\frac{4}{27}(\kappa - 2)(\kappa + 1)(2\kappa - 1).$$

The polynomial form of the $G_2$-model preserves $V_n$ if
$n = -3m_1 - m_2$ is a natural number.