# Quantum Argument Shifts in General Linear Lie Algebras

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# Outline

# Prior Research and Motivation



- Let $(e_n)_{n=1}^{d}$ be a linear basis of a complex Lie algebra $\mathfrak{g}$.
- The symmetric algebra $S\mathfrak{g}$ has a unique Poisson bracket extending the Lie bracket

$$\begin{array}{ccc} S\mathfrak{g} \times S\mathfrak{g} & \xrightarrow{\text{Poisson bracket}} & S\mathfrak{g} \\ \uparrow & & \uparrow \\ \mathfrak{g} \times \mathfrak{g} & \xrightarrow{\text{Lie bracket}} & \mathfrak{g} \end{array}.$$

# Prior Research and Motivation



Let $\overline{C}$ be the Poisson center of the Poisson algebra $S\mathfrak{g}$. Suppose that $\xi$ is an element of the dual space $\mathfrak{g}^*$ and let $\overline{\partial}_\xi = \sum_{n=1}^{d} \xi(e_n) \frac{\partial}{\partial e_n}$. The following is referred to as the argument shift method.

**Theorem (A. Mishchenko and A. Fomenko, 1978)**

*The subset* $\left\{ \overline{\partial}_\xi^n x : (n, x) \in \mathbb{N} \times \overline{C} \right\}$ *is Poisson commutative.*

# Prior Research and Motivation



- We obtained the $\boxed{\text{Poisson commutative}}$ subalgebra $\overline{C}_\xi$ generated by the elements $\overline{\partial}_\xi^n x$.
- The universal enveloping algebra $U\mathfrak{g}$ is considered as the quantisation of the symmetric algebra $S\mathfrak{g}$ and we have $\boxed{\operatorname{gr} U\mathfrak{g} = S\mathfrak{g}}$.
- Vinberg asked if the Poisson commutative subalgebra $\overline{C}_\xi$ can be quantised to the $\boxed{\text{commutative}}$ subalgebra $C_\xi$ of the universal enveloping algebra $U\mathfrak{g}$ with $\boxed{\operatorname{gr} C_\xi = \overline{C}_\xi}$.

# Prior Research and Motivation

- Vinberg's problem is solved by
  - Nazarov and Olshanski: (twisted) Yangians.
  - Tarasov: symmetrisation mapping.
- Vinberg's problem is also solved by the Feigin-Frenkel center
  - for regular elements $\xi$ (Feigin et al. and Rybnikov).
  - for simple Lie algebras of types A and C (Futorny, Molev and Molev, Yakimova).

## Motivation

The purpose of my talk is to quantise not only the algebra $\overline{C}_\xi$ but also the operator $\overline{\partial}_\xi$.

# Quantum Derivation of Algebra $U\mathfrak{gl}_d$



Let $e = \begin{pmatrix} e_1^1 & \cdots & e_d^1 \\ \vdots & \ddots & \vdots \\ e_1^d & \cdots & e_d^d \end{pmatrix}$ be a linear basis of the general linear Lie algebra $\mathfrak{gl}_d$ satisfying the commutation relations

$$[e_j^i, e_\ell^k] = \delta_\ell^i e_j^k - e_\ell^i \delta_j^k.$$

We define

$$\bar{\partial}_j^i = \frac{\partial}{\partial e_i^j}, \qquad \bar{\partial} x = \begin{pmatrix} \bar{\partial}_1^1 x & \cdots & \bar{\partial}_d^1 x \\ \vdots & \ddots & \vdots \\ \bar{\partial}_1^d x & \cdots & \bar{\partial}_d^d x \end{pmatrix}$$

for any element $x$ of the symmetric algebra $S\mathfrak{gl}_d$.

## Quantum Derivation of Algebra $U\mathfrak{gl}_d$



### Remark

The derivation

$$S\mathfrak{gl}_d \to M(d, S\mathfrak{gl}_d), \qquad x \mapsto \bar{\partial} x$$

is a unique linear mapping satisfying the following.

1. $\bar{\partial}\nu = 0$ for any scalar $\nu$.
2. $\bar{\partial}\,\mathrm{tr}(\xi e) = \xi$ for any numerical matrix $\xi$.
3. (Leibniz rule)

$$\bar{\partial}(xy) = (\bar{\partial}x)y + x(\bar{\partial}y)$$

for any elements $x$ and $y$ of the symmetric algebra $S\mathfrak{gl}_d$.

# Quantum Derivation of Algebra $U\mathfrak{gl}_d$



## Definition (Gurevich, Pyatov, and Saponov, 2012)

The quantum derivation

$$\boxed{U\mathfrak{gl}_d} \to M(d, U\mathfrak{gl}_d), \qquad x \mapsto \partial x$$

is a unique linear mapping satisfying the following.

1. $\partial \nu = 0$ for any scalar $\nu$.
2. $\partial \operatorname{tr}(\xi e) = \xi$ for any numerical matrix $\xi$.
3. (quantum Leibniz rule)

$$\partial(xy) = (\partial x)y + x(\partial y) + \boxed{(\partial x)(\partial y)}$$

for any elements $x$ and $y$ of the universal enveloping algebra $U\mathfrak{gl}_d$.

# Quantum Derivation of Algebra $U\mathfrak{gl}_d$



Let $C$ be the center of the algebra $U\mathfrak{gl}_d$. Suppose that $\xi$ is a numerical matrix and let $\partial_\xi = \operatorname{tr}(\xi\partial)$. The main theorem is the following.

**Theorem (I. and Sharygin, 2023)**

*The subset*

$$\left\{ \partial_\xi^n x : (n, x) \in \mathbb{N} \times C \right\} \tag{1}$$

*is commutative.*

**Corollary**

*The subalgebra $C_\xi$ generated by the subset (1) is the quantum argument shift algebra in the direction $\xi$.*

# Quantum Derivation of Algebra $U\mathfrak{gl}_d$



- We may assume that $\xi = \mathrm{diag}(z_1, \ldots, z_d)$ is diagonal and $(z_1, \ldots, z_d)$ is distinct considering the adjoint action of the general linear Lie group $GL_d$.
- Vinberg and Rybnikov showed that the quantum argument shift algebra in the direction $\xi$ is the centraliser of the set

$$\left\{ e_i^i, \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j} \right\}_{i=1}^{d}.$$

- The proof is carried out by showing that these elements commute with the quantum argument shift $\partial_\xi^n x$ by induction on the natural number $n$.

# Formula for Quantum Derivation

We have

$$C = \mathbb{C}\left[\operatorname{tr} e, \ldots, \operatorname{tr} e^d\right]$$

and the center $C$ of the universal enveloping algebra $U\mathfrak{gl}_d$ is the free commutative algebra on the elements

$$\operatorname{tr} e, \qquad \ldots, \qquad \operatorname{tr} e^d.$$

They are called the Gelfand invariants. We would like to calculate the quantum argument shift $\partial_\xi^n x$ for a central element $x$. It is necessary and even sufficient to calculate the quantum derivation $\partial(e^n)^i_j$.

# Formula for Quantum Derivation

I obtained the following formula for the quantum derivation.

## Definition

We define $f_{\pm}^{(m)}(x) = \frac{(x+1)^m \pm (x-1)^m}{2}$.

## Theorem (I, 2022)

*We have the formula*

$$\partial(e^n)^i_j = \sum_{m=0}^{n-1} \left( f_+^{(n-m-1)}(e)_j (e^m)^i + f_-^{(n-m-1)}(e)(e^m)^i_j \right)$$

*for any nonnegative integer n.*

The formula is used for the base case.

# Formula for Quantum Derivation



## Proof.

We assume the following form

$$\partial(e^n)^i_j = \sum_{m=0}^{n-1} \left( g^{(n-1)}_m(e)_j(e^m)^i + h^{(n-1)}_m(e)(e^m)^i_j \right),$$

where $g^{(n-1)}_m$ and $h^{(n-1)}_m$ are polynomials. We have

$$\partial(e^{n+1})^i_j = \sum_{k=1}^{d} \partial\left((e^n)^i_k e^k_j\right) = \cdots$$

by the quantum Leibniz rule and the commutation relations.

# Formula for Quantum Derivation

**Proof.**

We obtain the recursion formulae

1. $g_m^{(n)}(x) = g_m^{(n-1)}(x)x + h_m^{(n-1)}(x)$ for $0 \leq m < n$.
2. $g_n^{(n)}(x) = 1$ for $0 \leq n$.
3. $h_0^{(n)}(x) = \sum_{m=0}^{n-1} g_m^{(n-1)}(x)x^m$ for $0 \leq n$.
4. $h_m^{(n)}(x) = h_{m-1}^{(n-1)}(x)$ for $0 < m \leq n$.

The solution to them turn out to be

$$g_m^{(n)}(x) = f_+^{(n-m)}(x), \qquad h_m^{(n)}(x) = f_-^{(n-m)}(x).$$

□

# Formula for Quantum Derivation



*We write $\partial$ for the algebraic homomorphism*

$$U\mathfrak{gl}_d \to M(d, U\mathfrak{gl}_d), \qquad x \mapsto \operatorname{diag}(x, \dots, x) + \partial x$$

*from now on.* We write $C_\xi^{(n)}$ for the subalgebra generated by the subset $\bigcup_{m=0}^{n} \partial_\xi^m C$. We have

$$\partial_\xi\left(\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \cdots\right) = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \cdots \operatorname{tr}\left(\xi \prod_k f_-^{(n_k - m_k - 1)}(e)\right)$$

and the subalgebra $C_\xi^{(1)}$ is generated by the subset

$$C \cup \left\{\operatorname{tr}\left(\xi e^n\right)\right\}_{n=0}^{\infty}.$$

# Proof of Main Theorem

We show the base case $n = 1$. We are reduced to show

$$\left[\sum_{j\neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \operatorname{tr}(\xi e^n)\right] = \sum_{j\neq i} \frac{1}{z_i - z_j} \sum_{k=1}^{d} z_k \left[e_i^j e_j^i, (e^n)_k^k\right]$$
$$= \sum_{j=1}^{d} \left(-(e^n)_i^j e_j^i + e_i^j (e^n)_j^i\right) = 0.$$

To make the inductive step work it is sufficient to show

$$\left[\operatorname{ad} e_i^i, \partial_\xi\right] = \left[\left[\operatorname{ad} \sum_{j\neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi\right], \partial_\xi\right] = 0.$$

It can be done by computation.

# Generators of Quantum Argument Shift Algebra



$\bigcup_{n=0}^{\infty} C_\xi^{(n)}$ is the quantum argument shift algebra in the direction $\xi$. We have

$$\partial_\xi^2 \left(\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \cdots\right) = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \cdots$$

$$\sum_{k_1=-1}^{n_1-m_1-1} \sum_{k_2=-1}^{n_2-m_2-1} \cdots \operatorname{tr}\left(\xi \prod_\ell f_-^{(k_\ell)}(e) \partial \operatorname{tr}\left(\xi \prod_\ell f_-^{(n_\ell-m_\ell-k_\ell-2)}(e)\right)\right)$$

and the subalgebra $C_\xi^{(2)}$ is generated by the subset

$$C \cup \left\{\operatorname{tr}\left(\xi e^m \partial \operatorname{tr}\left(\xi e^n\right)\right) + \operatorname{tr}\left(\xi e^n \partial \operatorname{tr}\left(\xi e^m\right)\right)\right\}_{m,n=0}^{\infty}.$$

# Generators of Quantum Argument Shift Algebra

## Definition

We define the $m+n$ by $n$ integer matrix $P_n^{(m)}$ by

$$x^m f_+^{(n-j)}(x) = \sum_{i=1}^{m+n} \left(P_n^{(m)}\right)_j^i x^{i-1}$$

and let $P_n = P_n^{(0)}$.

## Definition

We define

$$\tau_\xi(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_j^i \,\mathrm{tr}\left(\xi e^{i-1} \xi e^{j-1}\right)$$

for any $m$ by $n$ integer matrix $x$.

# Generators of Quantum Argument Shift Algebra



## Definition

We define the $n$ by $n$ lower triangular integer matrix $\sigma(x)$ by

$$\sigma(x) = \begin{pmatrix} x_1^1 & 0 & \cdots & 0 \\ x_1^2 + x_2^1 & x_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n + x_n^1 & x_2^n + x_n^2 & \cdots & x_n^n \end{pmatrix}$$

for any $n$ by $n$ integer matrix $x$.

# Generators of Quantum Argument Shift Algebra





## Proposition

*We have $\tau_\xi(\sigma(x)) = \tau_\xi(x)$ for any square integer matrix $x$.*

## Proof.

It is equivalent to the conditions $\mathrm{tr}(\xi e^m \xi e^n) = \mathrm{tr}(\xi e^n \xi e^m)$. □

We have

$$\mathrm{tr}\left(\xi e^m \partial\, \mathrm{tr}(\xi e^n)\right) = \tau_\xi\left(P_n^{(m)}\right),$$

$$\mathrm{tr}\left(\xi e^m \partial\, \mathrm{tr}(\xi e^n)\right) + \mathrm{tr}\left(\xi e^n \partial\, \mathrm{tr}(\xi e^m)\right) = \tau_\xi\left(\sigma\begin{pmatrix} 0 & P_n^T \\ P_m & 0 \end{pmatrix}\right)$$

modulo $C_\xi^{(1)}$.

# Generators of Quantum Argument Shift Algebra



The following indicates the linear dependence of the generators.

## Theorem (I, 2023)

*We have*

$$\sigma \begin{pmatrix} 0 & P_m^T \\ P_{m+2n} & 0 \end{pmatrix} = \sum_{k=0}^{n} \left( \binom{2n-k}{k} + \binom{2n-k-1}{k-1} \right) P_{m+k}^{(m+k)},$$

$$\sigma \begin{pmatrix} 0 & P_m^T \\ P_{m+2n+1} & 0 \end{pmatrix} = \sum_{k=0}^{n} \binom{2n-k}{k} \left( P_{m+k+1}^{(m+k)} + P_{m+k}^{(m+k+1)} \right).$$

*for any nonnegative integers $m$ and $n$.*

# Generators of Quantum Argument Shift Algebra

We obtained the conclusion.

**Theorem (I, 2023)**

*The subalgebra $C_\xi^{(2)}$ is generated by the subalgebra $C_\xi^{(1)}$ and the subset*

$$\left\{ \tau_\xi\left(P_n^{(n)}\right), \tau_\xi\left(P_{n+1}^{(n)}\right) + \tau_\xi\left(P_n^{(n+1)}\right) : n = 1, 2, \ldots \right\}.$$

# Generators of Quantum Argument Shift Algebra



The generators are $\mathrm{tr}(\xi e)$, $\mathrm{tr}(\xi e^2)$, ... and

$$\begin{aligned}
&\mathrm{tr}(\xi^2 e),\\
&\mathrm{tr}(2\xi^2 e^2 + \xi e \xi e),\\
&\mathrm{tr}(\xi^2 e^3 + \xi e \xi e^2),\\
&\mathrm{tr}(2\xi^2 e^4 + 2\xi e \xi e^3 + \xi e^2 \xi e^2 + \xi^2 e^2),\\
&\mathrm{tr}(\xi^2 e^5 + \xi e \xi e^4 + \xi e^2 \xi e^3 + \xi^2 e^3),\\
&\mathrm{tr}(2\xi^2 e^6 + 2\xi e \xi e^5 + 2\xi e^2 \xi e^4 + \xi e^3 \xi e^3 + 4\xi^2 e^4 + \xi e \xi e^3),\\
&\mathrm{tr}(\xi^2 e^7 + \xi e \xi e^6 + \xi e^2 \xi e^5 + \xi e^3 \xi e^4 + 3\xi^2 e^5 + \xi e \xi e^4), \ldots.
\end{aligned}$$

They are mutually commutative.