Recursion operators and bi-Hamiltonian representations of cubic evolutionary (2+1)-dimensional systems

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Outline I

We study Lax pairs, recursion operators and bi-Hamiltonian representations of (2+1)-dimensional equations

$$F = f - u_{tt} g = 0 \quad \Longleftrightarrow \quad u_{tt} = \frac{f}{g} \tag{1}$$

where f and g are arbitrary smooth functions of $u_{t1}, u_{t2}, u_{11}, u_{12}, u_{22}$. We consider the equations $u_{tt} = f/g$ which are Lagrangian or become Lagrangian after multiplication by g, an integrating factor of the variational calculus. The Helmholtz conditions are $D_F^* = D_F$, i.e. the Fréchet derivative operator is self-adjoint. We solve completely the latter equation and obtain explicitly *all* (2+1)-dimensional equations (1) which have the Euler-Lagrange form:

$$
\begin{aligned}
&F \equiv \\
&a_1\Big\{u_{tt}(u_{11}u_{22}-u_{12}^2)-u_{t1}(u_{t1}u_{22}-u_{t2}u_{12})+u_{t2}(u_{t1}u_{12}-u_{t2}u_{11})\Big\} \\
&+a_2(u_{tt}u_{11}-u_{t1}^2)+a_3(u_{tt}u_{12}-u_{t1}u_{t2})+a_4(u_{tt}u_{22}-u_{t2}^2)+a_5u_{tt} \\
&+c_1(u_{t1}u_{12}-u_{t2}u_{11})+c_2(u_{t1}u_{22}-u_{t2}u_{12})+c_3(u_{11}u_{22}-u_{12}^2) \\
&+c_4u_{t1}+c_5u_{t2}+c_6u_{11}+c_7u_{12}+c_8u_{22}+c_9=0. \qquad (2)
\end{aligned}
$$

The form of equation (2) is exactly the same as obtained by E. Ferapontov et. al. The homotopy formula for this F, $L[u]=\int_0^1 uF(\lambda u)\,d\lambda$, yields the Lagrangian for equation (2) with the result

$$
\begin{aligned}
L = &\\
a_1\frac{u}{4}&\{u_{tt}(u_{11}u_{22} - u_{12}^2) - u_{t1}(u_{t1}u_{22} - u_{t2}u_{12}) + u_{t2}(u_{t1}u_{12} - u_{t2}u_{11})\}\\
+\frac{u}{3}&\{a_2(u_{tt}u_{11} - u_{t1}^2) + a_3(u_{tt}u_{12} - u_{t1}u_{t2}) + a_4(u_{tt}u_{22} - u_{t2}^2)\\
+&c_1(u_{t1}u_{12} - u_{t2}u_{11}) + c_2(u_{t1}u_{22} - u_{t2}u_{12}) + c_3(u_{11}u_{22} - u_{12}^2)\}\\
+\frac{u}{2}&(a_5u_{tt} + c_4u_{t1} + c_5u_{t2} + c_6u_{11} + c_7u_{12} + c_8u_{22}) + c_9u. \qquad (3)
\end{aligned}
$$

The two-component form of our equation $F = 0$ with F given in (2) reads

$$
\begin{aligned}
&u_t = v,\\
&v_t = \frac{1}{\Box}\{a_1(v_1^2u_{22} - 2v_1v_2u_{12} + v_2^2u_{11}) + a_2v_1^2 + a_3v_1v_2 + a_4v_2^2\\
&+c_1(v_2u_{11} - v_1u_{12}) + c_2(v_2u_{12} - v_1u_{22}) + c_3(u_{12}^2 - u_{11}u_{22})\\
&-c_4v_1 - c_5v_2 - c_6u_{11} - c_7u_{12} - c_8u_{22} - c_9\}
\end{aligned}
\tag{4}
$$

where

$$
\Box = a_1(u_{11}u_{22} - u_{12}^2) + a_2u_{11} + a_3u_{12} + a_4u_{22} + a_5. \tag{5}
$$

We modify the Lagrangian L given in (3) and skip some total derivative terms to obtain the Lagrangian for the two-component form (4) of our system

$$L = \left(u_t v - \frac{v^2}{2}\right)\{a_1(u_{11}u_{22} - u_{12}^2) + a_2 u_{11} + a_3 u_{12} + a_4 u_{22} + a_5\}$$
$$+u_t(c_1 u_1 - c_2 u_2)u_{12} + \frac{u_t}{2}(c_4 u_1 + c_5 u_2) - \frac{c_3}{3}u(u_{11}u_{22} - u_{12}^2)$$
$$-\frac{u}{2}(c_6 u_{11} + c_7 u_{12} + c_8 u_{22}) - c_9 u. \tag{6}$$

We define canonical momenta

$$\pi_u = \frac{\partial L}{\partial u_t} = v\{a_1(u_{11}u_{22} - u_{12}^2) + a_2u_{11} + a_3u_{12} + a_4u_{22} + a_5\}$$
$$+(c_1u_1 - c_2u_2)u_{12} + \frac{1}{2}(c_4u_1 + c_5u_2), \quad \pi_v = \frac{\partial L}{\partial v_t} = 0 \qquad (7)$$

which satisfy canonical Poisson brackets

$$[\pi_i(z), u^k(z')] = \delta_i^k \delta(z - z')$$

where $u^1 = u$, $u^2 = v$, $z = (z_1, z_2)$.
The Lagrangian (6) is degenerate because the momenta cannot be inverted for the velocities. Therefore, following the Dirac's theory of constraints, we impose (7) as constraints

$$\Phi_u = \pi_u - v\{a_1(u_{11}u_{22} - u_{12}^2) + a_2u_{11} + a_3u_{12} + a_4u_{22} + a_5\}$$
$$-(c_1u_1 - c_2u_2)u_{12} - \frac{1}{2}(c_4u_1 + c_5u_2), \quad \Phi_v = \pi_v.$$

Next we calculate Poisson brackets for the constraints

$$K_{11} = [\Phi_u(z_1, z_2), \Phi_{u'}(z'_1, z'_2)], \quad K_{12} = [\Phi_u(z_1, z_2), \Phi_{v'}(z'_1, z'_2)]$$
$$K_{21} = [\Phi_v(z_1, z_2), \Phi_{u'}(z'_1, z'_2)], \quad K_{22} = [\Phi_v(z_1, z_2), \Phi_{v'}(z'_1, z'_2)].$$

and obtain the following matrix of Poisson brackets

$$K = \begin{pmatrix} K_{11} & -\Box \\ \Box & 0 \end{pmatrix} \tag{8}$$

$$\begin{aligned} K_{11} &= a_1\{2(v_1 u_{22} - v_2 u_{12})D_1 + 2(v_2 u_{11} - v_1 u_{12})D_2 \\ &+ v_{11}u_{22} + v_{22}u_{11} - 2v_{12}u_{12}\} \\ &+ a_2(2v_1 D_1 + v_{11}) + a_3(v_2 D_1 + v_1 D_2 + v_{12}) + a_4(2v_2 D_2 + v_{22}) \\ &+ c_1(u_{11}D_2 - u_{12}D_1) + c_2(u_{12}D_2 - u_{22}D_1) - c_4 D_1 - c_5 D_2 \end{aligned} \tag{9}$$

which can be written in a skew-symmetric form.

The Hamiltonian operator, which determines the structure of the Poisson bracket, is the inverse to the symplectic operator $J_0 = K^{-1}$

$$J_0 = \begin{pmatrix} 0 & -K_{21}^{-1} \\ K_{12}^{-1} & K_{12}^{-1} K_{11} K_{12}^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\Box} \\ -\frac{1}{\Box} & \frac{1}{\Box} K_{11} \frac{1}{\Box} \end{pmatrix}, \tag{10}$$

$$J_0^{22} = \frac{1}{\Box} K_{11} \frac{1}{\Box} \tag{11}$$

with K_{11} defined by (9). Operator J_0 is Hamiltonian if and only if its inverse K is symplectic, which means that the volume integral $\Omega = \iiint_V \omega dV$ of $\omega = (1/2) du^i \wedge K_{ij} du^j$ should be a symplectic form, i.e. at appropriate boundary conditions $d\omega = 0$ modulo total divergence. Using (8), we obtain

$$\omega = \frac{1}{2} du \wedge K_{11} du - \Box du \wedge dv \tag{12}$$

Using K_{11} from (9), taking exterior derivative of (12) and skipping total divergence terms, we have checked that $d\omega = 0$ which proves that operator K is symplectic and hence J_0 defined in (10) is indeed a Hamiltonian operator. The corresponding Hamiltonian density H_1 is defined by $H_1 = u_t\pi_u + v_t\pi_v - L$ with the result

$$H_1 = \frac{1}{2}v^2\Box + \frac{c_3}{3}u(u_{11}u_{22} - u_{12}^2) + \frac{u}{2}(c_6u_{11} + c_7u_{12} + c_8u_{22}) + c_9u. \tag{13}$$

The Hamiltonian form of this system is

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix}. \tag{14}$$

Symmetry condition is the differential compatibility condition of equation (2) and the Lie equation $u_\tau = \varphi$, where φ is the symmetry characteristic and τ is the group parameter. It has the form of Fréchet derivative (linearization) of equation (2), $D_\tau F = M[\varphi] = 0$. We consider here case I of Ferapontov et. al. with $a_1 = 1$ when the equation (2) simplifies to

$$\Delta + a_5 u_{tt} + c_4 u_{t1} + c_5 u_{t2} + c_6 u_{11} + c_7 u_{12} + c_8 u_{22} + c_9 = 0 \quad (15)$$

where

$$\Delta = det \begin{bmatrix} u_{tt} & u_{t1} & u_{t2} \\ u_{1t} & u_{11} & u_{12} \\ u_{2t} & u_{21} & u_{22} \end{bmatrix}. \quad (16)$$

In the following, Δ can be eliminated by using equation (15) when necessary.

Operator M of symmetry condition becomes

$$\begin{aligned} M &= M_{12}D_t + M_{2t}D_1 + M_{t1}D_2 + a_5D_t^2 \\ &+ c_4D_tD_1 + c_5D_tD_2 + c_6D_1^2 + c_7D_1D_2 + c_8D_2^2 \end{aligned} \tag{17}$$

$$\begin{aligned} M_{12} &= (u_{11}u_{22} - u_{12}^2)D_t - (u_{t1}u_{22} - u_{t2}u_{12})D_1 + (u_{t1}u_{12} - u_{t2}u_{11})D_2 \\ M_{2t} &= -(u_{t1}u_{22} - u_{t2}u_{12})D_t + (u_{tt}u_{22} - u_{t2}^2)D_1 - (u_{tt}u_{12} - u_{t1}u_{t2})D_2 \\ M_{t1} &= (u_{t1}u_{12} - u_{t2}u_{11})D_t - (u_{tt}u_{12} - u_{t1}u_{t2})D_1 + (u_{tt}u_{11} - u_{t1}^2)D_2. \end{aligned} \tag{18}$$

We note the identities

$$\begin{aligned} u_{tt}M_{12} + u_{t1}M_{2t} + u_{t2}M_{t1} &= \Delta D_t, \\ u_{t1}M_{12} + u_{11}M_{2t} + u_{12}M_{t1} &= \Delta D_1, \\ u_{t2}M_{12} + u_{12}M_{2t} + u_{22}M_{t1} &= \Delta D_2 \end{aligned} \tag{19}$$

We apply our modification of A. Sergyeyev's method for constructing recursion operators which does not require previous knowledge of the Lax pair. For integrable equation (15), the symmetry condition $M[\varphi] = 0$, with the operator M given by (17), should be presented in the skew-factorized form

$$(A_1 B_2 - A_2 B_1)\varphi = 0 \tag{20}$$

where the commutator relations

$$[A_1, A_2] = 0, \quad [A_1, B_2] - [A_2, B_1] = 0, \quad [B_1, B_2] = 0 \tag{21}$$

should be satisfied on solutions of equation (15). It immediately follows that the following operators also commute on solutions

$$X_1 = \lambda A_1 + B_1, \quad X_2 = \lambda A_2 + B_2, \qquad [X_1, X_2] = 0 \tag{22}$$

and therefore constitute Lax representation for equation (15) with λ being a spectral parameter.

Symmetry condition (20) provides the recursion relations for symmetries

$$A_1\tilde{\varphi} = B_1\varphi, \quad A_2\tilde{\varphi} = B_2\varphi \tag{23}$$

where $\tilde{\varphi}$ satisfies symmetry condition $M[\tilde{\varphi}] = 0$ if and only if φ is also a symmetry, $M[\varphi] = 0$. The latter claim follows from the consequences of relations (21)

$$(A_1 B_2 - A_2 B_1)\varphi = [A_1, A_2]\tilde{\varphi} = 0,$$
$$(A_1 B_2 - A_2 B_1)\tilde{\varphi} = [B_2, B_1]\varphi = 0.$$

In the following, we will use obvious symmetries of the equation (15) generated by $X = \partial/\partial_t$, $X = \partial/\partial_1$, $X = \partial/\partial_2$ with the symmetry characteristics $\varphi = u_t$, $\varphi = u_1$, $\varphi = u_2$, respectively, which identically satisfy symmetry condition $M[\varphi] = 0$ with M defined in (17). Here we restrict ourselves to three-parameter equations where $a_1 = 1$.

We consider the three-parameter equation obtained by setting $c_6 = c_7 = c_8 = c_9 = 0$ in (15)

$$\Delta + a_5 u_{tt} + c_4 u_{t1} + c_5 u_{t2} = 0 \tag{24}$$

where Δ is defined in (16). The symmetry condition becomes

$$M[\varphi] = \{M_{12} D_t + M_{2t} D_1 + M_{t1} D_2 + a_5 D_t^2 + c_4 D_t D_1 + c_5 D_t D_2\}\varphi = 0. \tag{25}$$

It is identically satisfied by $\varphi = u_t$

$$M_{12}[u_{tt}] + M_{2t}[u_{t1}] + M_{t1}[u_{t2}] + a_5 u_{ttt} + c_4 u_{tt1} + c_5 u_{tt2} = 0 \tag{26}$$

where the square brackets denote values of operators. Combining the latter equation with the first equation from (19), we obtain

$$M_{12} u_{tt} + M_{2t} u_{t1} + M_{t1} u_{t2} + D_t(a_5 u_{tt} + c_4 u_{t1} + c_5 u_{t2}) = 0. \tag{27}$$

We use (27) in the identity transformation

$$M_{12}D_t = M_{12}u_{tt}\frac{1}{u_{tt}}D_t$$
$$= -\{M_{2t}u_{t1} + M_{t1}u_{t2} + D_t(a_5u_{tt} + c_4u_{t1} + c_5u_{t2})\}\frac{1}{u_{tt}}D_t. \tag{28}$$

Applying this to the symmetry condition (25) we transform it to the skew-factorized form (20)

$$\left\{(M_{2t} + c_4D_t)\left(D_1 - \frac{u_{t1}}{u_{tt}}D_t\right) + (M_{t1} + c_5D_t)\left(D_2 - \frac{u_{t2}}{u_{tt}}D_t\right)\right\}\varphi = 0. \tag{29}$$

with the definitions

$$A_1 = \frac{1}{u_{tt}}(M_{2t} + c_4D_t), \quad A_2 = \frac{1}{u_{tt}}(M_{t1} + c_5D_t)$$
$$B_1 = -D_2 + \frac{u_{t2}}{u_{tt}}D_t, \quad B_2 = D_1 - \frac{u_{t1}}{u_{tt}}D_t. \tag{30}$$

A straightforward check shows that all the integrability conditions (21) are identically satisfied on solutions of equation (24), together with their immediate consequences for the Lax pair (22) and recursion relations for symmetries (23).

If we choose B_i to be the second factors in (29), then A_i are defined by (29) only up to a common factor. It is interesting to note that choosing this factor to be $1/u_{tt}$, the same as the denominator in B_i, we obtain as a consequence that all the operators A_i and B_i automatically satisfy all the conditions (21). This turns out be the general property of all 1-parameter integrable equations.

The Lax pair is constituted by the operators

$$X_1 = \frac{1}{u_{tt}}(\lambda(M_{2t} + c_4 D_t) + u_{t2} D_t - u_{tt} D_2)$$
$$X_2 = \frac{1}{u_{tt}}(\lambda(M_{t1} + c_5 D_t) + u_{tt} D_1 - u_{t1} D_t). \tag{31}$$

The recursions for symmetries are given by

$$(M_{2t}+c_4 D_t)\tilde{\varphi} = (u_{t2} D_t - u_{tt} D_2)\varphi, \quad (M_{t1}+c_5 D_t)\tilde{\varphi} = (u_{tt} D_1 - u_{t1} D_t)\varphi. \tag{32}$$

We consider the three-parameter equation obtained by setting $a_5 = c_5 = c_8 = c_9 = 0$ in (15)

$$\Delta + c_4 u_{t1} + c_6 u_{11} + c_7 u_{12} = 0 \tag{33}$$

where Δ is defined in (16). The symmetry condition becomes

$$M[\varphi] = \{M_{12} D_t + M_{2t} D_1 + M_{t1} D_2 + c_4 D_t D_1 + c_6 D_1^2 + c_7 D_1 D_2\}\varphi = 0. \tag{34}$$

It is identically satisfied by $\varphi = u_1$. Combining the equation $M[u_1] = 0$ with the second equation from (19) we obtain

$$M_{12} u_{t1} + M_{2t} u_{11} + M_{t1} u_{12} + D_1(c_4 u_{t1} + c_6 u_{11} + c_7 u_{12}) = 0. \tag{35}$$

We use (35) in the identity transformation

$$M_{2t} D_1 = -\{M_{12} u_{t1} + M_{t1} u_{12} + D_1(c_4 u_{t1} + c_6 u_{11} + c_7 u_{12})\}\frac{1}{u_{11}} D_1. \tag{36}$$

Applying this to the symmetry condition (34) we transform it to the skew-factorized form (20)

$$\left\{(M_{12} + c_4 D_1)\left(D_t - \frac{u_{t1}}{u_{11}} D_1\right) + (M_{t1} + c_7 D_1)\left(D_2 - \frac{u_{12}}{u_{11}} D_1\right)\right\}\varphi = 0. \tag{37}$$

We define

$$\begin{aligned} A_1 &= \frac{1}{u_{11}}(M_{12} + c_4 D_1), \quad A_2 = \frac{1}{u_{11}}(M_{t1} + c_7 D_1) \\ B_1 &= -D_2 + \frac{u_{12}}{u_{11}} D_1, \quad B_2 = D_t - \frac{u_{t1}}{u_{11}} D_1 \end{aligned} \tag{38}$$

so that (37) takes the skew-factorized form (20).
The Lax pair and recursion relations for symmetries are obtained by using (38) in the formulas (22) and (23), respectively.

Now consider the three-parameter equation obtained by setting $a_5 = c_4 = c_6 = c_9 = 0$ in (15)

$$\Delta + c_5u_{t2} + c_7u_{12} + c_8u_{22} = 0 \tag{39}$$

where Δ is defined in (16). The symmetry condition becomes

$$M[\varphi] = \{M_{12}D_t + M_{2t}D_1 + M_{t1}D_2 + c_5D_tD_2 + c_7D_1D_2 + c_8D_2^2\}\varphi = 0. \tag{40}$$

It is identically satisfied by $\varphi = u_2$. We combine the equation $M[u_2] = 0$ with the second equation from (19) to obtain

$$M_{12}u_{t2} + M_{2t}u_{12} + M_{t1}u_{22} + D_2(c_5u_{t2} + c_7u_{12} + c_8u_{22}) = 0. \tag{41}$$

We use (41) in the identity transformation

$$M_{t1}D_2 = -\{M_{12}u_{t2} + M_{2t}u_{12} + D_2(c_5u_{t2} + c_7u_{12} + c_8u_{22})\}\frac{1}{u_{22}}D_2. \tag{42}$$

Applying this to the symmetry condition (40) we transform it to the skew-factorized form (20)

$$\left\{(M_{12} + c_5 D_2)\left(D_t - \frac{u_{t2}}{u_{22}} D_2\right) + (M_{2t} + c_7 D_2)\left(D_1 - \frac{u_{12}}{u_{22}} D_2\right)\right\}\varphi = 0. \tag{43}$$

We define

$$A_1 = \frac{1}{u_{22}}(M_{12} + c_5 D_2), \quad A_2 = \frac{1}{u_{22}}(M_{2t} + c_7 D_2)$$
$$B_1 = -D_1 + \frac{u_{12}}{u_{22}} D_2, \quad B_2 = D_t - \frac{u_{t2}}{u_{22}} D_2 \tag{44}$$

so that (43) takes the skew-factorized form (20). All the integrability conditions (21) are identically satisfied on solutions of equation (39).

The Lax pair and recursion relations for symmetries are obtained by using (44) in the formulas (22) and (23), respectively.

Here we present the notation convenient for bi-Hamiltonian systems and some remarks concerning second Hamiltonian operators.

$$\Box = u_{11}u_{22} - u_{12}^2 + a_5, \quad \Phi = v_1 u_{22} - v_2 u_{12}, \quad \chi = v_1 u_{12} - v_2 u_{11}. \tag{45}$$

The determinant Δ defined in (16) becomes

$$\Delta = v_t(\Box - a_5) - v_1\Phi + v_2\chi. \tag{46}$$

We introduce the operators

$$\Psi = \Phi D_1 - \chi D_2, \Longrightarrow D_1\Phi - D_2\chi = -\Psi^T, \quad \hat{\Psi} = \Psi - c_4 D_1 - c_5 D_2 \tag{47}$$

where T denotes transposed operator. We define

$$\Gamma = v_2 D_1 - v_1 D_2, \quad \Upsilon = u_{12} D_1 - u_{11} D_2, \quad \Theta = u_{22} D_1 - u_{12} D_2$$
$$\tilde{\Gamma} = a_5\Gamma + c_4\Upsilon + c_5\Theta. \tag{48}$$

First-order operators (18) entering the symmetry condition read

$$M_{12} = (\Box - a_5)D_t - \Psi, \quad M_{2t} = -\Phi D_t + v_t\Theta - v_2\Gamma$$
$$M_{t1} = \chi D_t - v_t\Upsilon + v_1\Gamma. \tag{49}$$

In the following section, we only show that the second Hamiltonian operator $J_1 = RJ_0$ is skew symmetric, $J_1^T = -J_1$ skipping the check of the Jacobi identities and compatibility of the two Hamiltonian operators J_0 and J_1.

The resulting bi-Hamiltonian system has the form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} \tag{50}$$

where J_0 and J_1 are the first and second Hamiltonian operators, respectively, while H_1 and H_0 are the corresponding Hamiltonian densities.

According to (46) the equation (24) in the two-component form becomes

$$u_t = v, \quad v_t = \frac{(\Psi[v] - c_4 v_1 - c_5 v_2)}{\Box} \equiv \frac{\hat{\Psi}[v]}{\Box}. \qquad (51)$$

We will need the inverse operator $\tilde{\Gamma}^{-1}$ which can make sense merely as a *formal* inverse. Thus, the relations involving $\tilde{\Gamma}^{-1}$ are also formal. The proper interpretation of the inverse operators and relations involving them requires the language of differential coverings.
We specify the inverse of $\tilde{\Gamma}$ by the property $\tilde{\Gamma}^{-1}\tilde{\Gamma} = I$ where I is the unit (identity) operator. A detailed example of constructing such an inverse operator was given in our previous paper.

According to the definitions (30) and (49), we have

$$A_1 = \frac{1}{v_t}\{-(\Phi - c_4)D_t + v_t\Theta - v_2\Gamma\}, \quad B_1 = \frac{1}{v_t}(v_2 D_t - v_t D_2)$$
$$A_2 = \frac{1}{v_t}\{(\chi + c_5)D_t - v_t\Upsilon + v_1\Gamma\}, \quad B_2 = \frac{1}{v_t}(v_t D_1 - v_1 D_t). \tag{52}$$

Recursion relations (23) become

$$-(\Phi - c_4)\tilde{\psi} + (v_t\Theta - v_2\Gamma)\tilde{\varphi} = v_2\psi - v_t D_2\varphi$$
$$(\chi + c_5)\tilde{\psi} + (-v_t\Upsilon + v_1\Gamma)\tilde{\varphi} = v_t D_1\varphi - v_1\psi \tag{53}$$

where φ and $\tilde{\varphi}$ are symmetry characteristics for the original and transformed symmetry, respectively, and $\psi = \varphi_t$, $\tilde{\psi} = \tilde{\varphi}_t$. The subscripts denote partial derivatives.

Combining the two equations in (53) we eliminate $\tilde{\psi}$ with the result

$$\tilde{\Gamma}\tilde{\varphi} = \hat{\Psi}\varphi - \Box\psi \iff \tilde{\varphi} = \tilde{\Gamma}^{-1}(\hat{\Psi}\varphi - \Box\psi). \tag{54}$$

Utilization of (54) in (53) yields only one independent equation

$$\tilde{\psi} = \frac{1}{\Box(\Phi - c_4)}\{(\Phi - c_4)\Psi\tilde{\Gamma}^{-1}\hat{\Psi} - v_2\hat{\Psi} + \hat{\Psi}[v]D_2\}\varphi$$
$$-\frac{1}{\Box}\Psi\tilde{\Gamma}^{-1}\Box\psi. \tag{55}$$

Recursion relations (54) and (55) can be written in the form of a matrix recursion operator R

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} = R \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{56}$$

where

$$R_{11} = \tilde{\Gamma}^{-1}\hat{\Psi}, \quad R_{12} = -\tilde{\Gamma}^{-1}\Box, \quad R_{22} = -\frac{1}{\Box}\Psi\tilde{\Gamma}^{-1}\Box$$
$$R_{21} = \frac{1}{\Box(\Phi - c_4)}\{(\Phi - c_4)\Psi\tilde{\Gamma}^{-1}\hat{\Psi} - v_2\hat{\Psi} + \hat{\Psi}[v]D_2\} \tag{57}$$

The first Hamiltonian operator (10) for equation (24) due to (9) takes the form

$$J_0 = \begin{pmatrix} 0, & \frac{1}{\Box} \\ -\frac{1}{\Box}, & \frac{1}{\Box}(\hat{\psi} - \psi^T)\frac{1}{\Box} \end{pmatrix} \tag{58}$$

and the corresponding Hamiltonian density (13) becomes

$$H_1 = v^2\Box/2. \tag{59}$$

The second Hamiltonian operator obtained by the formula $J_1 = RJ_0$ has the form

$$J_1 = \begin{pmatrix} \tilde{\Gamma}^{-1}, & \tilde{\Gamma}^{-1}\psi^T\frac{1}{\Box} \\ \frac{1}{\Box}\psi\tilde{\Gamma}^{-1}, & \frac{1}{\Box}(\psi\tilde{\Gamma}^{-1}\psi^T - \Gamma)\frac{1}{\Box} \end{pmatrix}. \tag{60}$$

The operator J_1 in (60) is manifestly skew symmetric, same as J_0 in (58).

The remaining task is to find the Hamiltonian density H_0 corresponding to the new Hamiltonian operator (60) according to the formula

$$J_1\begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{\hat{\Psi}[v]}{\Box} \end{pmatrix} \tag{61}$$

where (51) has been used. We assume that H_0 does not depend on partial derivatives of v, so that $\delta_v H_0 = H_{0,v}$.

The first line of equation (61) with J_1 defined in (60)

$$\tilde{\Gamma}^{-1}\left(\delta_u H_0 + \Psi^T\frac{H_{0,v}}{\Box}\right) = v \iff \delta_u H_0 = -\Psi^T\frac{H_{0,v}}{\Box} + c_4\chi + c_5\Phi \tag{62}$$

being used in the second line of this equation

$$\frac{1}{\Box}\Psi\tilde{\Gamma}^{-1}\left(\delta_u H_0 + \Psi^T\frac{H_{0,v}}{\Box}\right) - \frac{1}{\Box}\Gamma\frac{H_{0,v}}{\Box} = \frac{\hat{\Psi}[v]}{\Box}$$

implies $\Gamma[H_{0,v}/\Box] = c_4v_1 + c_5v_2 \iff$
$v_2D_1[H_{0,v}/\Box] - v_1D_2[H_{0,v}/\Box] = c_4v_1 + c_5v_2$, or finally

$$H_0 = \Box\{(c_5z_1 - c_4z_2)v + F(v)\} + h[u] \tag{63}$$

where $f(v)$ is an arbitrary smooth function belonging to the kernel of Γ, F is the antiderivative for f, $\Box = u_{11}u_{22} - u_{12}^2 + a_5$ and $h[u]$ is a function only of u and its partial derivatives.

H_0 in (63) should satisfy the second equation in (62) which yields

$$\begin{aligned} \delta_u H_0 &= (c_5z_1 - c_4z_2 + f(v))(v_{11}u_{22} + v_{22}u_{11} - 2v_{12}u_{12}) \\ &+ f'(v)(v_1^2u_{22} + v_2^2u_{11} - 2v_1v_2u_{12}) + 2(c_4\chi + c_5\Phi). \end{aligned} \quad (64)$$

Calculating directly the variational derivative $\delta_u H_0$ from H_0 in (63) and comparing it with (64) we obtain $\delta_u h[u] = 0$. In the final result we skip the "null Hamiltonian" $h[u]$

$$H_0 = \{F(v) + (c_5z_1 - c_4z_2)v\}(u_{11}u_{22} - u_{12}^2 + a_5). \quad (65)$$

Bi-Hamiltonian representation of the $(a_5c_4c_5)$-parameter system (51) has the form (50) with J_0 defined in (58), H_1 in (59), J_1 in (60), H_0 in (65) and the recursion operator R determined by (57).

We have obtained the general form of Euler-Lagrange evolutionary equations in $(2+1)$ dimensions containing only second order partial derivatives of the unknown. Their Lagrangians have also been constructed. We have converted these equations into two-component evolutionary form and obtained Lagrangians for these systems. The Lagrangians are degenerate because the momenta cannot be inverted for the velocities. Applying to these degenerate Lagrangians the Dirac's theory of constraints, we have obtained the Hamiltonian operator J_0 for each such system together with the Hamiltonian density H_1. We have explicitly demonstrated how the presentation of a symmetry condition in the skew-factorized form supply Lax pairs and recursion relations without the previous knowledge of Lax pairs.

In particular, we have shown how the symmetry condition for three-parameter cubic equations can be converted to a skew-factorized form and obtained Lax pair and recursion relations for such an equation.
We have derived a recursion operator in a 2×2 matrix form for the first three-parameter two-component system. Composing the recursion operator R with the Hamiltonian operator J_0 we have obtained the second Hamiltonian operator $J_1 = RJ_0$. We have found the Hamiltonian density H_0 corresponding to J_1 for this system, thus ending up with a new bi-Hamiltonian cubic system in $(2+1)$ dimensions. Similar results could be obtained for the other two cubic three-parameter systems.

Thank you very much for your attention.