Reciprocal transformations and deformations of integrable hierarchies

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R-matrix formalism

A systematic way to construct the operators A_k for a given L so that we obtain self-consistent compatibility conditions $L_{t_k} = [A_k, L]$ is provided by the *R-matrix formalism.*

Given a Lie algebra $\mathfrak{G}$ with the commutator $[,]$, a linear map $R : \mathfrak{G} \to \mathfrak{G}$ such that the bracket

$$[a,b]_R := [Ra,b] + [a,Rb] \tag{1}$$

is another Lie bracket on $\mathfrak{G}$ is called the classical R-matrix.

R-matrix formalism II

A sufficient condition for R to be a classical R-matrix is to satisfy the following so-called Yang-Baxter equation, YB(α),

$$[Ra, Rb] - R\,[a, b]_R + \alpha\,[a, b] = 0, \tag{2}$$

where α is a number from the ground field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). There are only two substantially different cases, namely $\alpha \neq 0$ and $\alpha = 0$.

Algebra of pseudo-differential operators

$$\mathfrak{A} = \left\{ L = \sum_{i=-\infty}^{N} h_i D^i \right\}, \qquad (3)$$

where D is (informally) the total x-derivative, where x is the independent variable.

The noncommutative multiplication in $\mathfrak{A}$ is defined using the (generalized) Leibniz rule

$$D^k \circ f = \sum_{i=0}^{\infty} \frac{k(k-1)\cdots(k-i+1)}{i!} D^i(f) D^{k-i} \qquad (4)$$

$$[A, B] := A \circ B - B \circ A \qquad A, B \in \mathfrak{A}.$$

Algebra of pseudo-differential operators II

Consider the following decomposition of $\mathfrak{A}$:

$$\mathfrak{A} = \mathfrak{A}_{\geqslant k} \oplus \mathfrak{A}_{<k} := \left\{ \sum_{i \geqslant k} u_i D^i \right\} \oplus \left\{ \sum_{i<k} a_i D^i \right\}. \qquad (5)$$

Then, $\mathfrak{A}_{\geqslant k}$ and $\mathfrak{A}_{<k}$ are Lie subalgebras of $\mathfrak{A}$ only for $k = 0, 1, 2$, and we have the classical R-matrices

$$R_k = \frac{1}{2}(P_{\geqslant k} - P_{<k}) = P_{\geqslant k} - \frac{1}{2} = \frac{1}{2} - P_{<k}, \qquad (6)$$

where $P_{\geqslant k}$ and $P_{<k}$ are projections onto $\mathfrak{A}_{\geqslant k}$ and $\mathfrak{A}_{<k}$, respectively.

Fractional powers

Consider an element L from $\mathfrak{A}$ of the form

$$L = u_N D^N + u_{N-1} D^{N-1} + u_{N-2} D^{N-2} + \ldots, \quad (7)$$

where $N > 0$. Then its N-th root

$$L^{\frac{1}{N}} = a_1 D + a_0 + a_{-1} D^{-1} + a_{-2} D^{-2} + \ldots,$$

where the coefficients a_i are **local** (depend on u_i and a finite number of their x-derivatives) and can be constructed recursively from the equality

$$\left(L^{\frac{1}{N}}\right)^N = \overbrace{L^{\frac{1}{N}} \circ \ldots \circ L^{\frac{1}{N}}}^{N \text{ times}} = L.$$

Fractional powers and Lax hierarchies on $\mathfrak{A}$

The fractional powers $L^{\frac{n}{N}} = \overbrace{L^{\frac{1}{N}} \circ \ldots \circ L^{\frac{1}{N}}}^{n \text{ times}}$ generate the following Lax hierarchies related to classical R-matrices (6):

$$L_{t_n} = \left[R_k(L^{\frac{n}{N}}), L\right] = \left[\left(L^{\frac{n}{N}}\right)_{\geqslant k}, L\right] = -\left[\left(L^{\frac{n}{N}}\right)_{<k}, L\right],$$
$$k = 0, 1, 2, \quad n = 1, 2, \ldots, \quad n \neq 0 \bmod N \tag{8}$$

Key result: commutativity of the flows:

$$(L_{t_n})_{t_m} = (L_{t_m})_{t_n}$$

follows from the YBE for R_k.

More on Lax hierarchies on $\mathfrak{A}$

The Lax hierarchies involving *finitely many* dynamical variables u_i are obtained within this approach by choosing a special form of L.

$$k = 0,1,2 : L = \sum_{j=0}^{N} u_j D^j, \tag{9}$$

$$k = 1,2 : L = \sum_{j=0}^{N} u_j D^j + D^{-1} \circ u_{-1}, \tag{10}$$

$$k = 2 : L = \sum_{j=0}^{N} u_j D^j + D^{-1} \circ u_{-1} + D^{-2} \circ u_{-2}, \tag{11}$$

It can be shown that without loss of generality we can set $u_N = 1$ in (9) and (10).

Is there more than that?

Błaszak and Szablikowski (*J. Math. Phys.* 2006) found the following deformations of the above R-matrices R_k:

$$R'_k = P_{\geqslant k} - \frac{1}{2} + \epsilon P_{k-1}(\cdot)D^k = \frac{1}{2} - P_{<k} + \epsilon P_{k-1}(\cdot)D^k, \ i.e.,$$

$$\begin{aligned} R'_k(L) &= P_{\geqslant k}(L) - \tfrac{1}{2}L + \epsilon P_{k-1}(L)D^k \\ &= \tfrac{1}{2}L - P_{<k}(L) + \epsilon P_{k-1}(L)D^k, \quad L \in \mathfrak{A}. \end{aligned}$$

Here $P_k = P_{\geqslant k} - P_{\geqslant k-1}$ and ϵ is an arbitrary constant.

Deformed Lax hierarchies

$$L_{t_n} = \left[R'_k(L^{\frac{n}{N}}), L\right] = \left[\left(L^{\frac{n}{N}}\right)_{\geqslant k} + \epsilon P_{k-1}(L^{\frac{n}{N}})D^k, L\right]$$

$$= -\left[\left(L^{\frac{n}{N}}\right)_{<k} - \epsilon P_{k-1}(L^{\frac{n}{N}})D^k, L\right],$$

$$k = 0, 1, 2, \quad n = 1, 2, \dots, \quad n \not\equiv 0 \text{ mod } N \tag{12}$$

However, for $k = 0$ and $k = 2$ Błaszak & Szablikowski have shown that these hierarchies can be reduced to known ones, i.e., the deformations in question are trivial.

Problem: is this true also for $k = 1$?

Trivializing transformation for $k = 1$

Consider the *reciprocal transformation* from x and t_i, $i = 1, 2, \ldots, i \not\equiv 0 \text{ mod } N$ to new independent variables z and τ_i, $i = 1, 2, \ldots$, where $\tau_i = t_i$, $i = 1, 2, \ldots, i \not\equiv 0 \text{ mod } N$ and z is defined by the formula

$$dz = (u_N)^{-1/N} dx + \epsilon \sum_{q=1, q \not\equiv 0 \text{ mod} N}^{\infty} (u_N)^{-1/N} P_0(L^{q/N}) dt_q. \tag{13}$$

Trivializing transformation for $k = 1$: continued

Introduce new dependent variables v_i related to u_i by means of the formulas

$$\tilde{L} = L|_{D=(u_N)^{-1/N}\tilde{D}} = \sum_{i=-\infty}^{N} u_i((u_N)^{-1/N}\tilde{D})^i$$
$$\equiv \tilde{D}^N + \sum_{i=-\infty}^{N-1} v_i \tilde{D}^i \tag{14}$$

(informally, $\tilde{D}$ is the total z-derivative) and

$$v_N = (u_N)^{1/N}. \tag{15}$$

Main result

Theorem 1 *The above transformation sends the hierarchy*

$$L_{t_n} = \left[R_1'(L^{\frac{n}{N}}), L\right] = \left[\left(L^{\frac{n}{N}}\right)_{\geqslant 1} + \epsilon P_0(L^{\frac{n}{N}})D, L\right]$$

$$k = 0, 1, 2, \quad n = 1, 2, \ldots, \quad n \not\equiv 0 \text{ mod } N$$

into the undeformed $k = 1$*-hierarchy for* v_i*,*

$$\tilde{L}_{\tau_q} = [P_{\geq 1}(\tilde{L}^q), \tilde{L}], \quad q = 1, 2, \ldots, q \not\equiv 0 \text{ mod } N,$$

for $\tilde{L} = \tilde{D}^N + \sum_{i=-\infty}^{N-1} v_i \tilde{D}^i$ *along with a hierarchy of equations for* v_N*:*

$(v_N)_{\tau_q} = -\epsilon \tilde{D}(P_0(\tilde{L}^{q/N}))$, $q = 1, 2, \ldots,$ $q \not\equiv 0 \text{ mod } N$.

Example: Extended Broer–Kaup hierarchy

Consider the extended Broer–Kaup system with $L = uD + v + D^{-1} \circ w$. From

$$L_{t_i} = [P_{\geq 1}(L^i) + \epsilon P_0(L^i)D, L]$$

with $i = 1, 2$ we obtain

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_1} = \begin{pmatrix} \epsilon u_x v - \epsilon u v_x \\ u v_x + \epsilon v v_x \\ u_x w + u w_x + \epsilon v_x w + \epsilon v w_x \end{pmatrix}$$

Extended Broer–Kaup system

$$
\begin{aligned}
u_{t_2} &= \epsilon u_x v^2 - 2\epsilon u v v_x - 2\epsilon u^2 w_x - \epsilon u^2 v_{xx} \\
v_{t_2} &= 2u u_x w + 2u v v_x + 2u^2 w_x + u u_x v_x + u^2 v_{xx} \\
&+\epsilon v^2 v_x + 2\epsilon u v_x w + \epsilon u v_x^2 \\
w_{t_2} &= 2u_x v w + 2u v_x w + 2u v w_x - u_x^2 w - 3u u_x w_x \\
&-u u_{xx} w - u^2 w_{xx} + 2\epsilon u_x w^2 \\
&+2\epsilon v v_x w + \epsilon u_x v_x w + \epsilon v^2 w_x + 4\epsilon u w w_x \\
&+\epsilon u v_x w_x + \epsilon u v_{xx} w.
\end{aligned}
$$

Upon setting $\epsilon = 0$ and $u = 1$ we recover from the second (t_2) flow the standard Kaup–Broer system.

Transformation to the Broer–Kaup hierarchy

By Theorem 1, pass from x and $t_{1,2}$ to z and $\tau_{1,2}$ defined by the formulas $\tau_1 = t_1$, $\tau_2 = t_2$, and

$$dz = (1/u)dx + \epsilon(v/u)dt_1 + (\epsilon/u)(2wu + uv_x + v^2)dt_2$$

(we ignore here the times t_i with $i \neq 1, 2$).

We have $\partial_x = (1/u)\partial_z$, so L goes into

$$\tilde{L} = \left(uD + v + D^{-1} \circ w\right)_{D=(1/u)\tilde{D}} = \tilde{D} + v + \tilde{D}^{-1} \circ r,$$

where $r = wu$.

Transformation to the Broer–Kaup hierarchy II

We readily see that the transformed hierarchy

$$\tilde{L}_{\tau_q} = [P_{\geq 1}(\tilde{L}^q), \tilde{L}], \quad q = 1, 2, \ldots,$$

is nothing but the *standard* Broer–Kaup hierarchy for v, r with the independent variables z and τ_i, and we have separated equations for u which are linear on the background of this hierarchy:

$$u_{\tau_q} = -\epsilon \tilde{D}(P_0(\tilde{L}^{q/N})),\ q = 1, 2, \ldots$$

Dispersionless limit

Theorem 1 remains valid if we replace the algebra $\mathfrak{A}$ by its dispersionless (or *quasiclassical*) limit: $D \to p$, $[,] \to \{,\}$, $\{,\}$ is the Poisson bracket:

$$\{f, g\} = D(f)\frac{\partial g}{\partial p} - D(g)\frac{\partial f}{\partial p}.$$

Now $L = \sum_{i=-\infty}^{N} u_i p^i$ is a function rather than an operator, the reciprocal transformation is the same: from x and t_i we pass to z and τ_i, where $\tau_i = t_i$, $i = 1, 2, \ldots$, where z is defined by

$$dz = (u_N)^{-1/N} dx + \epsilon \sum_{q=1, q \not\equiv 0 \ \mathrm{mod} N}^{\infty} (u_N)^{-1/N} P_0(L^{q/N}) dt_q.$$

Dispersionless limit II

New dependent variables v_i are now related to u_i by means of the formulas

$$\tilde{L} = L|_{p=(u_N)^{-1/N}\tilde{p}} = \sum_{i=-\infty}^{N} u_i((u_N)^{-1/N}\tilde{p})^i$$

$$\equiv \tilde{p}^N + \sum_{i=-\infty}^{N-1} v_i\tilde{p}^i,$$

i.e.,

$$v_i = u_i(u_N)^{-i/N}, \qquad (16)$$

and

$$v_N = (u_N)^{1/N}. \qquad (17)$$

Main result for the dispersionless case

Theorem 2 *The above transformation sends*

$$L_{t_n} = \{R_1'(L^{\frac{n}{N}}), L\} = \{\left(L^{\frac{n}{N}}\right)_{\geqslant 1} + \epsilon P_0(L^{\frac{n}{N}})p, L\}$$

$$k = 0, 1, 2, \quad n = 1, 2, \ldots, \quad n \not\equiv 0 \text{ mod } N$$

into the undeformed $k = 1$-*hierarchy for* v_i,

$$\tilde{L}_{\tau_q} = \{P_{\geq 1}(\tilde{L}^q), \tilde{L}\}', \quad q = 1, 2, \ldots, q \not\equiv 0 \text{ mod } N,$$

for $\tilde{L} = \tilde{p}^N + \sum_{i=-\infty}^{N-1} v_i \tilde{p}^i$, *where*

$$\{f, g\}' = \tilde{D}(f)\frac{\partial g}{\partial \tilde{p}} - \tilde{D}(g)\frac{\partial f}{\partial \tilde{p}},$$

along with a hierarchy for v_N*:*

$$(v_N)_{\tau_q} = -\epsilon \tilde{D}(P_0(\tilde{L}^{q/N})),\ q = 1, 2, \ldots,\ q \not\equiv 0 \text{ mod } N.$$

Conclusions

All integrable hierarchies constructed using the deformed R-matrices on $\mathfrak{A}$ given for $k = 0, 1, 2$ by

$$\begin{aligned} R'_k(L) &= P_{\geqslant k}(L) - \tfrac{1}{2}L + \epsilon P_{k-1}(L)D^k \\ &= \tfrac{1}{2}L - P_{<k}(L) + \epsilon P_{k-1}(L)D^k, \qquad L \in \mathfrak{A}, \end{aligned}$$

can be reduced to linear extensions of the undeformed ($\epsilon = 0$) hierarchies using suitable changes of dependent and independent variables. The same holds for their dispersionless counterparts.