Multiplicative Kernels, Non-Abelian Abel th. and around

Talk 09.03.2022

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Joseph Krasil'shchik's Seminar on Geometry of Differential Equations

based on arXiv:2102.09511

(joint with V. Golyshev, A. Mellit and D. van Straten)

and on the ongoing project with I. Gaiur an D. van Staten

CY PF equations

Motivations: To find analogues of the Apéry - Beukers - Zagier list (6 very special 2nd order DE) $L\varphi = 0$

$$L = \theta^n + t P_1(\theta) + t^2 P_2(\theta) + \dots + t^r P_r(\theta)$$

$P_i(\theta)$ - polynomials of degree n and $\theta \equiv t\, d/dt$

with the MUM property at $t=0$ (maximal unipotent monodromy)

We shall call such operators

Calabi-Yau operators

They arised as equations on periods of CY varieties:

If M_t · a family of CY r-folds $\omega_r \in \Omega^r(M_t)$ - holomorphic top form

Then (GM connection theory):

periods $\varphi = \int_{\sigma_t} \omega_t$

satisfies Picard-Fuchs equations

$r = 1$ $M_t = \mathcal{E}_t : y^2 = x(x-1)(x-t)$

1-D CY-variety and the PF:

$$\theta^2\varphi + \frac{t}{t-1}\theta\varphi + \frac{t}{t-1}\varphi = 0$$

The period is given by

$$\varphi(t) = \int^{\infty} \frac{dx}{\sqrt{x(x-1)(x-t)}}$$

$r=2$ CY are K3-surfaces and PF DO has order 3

For example, if K3-family of quartics in $\mathbb{P}^3$

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 - t^{-1}x_1x_2x_3x_4 = 0$$

PF-equation is:

$$L\varphi = \theta^3\varphi + 4t(4\theta+1)(4\theta+2)(4\theta+3)\varphi = 0$$

$H_4(X,Y)$ $Y=\alpha X$

For $r=3$ and CY with $h^{2,1}=1$ the PFs have order 4

$$x_1^5+x_2^5+x_3^5+x_4^5+x_5^5-t^{-1}x_1x_2x_3x_4x_5 = 0 \subset \mathbb{P}^4$$
– Dwork quintic CY threefold and PF:

$$L\varphi = \theta^4\varphi - 5t(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4) = 0.$$

2. Abel theorem.

0° Addition laws:

$(\mathbb{R}^*_t, \cdot)$ $\quad \omega := \frac{du}{u}$

$$\int_1^x \frac{du}{u} + \int_1^y \frac{du}{u} = \int_1^z \frac{du}{u}$$

$$= \ln x + \ln y = \ln z = \ln(xy)$$

$$z = xy$$

$(\mathbb{R}, +)$ $\quad \omega = du$

$$\int_0^x du + \int_0^y du = \int_0^z du \Rightarrow$$

$$\text{“} x + y = z \text{.”}$$

Euler formulae:

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

$$= \sin u \sqrt{1-\sin^2 v} + \sqrt{1-\sin^2 u}\, \sin v$$

$$\Rightarrow \int_0^x \frac{dt}{\sqrt{1-t^2}} + \int_0^y \frac{dt}{\sqrt{1-t^2}} = \int_0^z \frac{dt}{\sqrt{1-t^2}}$$

$$z = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\arcsin(x) + \arcsin(y) = \arcsin\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

Euler (1750): addition law

(Landen transform for elliptic functions for arc length of lemniscata computarisers)

$$\int_0^x \frac{dt}{\sqrt{1-t^4}} + \int_0^y \frac{dt}{\sqrt{1-t^4}} = \int_0^z \frac{dt}{\sqrt{1-t^4}} \Rightarrow$$

$$z = \frac{x\sqrt{1-y^4} + y\sqrt{1-x^4}}{1+x^2y^2}$$ - algebraic expression.

Abel theorem (1828)

Generalization of the Euler formula

$\omega = \underbrace{R(x,y)}\,dx$; $x, y \in C = \{F(x,y)=0\}$

$y = y(x)$ - algebraically

Abel works with hyperelliptic C

$\exists$ P such a number that $\forall N \geq P$

$$\int_0^{x_1} \omega + \ldots + \int_0^{x_N} \omega = \int_0^{y_1} \omega + \ldots + \int_0^{y_P} \omega + E(x,y)$$

rat et log rat

$y_i = y_i(x_i)$ algebraically depends.

Later $\Longrightarrow$ Jacobi, Riemann, Clebsch

$C = \{F(x,y) = 0\}$ - compact curve

ω - 1-form on the curve

The Abel theorem can be reformulated as a form of linear equivalency for divisors:

$$\sum x_i - a = \sum y_i - a$$

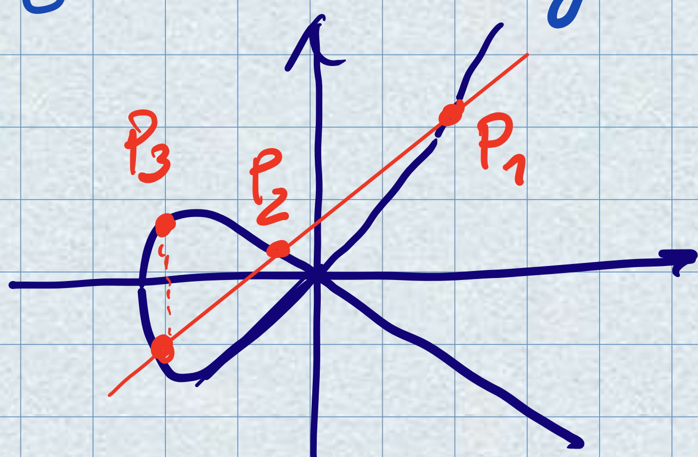
(which means that the reducibility of # of $\int$ can be seen as a reflection of Abel-Jacobi:

$$AJ: \ Sym^N(C) \to Jac(C)$$

$$N \geq p = g(C)$$

The classical Abel result is specified for $\mathcal{E}: y^2 = x^3 + g_1 x + g_2$

The elliptic curve $\mathcal{E}$ - abelian variety: the "addition law" geometrically



P_1 " $+$ " $P_2 = P_3$

$$\omega = \frac{dx}{y}$$

$$\int_{P_0}^{P_1} \omega + \int_{P_0}^{P_2} \omega = \int_{P_0}^{P_3} \omega \pmod{\Gamma}$$

$$\mathcal{E} = \mathbb{C}/\Gamma$$

Picard variety $\mathrm{Pic}_0(\mathcal{E})$ - abelian group:

$$\mathcal{O}(P_1 - P_0) \otimes \mathcal{O}(P_2 - P_0) \simeq \mathcal{O}(P_3 - P_0)$$

iff "$P_1 + P_2 = P_3$

Non-abelian Abel (in our interpretation) is a passage:

1st order DO, Line bundles with connection $\Rightarrow$ Any order DO vector bundles with connection

start with LHS: suppose that our 1-form ω - connection form

$$\frac{d\psi}{dt} = \psi\omega \Rightarrow \psi(t) = e^{\int_0^t \omega}$$

and

$$\psi(x)\psi(y) = e^{\int_0^x \omega} \cdot e^{\int^y \omega} =$$

$$= e^{\int^x \omega + \int^y \omega} = e^{\int^{x+y} \omega}$$

One can write this product in the form with a Kernel

$$\psi(x)\psi(y) = \int K(x,y,z)\,\psi(z)\,dz$$

(This is a form of Abel th.!)

In general we may hope to have

$$\phi(x_1)\phi(x_2)\dots\phi(x_N) = \int K(x_1,\dots,x_N; y_1,\dots y_p)$$

$\phi(y_1)\dots\phi(y_p)\,dy_1\,dy_2\dots dy_p$ for some "kernel function" of $N+P$ variabl.

Let us consider our "toy model" and find the kernel:

$$L = \frac{d}{dx} - \lambda \Rightarrow \phi_\lambda = e^{\lambda x}$$

$$\phi_\lambda(x)\phi_\lambda(y) = \int \delta(x+y-z)\,\phi_\lambda(z)\,dz$$

$$= \oint \frac{e^{\lambda z}}{x+y-z}\,dz$$

Namely

$$e^x \cdot e^y = \left(\sum_n \frac{x^n}{n!}\right)\left(\sum_m \frac{y^m}{m!}\right) =$$

$$= \sum_n \sum_m \frac{z^{m+n}}{(m+n)!} \frac{1}{(n!\,m!)} \cdot \left(\frac{x}{z}\right)^m \left(\frac{y}{z}\right)^m (m+n)!$$

$$= \sum_{n,m} \binom{m+n}{n} \left(\frac{x}{z}\right)^n \left(\frac{y}{z}\right)^m \frac{z^{m+n}}{(m+n)!} =$$

$$= \frac{1}{2\pi i} \oint \frac{1}{1 - \frac{x+y}{z}} e^z \frac{dz}{z} = \frac{1}{2\pi i} \oint \frac{e^z dz}{z-(x+y)}$$

$$\boxed{K(x+y|z) = \frac{1}{z-(x+y)}}$$

"Conjecture" ALL 1^{st} od. i. kernels are δ

$L = x\partial_x - \lambda$, $\phi = x^\lambda$ $K(x,y,z) = \delta(xy-z)$

2nd od. Observation: take the modification for $\phi(x) = \mathcal{J}_N(x) = \sum \frac{x^n}{(n!)^N}$

which is a solution of the equation

$$(\Theta^N - x)\,\mathcal{J}_N(x) = 0$$

($N=2$ - a version of the Bessel)

Theorem (known for $N=2$ since XIX)

$$f_N(x)\, g_N(y) = \oint \underbrace{H_N\left(\frac{x}{z}, \frac{y}{z}\right)}\, f_N(z) \frac{dz}{z}$$

$$\boxed{H_N(x,y) = \sum \binom{n+m}{n}^N x^m y^n}$$

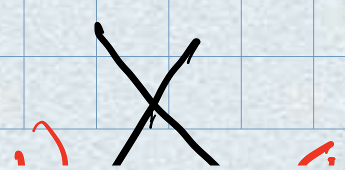
generating function.

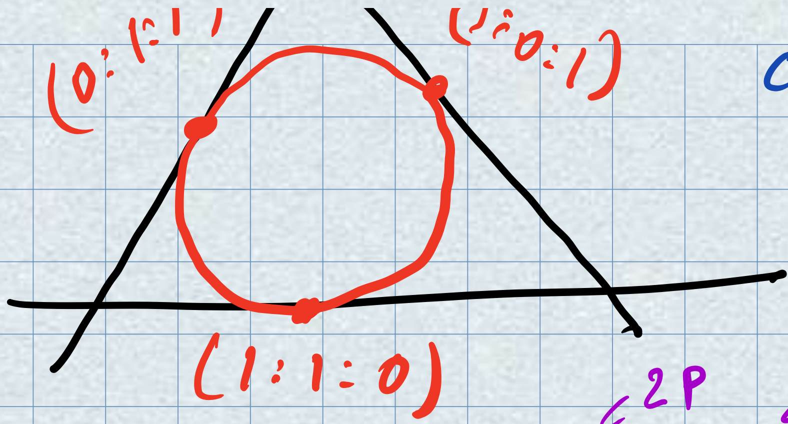
For $N=2$ this gen. function has a beatiful algebraic form

$$H_2(x,y) = \frac{1}{\sqrt{\Delta(x,y,1)}}, \text{ where}$$

$$\Delta(x,y,z) = x^2+y^2+z^2-(2xy+2xz+2yz)$$

fully symmetric. Its zero locus in $\mathbb{P}^2$ is the circle tangent to coordinates lines:





change variables

$x \to u^2$
$y \to v^2$
$z \to w^2$

$$\Delta(u^2, v^2, w^2) = (u+v+w)(-u+v+w)(u-v+w)(u+v-w)$$

$2p \quad 2(p-u) \quad \ldots$

$$\Delta(u^2, v^2, w^2) = 16\, S^2\left({}^{u}\triangle^{v}_{w}\right)$$

("Heron configuration")

Famous Sonin - Gegenbauer formula

- very well-known in th. of special function and representations (N. Ya. Vilenkin 19..)
- $\Delta(x,y,z)$ is a particular case of

Kontsevich generalized polynomials

Consider the following 2nd order DE

$$0 = L\phi = [\partial f \partial + (t+\lambda)]\phi = f\partial^2\phi + f'\partial\phi + (t+\lambda)\phi$$

where $f = t^3 + at^2 + bt + c$, then

for solutions ϕ of $L\phi = 0$

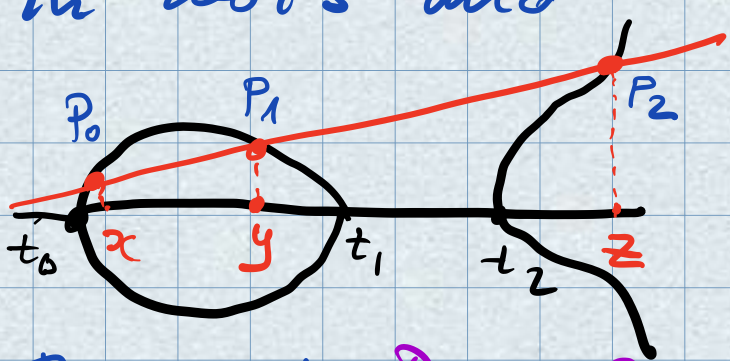
$$\phi(x)\phi(y) = \oint K(x,y,z)\phi(z)\frac{dz}{z}$$ with

$$K(x,y,z) = \frac{b}{\sqrt{P_{a,b,c}(x,y,b/z)}},$$

$$P_{a,b,c} = \mathrm{Disc}\left[f(t) - (t-x)(t-y)(t-z)\right]$$
$$= (b-(xy+yz+zx))^2 - 4(x+y+z+a)(xyz+c)$$

(For so-called special Heun
$f(t) = t(t-1)(t-\mu) \Rightarrow P_\mu(x,y,z)$)

The kernel depends in a universal way on the elliptic curve $u^2 = f(t)$ which is branched in roots and taking the secant line



The points P_0, P_1, P_2 are colinear $\Leftrightarrow$ "addition law" for $u^2 = f(t)$

A fantastic results for the "Kontsevich polynomials" (EFK)

Consider an operator H_x acting in some appr. Hilbert space of fun's ϕ:

$$(H_x \phi)(y) := \frac{2}{\pi} \int_{\mathbb{C}} \frac{\phi(z)\, dz\, d\bar{z}}{|P_\mu(x,y,z)|}$$

it is compact, self-adjoint and $[H_x, H_{x'}] = 0$ Such operators are $\mathbb{C}$-analogues of Hecke operators over $\mathbb{F}_p$ (initial Kontsevich constr.)

Buchshtaber addition laws in formal two-valued groups

If R - commutative ring $\Rightarrow$ formal group law is a power series $\Phi(u,v) \in R[[u,v]]$:

- $\Phi(u,v) = u + v + \text{(higher terms)}$
- $\Phi(u, \Phi(v,w)) = \Phi(\Phi(u,v), w)$ - "associativity"

We will restrict to commutative laws $\Phi(u,v) = \Phi(v,u)$

$\Phi(u,v) = u+v$ - additive
$\Phi(u,v) = u+v+uv$ - multiplicative
"Elliptic" formal group law:

$$\int_0^u \frac{dt}{\sqrt{1-t^4}} + \int_0^v \frac{dt}{\sqrt{1-t^4}} = \int_0^{\Phi(u,v)} \frac{dt}{\sqrt{1-t^4}}$$

<u>Theorem</u> (Buchshtaber) (1990-93)
Two-valued algebraic groups, obtained by constructions of square of moduli of formal group with the addition law based on BA-functions on $\mathcal{E}$ are classified by zero locus of the following fully symmetric polynomial in 3 variables

$$F(x,y,z) = (x+y+z-a_2 xyz)^2 - 4(1+g_3 xyz)(xy+yz+xz+a_1 xyz)$$

$a_1 = \wp(\alpha)$, $a_2 = 3\wp(\alpha)^2 - \frac{g_2}{4}$, $g_3 = \frac{1}{4}(4\wp^3(\alpha) - g_2\wp(\alpha) - g_3)$, $\alpha \in \mathcal{E}: v^2 = 4u^3 - g_2 u - g_3$.

Theorem (Buchshtaber-Veselov, 2019)

The formal multiplication law given by $F(x, y, z) = 0$ can be reduced to $X \pm Y \pm Z = 0$ by the following change of variables $x = \frac{1}{\wp(X) + \wp(\alpha)}$, $y = \frac{1}{\wp(Y) + \wp(\alpha)}$; $z = \frac{1}{\wp(Z) + \wp(\alpha)}$

The corresponding group is the abelian "coset" group $\mathcal{E}/\sigma = \mathbb{P}^1$

$\mathcal{E}$: $v^2 = P(u) = u^3 + a_1 u^2 + a_2 u + a_3$

σ: $v \to -v$

Link to the Kontsevich family:

- $a_1 = a_2 = a_3 = 0$ ($\wp(\alpha) = 0$, $g_2 = g_3 = 0$)

$P_{0,0,0}(x, y, z) = \boxed{x^2 + y^2 + z^2 - (2xy + 2yz + 2zx)}$

"Heron configuration"

$\Phi(u, v) = u + v$ - cohomology theory (Buchshtaber - Novikov)

- $\wp(\alpha)\neq 0,\ a_2=a_3=0\ \left(\wp(\alpha)=\frac{3}{\sqrt{g_2^3}}\left(\frac{2}{\sqrt{3}}+\frac{g_3}{\sqrt{g_2}}\right)\right)$

$\Phi(u,v)=u+v-q\,uv$ ~ K-theory

the corresponding family

$F(x,y,z)=x^2+y^2+z^2-2xy-2xz-2yz-q^2xyz$

- Cayley cubic in $\mathbb{A}^3$ (4 conic singularities)

- $a_1=0,\ a_2=-g_2/4,\ a_3=-g_3/4$

$F(x,y,z)=P_{0,g_2,g_3}(x,y,z)$

Addition law: $P_{0,g_2,g_3}(x,y,z)=0$

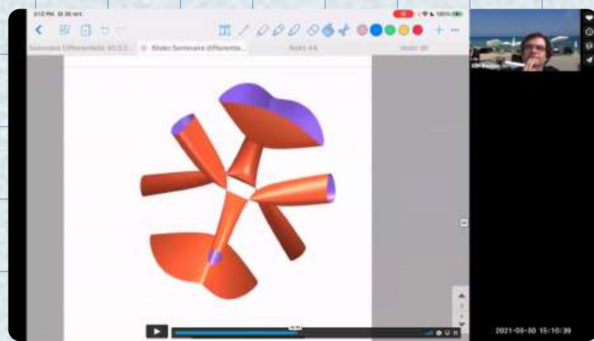
$x=\wp(X),\ y=\wp(Y)$

$z=\wp(X\pm Y)$:

$$\boxed{\left(xy+yz+xz+g_2/4\right)^2=4(x+y+z)\left(xyz-\tfrac{1}{4}g_3\right)}$$

$\mathcal{E}:\ \xi^2=t^3-\tfrac{1}{4}g_2t-\tfrac{1}{4}g_3$

The quartic surface is K3 with $6A_1$ and $3D_4$ singularities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