"Master formula" & "multiplicative kernels"

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(part 2)

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Apéry - Beukers - Zagier equation

Let $A, B \in \mathbb{C}$;

$f(t) := t^3 + At^2 + Bt$, $\partial = \frac{\partial}{\partial t}$

$L := f\partial^2 + f'\partial + t$

(a particular case of a construction in Part I)

Def. $D2$ - equation:

$\forall \lambda \in \mathbb{C}$ $\quad \underline{(L - \lambda)\varphi(t) = 0}$

Recursive approach to solution:

For fixed $A, B \in \mathbb{C}$ a sequence $\{b_n(\lambda)\}_{n\in\mathbb{N}}$, $b_n(\lambda) \in \mathbb{C}[\lambda]$

is defined as:

$$b_0 = 1, \quad b_{n+1}(\lambda) = \frac{1}{B(n+1)^2} \cdot \left\{ \left[\lambda - A\, n(n+1)\right] b_n(\lambda) - n^2 b_{n-1}(\lambda) \right\}$$

Then

$$\varphi(\lambda) := \sum_{n=0}^{\infty} b_n(\lambda)\, t^n$$

satisfies the D2.

Remark: $\deg b_n(\lambda) = n$

Multiplicative law and recursion:

Define numbers c_{kem}:

$$\boxed{b_k(\lambda) \cdot b_e(\lambda) = \sum_{m=0}^{\infty} c_{kem}\, b_m(\lambda)}$$

We obtain that

$$\sum_{k,l,m=0}^{\infty} C_{klm}\, x^k y^l z^m = \frac{B}{\sqrt{P(x,y,Bz)}}$$

where $P(x,y,z) = \mathrm{discr}\,[f(t) - (t-x)(t-y)(t-z)]$

the "Kontsevich polynomial":

$$P(x,y,z) = [B - (xy+yz+zx)]^2 - 4xyz\,(x+y+z+A)$$

Integral representation

Define a "residue" pairing in $\mathbb{C}((t))$ using

$$\oint x(t)\,dt := \operatorname*{Res}_{t=0} x(t)\,dt$$

$x(t) \in \mathbb{C}((t))$

Then the pairing

$$\langle x(t), y(t)\rangle = \oint x(t)\,y(t)\,dt$$

identifies

$$(\mathbb{C}[[t]])^* \simeq t^{-1}\mathbb{C}[t^{-1}]$$

and we have

$$\partial^* = -\partial, \quad (t\partial)^* = -(t\partial+1)$$

$$(t^2\partial+t)^* = -(t^2\partial+t) \text{ and}$$

$$\mathcal{L}^* = \mathcal{L} \quad \text{- self-adjoint}$$

Put

$$K(x,y,z) = \frac{B z^{-1/2}}{\sqrt{P(x,y,Bz^{-1/2})}} \in z^{-1}\mathbb{C}[z^{-1}][[x,y]]$$

<u>Thm</u> 1° $L_x K = L K = L_z K$;

2° $L_x P^{-1/2} = L_y P^{-1/2} = L_z P^{-1/2}$

+ invariance $\varphi \to t^{-1}\varphi(Bt^{-1})$ and $\mathcal{L}$

Consider $\forall \lambda \in \mathbb{C}$ the function

$$\underline{\psi_\lambda(x,y) = \int K(x,y,z)\,\varphi_\lambda(z)\,dz}$$

Prop. $L_x \psi_\lambda = L_y \psi_\lambda = \lambda \psi_\lambda$

and therefore

$$\underline{\int K(x,y,z)\,\varphi_\lambda(z)\,dz = k(\lambda)\,\varphi_\lambda(x)\,\varphi_\lambda(y)}$$

take $x=0$

$$\int K(0,y,z)\,\varphi_\lambda(z)\,dz = k(\lambda)\,\varphi_\lambda(0)\,\varphi_\lambda(y)$$

$\varphi_\lambda(0) = 1 \Rightarrow K(0,y,z) = \frac{1}{z-y}$

(Cauchy integral) $k(\lambda) = 1$

$$\boxed{\int K(x,y,z)\,\varphi_\lambda(z)\,dz = \varphi_\lambda(x)\,\varphi_\lambda(y)}$$

Again Bessel! (Sonin-Gegenbauer)

$A = B = 0$ (degenerate case)

D2: $(t^3 \partial^2 + 3t^2 \partial + t - \lambda)\varphi = 0$

After corresponding change of variables ($t = -\frac{4\lambda}{x^2}$ → Bessel!)

D2: $(L - \lambda)\underbrace{t^{-1} J_0\left(2\sqrt{\frac{-\lambda}{t}}\right)}_{\varphi_\lambda} = 0$

Sonin-Gegenbauer:

$$J_0(k\sqrt{x})J_0(k\sqrt{y}) = \frac{2}{\pi}\int_{(\sqrt{x}-\sqrt{y})^2}^{(\sqrt{x}+\sqrt{y})^2} J_0(k\sqrt{z})K(x,y,z)\,dz$$

$N=2$

$$K(x,y,z) = (2xy + 2yz + 2zx - x^2 - y^2 - z^2)^{-1/2}$$

$\varphi_\lambda(x) = J_0(\sqrt{x})$

A hypergroup is a very useful generalization of a group

(Dunkl, Spector, ..., Grishin Litvinov)

Def. A hypergroup is a locally compact $\mathcal{K}$ space with Borel measures $\mathcal{M}(\mathcal{K})$ form a $*$-algebra with axioms:

- Closure: $\delta_x * \delta_y$, $x \in \mathcal{K}$, $y \in \mathcal{K}$ is always cmpct. supp. probability measure, which continuously varies with x, y.
- Associativity $\mathcal{M}(\mathcal{K})$ - associative algebra
- Identity & Existence of it $\exists e \in \mathcal{K}$ s.t. δ_e is identity
- Inverses & Existence: $\forall x \in \mathcal{K}$ $\exists! \; x^*$ s.t. e is contained in the support of $\delta_x * \delta_{x^*}$.

Moreover, $(\delta_x)^* = \delta_{x^*}$

The axioms neither precise nor stand.
(taken from Wildberger (2014))

Symmetric space $X = G/K$ - compact s/gr.

$\mathcal{K}(X)$ - canonical hypergroup
(spherical hypergroup)
set of K-orbits on X.

The algebra structure on the measures on $\mathcal{K}(X)$ is induced from the convolution structure of K-invariant measures on X (in fact K-biinvariant measures on G). The resulting hypergroup is independent on the base pt. O.

The main point is that $\mathcal{K}(X)$ is a commutative hypergroup

Such hypergroups have a Character Theory (much likely as for abelian locally cmpct. groups; there is a notion of dual object, Fourier tr. Haar measure & Plancherel

Att. dual object of cmpct. hyper-group is NOT always a H.G.

Simpliest examples: $X = \mathbb{R}^2, S^2, H^2$
hyperbolic

For each of this $K = S^1$ (fixing pt. O)

$$\mathcal{K}(X) \underset{\text{homeo}}{\simeq} \begin{cases} \mathbb{R}^+ , & \mathbb{R}^2 \\ [0,\pi] , & S^2 \\ \mathbb{R}^+ , & H^2 \end{cases}$$

Describe the algebra (convolution) structure for $\mathbb{R}^2$

Two circles C_{r_1}, C_{r_2} may be convolved by identifying each with the unique probability measure (rotationally inv.)

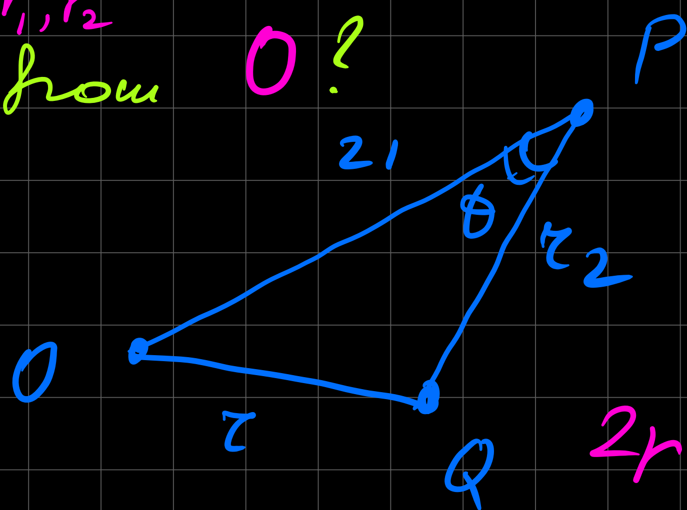
Take randomly a point P : $\rho(O,P)=r_1$

Q : $\rho(P,Q)=r_2$ and ask :

what is the probability density

$P_{r_1,r_2}(r)$ for Q to be of distance r

from O ?



$\theta=\angle OPQ$

$r^2=r_1^2+r_2^2-$
$-2r_1r_2\cos\theta$

$2r\,dr=2r_1r_2\sin\theta\,d\theta$

Due to the rotational symmetry of

$P_{r_1r_2}(r)dr$ it will be measure of

that piece of circle around P

for which $\angle OPQ\in[\theta,\theta+d\theta]$

or $\frac{d\theta}{\pi}\Rightarrow$ for

$$|r_1 - r_2| \le r \le r_1 + r_2$$

$$P_{r_1, r_2}(r) = \frac{}{\pi\, 2 r_1 r_2 \sin\theta} = \frac{2r}{4\pi S_{\Delta OPQ}} =$$

= (Heron)

$$P_{r_1 r_2}(r) = \frac{2r}{\pi\left[(r^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - r^2)\right]^{1/2}} \cdot$$

$\cdot\ \chi[|r_1 - r_2|, r_1 + r_2]$ — characteristic function of the interval.

The function $P_{r_1 r_2}(r)$ defines the convolution structure of rotational invariant measures on $\mathbb{R}^2$. Identifying $C \sim r \in \mathbb{R}^+$ one can transfer this structure to $\mathcal{M}(\mathbb{R}^+)$ - Borel measures on $\mathbb{R}^+$:

$$\delta_{r_1} * \delta_{r_2} = \underbrace{p_{r_1 r_2}(r)\, dr}$$

for Dirac measures δ_{r_i} at r_i

$M(K)$ is a $*$-algebra of commutative spherical hypergroup. $x^* = x)$

Now, let K such a hypergroup

<u>Def.</u> A <u>character</u> K is a bounded function $\varphi : K \to \mathbb{C}$:

1° $\varphi(x)\varphi(y) = \int_K \delta_x * \delta_y \, \varphi(z) \quad \forall x, y \in K$

2° $\varphi(x^*) = \overline{\varphi(x)} \quad \forall x \in K$

$K^\vee$ - character set of K.

As in general there <u>does not appear a complete duality between</u> $K, K^\vee$ there are many interesting cases when

$\mathcal{K}^\vee$ is a hypergroup.

When $\mathcal{K}$ is the spherical hypergroup of a symmetric space X a character $\varphi \in \mathcal{K}^\vee$ may be regarded as a function on X constant on K-orbits

$\Rightarrow$ spherical harmonic analysis (Helgason) $\Rightarrow$ Fourier transform = spherical transform

S^2, $T_x S^2 \simeq \mathbb{R}^2$ $K = S^1$. Orbits of $K = S^1$ on $\mathbb{R}^2$ - hypergroup $\mathcal{K}(\mathbb{R}^2, S^1)$

But S^1 acts on $(\mathbb{R}^2)^*$: $(rf)(v) = f(r^{-1}v)$

$v \in \mathbb{R}^2$, $f \in (\mathbb{R}^2)^*$, $r \in S^1$

The hypergroup $\mathcal{K}((\mathbb{R}^2)^*, S^1) =$

$= \mathcal{K}(\mathbb{R}^2, S^1)^\vee$

This means that every orbit $\mathcal{O} \subset (\mathbb{R}^2)^*$ determines a character $\chi_{\mathcal{O}} \in \mathcal{K}(\mathbb{R}^2, S^1)$

and vise versa.

If $\mu_{\mathcal{O}}$ - invariant probability measure on $\mathcal{O}$ and $\mu_{\mathcal{L}}$ - invariant probabity measure on $\mathcal{L} \subset \mathbb{R}^2$ (an orbit)

then $$\chi_{\mathcal{O}}(\mathcal{L}) = \int_{\mathcal{O}}\int_{\mathcal{L}} e^{if(x)} d\mu_{\mathcal{L}}(x) d\mu_{\mathcal{O}}(f)$$

By choosing $x_0 \in \mathcal{L}$, $f_0 \in \mathcal{O}$ (any) and S^1-invariance $\quad f \in (\mathbb{R}^2)^*$

$$\chi_{\mathcal{O}}(\mathcal{L}) = \int_{\mathcal{O}} e^{if(x_0)} d\mu_{\mathcal{O}}(f) =$$

$$= \int_{\mathcal{L}} e^{if_0(x)} d\mu_{\mathcal{L}}(x).$$ In other words, characters of $\mathcal{K}(\mathbb{R}^2, S^1)$ are given by Fourier transforms of measures on orbits in $(\mathbb{R}^2)^*$

Multiplication of characters corresponds on the Fourier transform side to Convolution of the corresp. orbits in $(\mathbb{R}^2)^*$

Since the actions of S^1 on $\mathbb{R}^2$ and $(\mathbb{R}^2)^*$ are isomorphic $\Rightarrow$

$$\underline{\mathcal{K}(\mathbb{R}^2, S^1) \simeq \mathcal{K}(\mathbb{R}^2, S^1)^\vee}$$

Any character of $\mathcal{K}(\mathbb{R}^2)$ was given by a FT of a circle $C_r \subset (\mathbb{R}^2)^*$ with radius $r \geq 0$. To define the FT of the invariant probability measure μ_r on this circle we fix a specific direction in $(\mathbb{R}^2)^*$ say the axe z and consider the orthogonal push-down of μ_r on to the axe: this is the measure on $[-r, r]$ given by $\dfrac{dz}{2\pi\sqrt{r^2 - z^2}}$

$$\hat{\mu}_r(p) = \frac{1}{2\pi}\int_{-r}^{r} e^{ipz}\frac{dz}{\sqrt{r^2-z^2}} =$$

$$= \frac{1}{2\pi}\int_{-1}^{1} e^{irpw}\frac{dw}{\sqrt{1-w^2}} = \underline{\underline{J_0(rp)}}$$

Bessel!

$$\left(J_0(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\, x^{2m}}{2^{2m}\, m!\, \Gamma(m+1)};\right)$$

<u>Résumé</u>:

$$\mathfrak{X}(\mathbb{R}^2) \simeq \mathbb{R}^+\{\rho \geq 0\}$$

$$\mathfrak{X}(\mathbb{R}^2)^\vee \simeq \mathbb{R}^+\{r \geq 0\}$$

$\forall r \in \mathbb{R}^+$ the corresponding character is $\varphi_r(p) = J_0(rp)$

The multiplication law $\Longleftrightarrow$ multiplication of characters = Bessel functions:

$$\varphi_{r_1}\varphi_{r_2} = \int_{|r_1-r_2|}^{(r_1+r_2)} \frac{2r}{\pi\sqrt{(r^2-(r_1-r_2)^2)((r_1+r_2)^2-r^2)}}\,\varphi_r\, dr$$

"Master-formula"

First, recall the approach of Kontsevich and Odesskii:

Let us consider polynomials:

$P_0(\lambda), P_1(\lambda), \ldots \in \mathbb{C}[\lambda]$, $P_0(\lambda) = 1$

$\deg P_i = i$;

"structure constants":

$$\underline{P_i(\lambda)\, P_j(\lambda)} = \sum_{k=0}^{\infty} c_{ij}^k \, \underline{P_k(\lambda)}$$

Consider the following "generating functions":

$$f_\lambda(x) = \sum_{i=0}^{\infty} P_i(\lambda)\, x^i = 1 + O(\lambda)$$

and the following form:

$$\boxed{K(x,y\,|\,z) = \sum_{i,j,k} c_{ij}^k \frac{x^i y^j}{z^{k+1}} dz}$$

<u>Theorem</u> (MK, AO):

$$f(x)f(y) = \frac{1}{2\pi i}\oint K(x,y|z)\,f(z)$$

Can be generalized for $\mathbb{C}[\lambda_1, \lambda_2 \ldots \lambda_g]$

"Master formula" - another approach to the multiplication kernels.

Consider a Spectral Problem:

$$\mathcal{L}_x \Psi = \lambda \Psi$$

λ - spectral parameter

This approach always work for rank 2 differential system ($\mathcal{L}_x$ - second order operator)

- We shall assume that there is an analytic solution at $x=0$ (MUM/regular)

Choose a solution

$$\varphi_0(\lambda, x) = 1 + \sum_{i=1}^{\infty} P_i(\lambda) x^i, \text{ where}$$

$P_i(\lambda)$ polynomials, $\deg P_i = i$

"Spectral theorem":

$$\boxed{P_i(\mathcal{L}_y)\, \varphi_0(\lambda, y) = P_i(\lambda)\, \varphi_0(\lambda, y)}$$

"The Master formula" (empiric) in general

$$\boxed{\begin{aligned} \varphi_0(\mathcal{L}_y; x)\, \varphi_0(\lambda, y) &= \sum_{i=0}^{\infty} x^i P_i(\mathcal{L}_y)\, \varphi_0(\lambda, y) \\ &= \sum_{i=0}^{\infty} x^i P_i(\lambda)\, \varphi_0(\lambda; y) = \varphi_0(\lambda, x)\, \varphi_0(\lambda, y) \end{aligned}}$$

"Proof" of the concept:

we start with the equation

$$\frac{d}{dx}\psi = \lambda \psi \quad \text{analytic sol. at } x=0$$

$$\varphi_0(\lambda,x) = e^{\lambda x} = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n x^n}{n!}$$

Master-formula: $e^{x\frac{d}{dy}} \to$ "shift"

$$\varphi_0\left(\frac{d}{dy};x\right)\varphi_0(\lambda,y) = e^{x\frac{d}{dy}}\varphi_0(\lambda,y) =$$

$$= \varphi_0(\lambda, y+x) = \oint \frac{dz}{z-(x+y)}\varphi_0(\lambda,z).$$

Weyl algebra and integral operators

We shall adopt the Cauchy formula:

$$\varphi(y) = \oint \frac{dz}{z-y}\varphi(z), \quad \frac{d^k}{dz^k}(\varphi) =$$

$$= \oint \frac{k!\,dz}{(z-y)^{k+1}}$$

We introduce the following operators

$$y^j \cdot \frac{d^k}{dz^k} \circ = \oint y^j \frac{k!\,dz}{(z-y)^{k+1}} \circ$$

Their kernels: $\underrightarrow{K\left(y^j \frac{d^k}{dz^k}\right)} = \frac{k!}{(z-y)^{k+1}}$

Lemma (composition):

$$K\left(y^j \frac{d^k}{dy^k} \circ y^r \frac{d^l}{dy^l}\right) = K\left(y^j \frac{d^k}{dy^k}\right) * K\left(y^r \frac{d^l}{dy^l}\right)$$

$$= y^j \frac{d^k}{dy^k} K\left(y^r \frac{d^l}{dy^l}\right).$$

$$K(\mathcal{L}_y) = K(\mathcal{L}_y \circ 1) = \mathcal{L}_y K(1) =$$

$$= \mathcal{L}_y \left[\frac{1}{z-y}\right]$$

This lemma gives us a passage from the Weil algebra $\mathbb{C}[y, \partial_y] / [\partial_y, y] = 1$ to the formal algebra of elementary kernels

Proof of concept: Exponential

$$K \cdot \exp\left(x \frac{d}{dy}\right) := \sum_{n=0}^{\infty} \frac{K\left(\frac{d^n}{dy^n}\right) x^n}{n!} =$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{(z-y)^{n+1}} = \frac{1}{z-y-x};$$

$\mathfrak{sl}_2$ - syst.

SOV and Master-formula

Consider DE: $\left[\mathcal{L}_t - \sum_{i=1}^{g} \lambda_i t^{i-1}\right]\psi = 0$!

$$\mathcal{L}_t := f(t)\partial^2 + f'(t)\partial + \frac{(g+1)^2}{4} t^g$$

$$f(t) = t(t-1)\prod_{i=1}^{g}(t-u_i)$$ $\deg f(t) = g+2$

$g+3$ punctures on $\mathbb{P}^1(0, 1, \infty, u_i)$

$t=0$

MUM

Let $\varphi(t)$ will be a solution.

Consider the following function

$$\Phi(\lambda_1, \dots, \lambda_g, x_1, \dots, x_g) = \prod_{i=1}^{g} \varphi(x_i).$$

multispectral problem

Théorèm (Enriquez, VR)

The set of operators

$$\mathcal{M}_i := \sum_{j=1}^{g} \frac{\prod_{k\neq j}(t-x_k)\big|_{t=0}}{\prod_{k\neq j}(x_j-x_k)} \mathcal{L}_{x_j}.$$

Consists of pair-wise commuting:

$\underline{[\mathcal{M}_i, \mathcal{M}_j] = 0}$. The function

$\Phi(x_1, \dots, x_g)$ is the joint eigen-function

$$\boxed{\mathcal{M}_i \Phi = \lambda_i \Phi}$$

λ_i - joint spectrum

Example Five singular points:

$g=2$, $\mathcal{L}_t = f\partial^2 + f'\partial + \frac{9}{4}t^2$

$f(t) = t(t-1)(t-u_1)(t-u_2)$

Commutative family is

$$\{(\mathcal{M}_1, \mathcal{M}_2)\} \quad \begin{cases} \mathcal{M}_1 = \frac{y}{x-y}\mathcal{L}_x + \frac{x}{y-x}\mathcal{L}_y \\ \mathcal{M}_2 = \frac{1}{x-y}\mathcal{L}_x + \frac{1}{y-x}\mathcal{L}_y, \end{cases}$$

Quantum Schlesinger hamiltonians;

Link to the Master Formula:

Choose an analytic solution $\varphi_{(t=0)}$ at

Multiplication formula:

$$\varphi(x_0, \bar\lambda)\,\varphi(x_1, \bar\lambda)\cdots\varphi(x_g, \bar\lambda) =$$

$$= \int K(x_0, x_1, \dots, x_g \mid y_1, \dots, y_g)\,\varphi(y_1, \bar\lambda)\cdots\varphi(y_g, \bar\lambda)\,dy_1\cdots dy_g =$$

$$= \int K(x_0, x_1, \dots, x_g \mid y_1, \dots, y_g)\,\Phi(\bar y)\,d\bar y$$

Solution: $\varphi(\lambda_1, \dots, \lambda_g, x_0)$, Master formula!

$$\varphi(\mu_1, \dots, \mu_g, x_0)\, \Phi_g(x_1, \dots, x_g) =$$

$$= \varphi(\mu_1, \dots, \mu_g, x_0) \prod_{i=1} \varphi(x_i) = \prod_{i=1} \varphi(x_i)$$

order 2

Hence, the Kernel

$$K(x_0, x_1, \dots, x_g, y_1, \dots, y_g) \in$$

$$\mathbb{C}\left[x_0, x_1, \dots, x_g, \frac{1}{x_1 - y_1}, \frac{1}{x_2 - y_2}, \dots, \frac{1}{x_g - y_g}\right]$$

A - nc, 1, ass. $\quad \forall\, a \neq 0\ a \in A$ $\exists\, a^{-1} a = 1$ or $a a^{-1} = 1$ ...

$\mathcal{R}(A) \Rightarrow$ Skew-field.

analogue of fractions.

$B_i \subset A \quad B_i$ - sub/alg. nc.

n sub. $1 \le i \le n$.

$b_i \in B_i \quad b_j \in B_j \Rightarrow [b_i, b_j] = 0$

$$M = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \begin{array}{l} \in B_1 \\ \dots \\ \in B_n \end{array}$$

Cartier - Foata matrix

$\Rightarrow$ det M in the usual way!

$$\begin{pmatrix} b_{10} & - & - & - & - \\ b_{20} & - & - & - & \\ \vdots & & & & \\ b_{n0} & & & & \end{pmatrix} \quad n \times (n+1)$$

$\underbrace{M_0}, M_1 \dots M_n$

$\exists M_0^{-1} \qquad \underbrace{H_i := M_0^{-1} M_i}$

Th. (B.E.V.R. 2003, Duke ..)

$[H_i, H_j] = 0 \Rightarrow$ Poisson version

$\underline{\text{IS}} \qquad H = \sum p_i^2 + V(\ ) \rightsquigarrow$ natural metric

$\underline{\text{Stäckel}} \Rightarrow$ SoV . ...