Classification of irreducible $\mathfrak{sl}_n$-valued zero curvature representations

Peter Sebestyén

Mathematical Institute, Silesian University in Opava,
Czech Republic

`e-mail: Peter.Sebestyen@math.slu.cz`

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Preliminaries

We consider a system of nonlinear differential equations

$$F^\ell(t, x, u^k, \dots, u^k_I, \dots) = 0. \tag{1}$$

in two independent variables t and x, a finite number of dependent variables u^k and their derivatives u^k_I, where I denotes a finite symmetric multiindex over t and x. Moreover, we consider an infinite-dimensional jet space J^∞ such that t, x, u^k, u^k_I are local jet coordinates on it. We have two distinguished vector fields on J^∞:

$$D_t = \frac{\partial}{\partial t} + \sum_{k,I} u^k_{It} \frac{\partial}{\partial u^k_I}, \qquad D_x = \frac{\partial}{\partial x} + \sum_{k,I} u^k_{Ix} \frac{\partial}{\partial u^k_I},$$

which are called *total derivatives.*

Let $D_I = D_{i_1} \cdots D_{i_\kappa}$. Then $\sum_{\ell,I} D_I F^\ell = 0$ represent an equation manifold $\mathcal{E}$ as a submanifold of J^∞.

Let $\mathfrak{g}$ be a matrix Lie algebra. A $\mathfrak{g}$-valued *zero curvature representation* (ZCR) for equations (1) is a pair (A, B) of $\mathfrak{g}$-valued functions on J^∞, which satisfy

$$D_t A - D_x B + [A, B] = 0, \tag{2}$$

when restricted to $\mathcal{E}$.

By (2) we mean that there exists $\mathfrak{g}$-valued functions K_ℓ^I on J^∞ such that

$$D_t A - D_x B + [A, B] = \sum_{\ell, I} D_I F^\ell \cdot K_\ell^I. \tag{3}$$

Let G be the connected and simply connected Lie group associated with $\mathfrak{g}$. Then for every G-valued function W we define the *gauge transformation* of ZCR (A, B) by the formulas

$$\begin{aligned} A^W &:= D_x W \cdot W^{-1} + W \cdot A \cdot W^{-1}, \\ B^W &:= D_t W \cdot W^{-1} + W \cdot B \cdot W^{-1}. \end{aligned}$$

(A^W, B^W) is a ZCR again, and we say that it is *gauge equivalent* to (A, B).

We consider the differential operator $\widehat{D}_I$ defined on J^∞ by

$$\widehat{D}_x M = D_x M - [A, M], \qquad \widehat{D}_t M = D_t M - [B, M],$$

where M is arbitrary $\mathfrak{g}$-matrix, $\widehat{D}_I = \widehat{D}_{i_1} \cdots \widehat{D}_{i_\kappa}$

Definition 1 ([Marvan],[Sakovich]) Let $\mathfrak{g}$-matrices K_ℓ^I satisfy

$$D_t A - D_x B + [A, B] = \sum_{\ell, I} D_I F^\ell \cdot K_\ell^I.$$

Put

$$C_\ell = \sum_I (-\widehat{D})_I K_\ell^I.$$

Then C_ℓ, restricted to $\mathcal{E}$, is the *characteristic element* for ZCR (A, B).

Proposition 1 ([Marvan],[Sakovich]) Gauge equivalent ZCR's have *conjugate* characteristic elements.

Definition 2 If a ZCR (A, B) is gauge equivalent to another ZCR with values in a proper subalgebra of $\mathfrak{g}$, then we say that the ZCR is *reducible.* Otherwise it is said to be *irreducible.* A ZCR gauge equivalent to zero is called *trivial.* A ZCR gauge equivalent to $(A, 0)$ or $(0, B)$ is called *quasi-trivial.*

Normal forms of irreducible $\mathfrak{sl}_n$-valued ZCR

Consider an $\mathfrak{sl}_n$-valued ZCR (A, B). Using Definition 1 we compute its characteristic element C. Following Proposition 1, we consider the stabilizer group H_J of the Jordan normal form J with respect to conjugation. The stabilizer H_J is a proper subgroup of SL_n. We compute its action on the matrix A and find the corresponding normal forms.

We denote

$$\begin{aligned} \widehat{A} := A^W &= D_x W \cdot W^{-1} + W \cdot A \cdot W^{-1} \\ \widehat{a}^i_j &= D_x w^i_s \cdot \bar{w}^s_j + w^i_s \cdot a^s_r \cdot \bar{w}^r_j, \end{aligned}$$

where $\widehat{a}^i_j$ (resp. w^i_j) are elements of $\widehat{A}$ (resp. $W \in H_J$) and $\bar{w}^i_j$ are elements of $W^{-1} \in H_J$.

We solve algebraic equations

$$\widehat{a}^i_j = c, \quad (c = 0 \text{ or } 1).$$

Normal forms of irreducible $\mathfrak{sl}_2$-valued ZCR

([Marvan])

Normal form of C	Normal form of A
$R_1 = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}$	$M_1 = \begin{pmatrix} a_1 & a_3 \\ 1 & -a_1 \end{pmatrix}$
$R_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$M_2 = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}$

Table 1.

Normal forms of irreducible $\mathfrak{sl}_3$-valued ZCR

Normal form of C	Stabilizer W	Normal forms of A
$\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & -\lambda_1-\lambda_2\end{pmatrix}$ $\lambda_1 \neq \lambda_2$	$\begin{pmatrix}w_1 & 0 & 0\\ 0 & w_2 & 0\\ 0 & 0 & (w_1 w_2)^{-1}\end{pmatrix}$	$\begin{pmatrix}. & . & .\\ 1 & . & .\\ . & 1 & .\end{pmatrix}$
$\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & -2\lambda\end{pmatrix}$ $\lambda \neq 0$	$\begin{pmatrix}w_{11} & w_{12} & 0\\ w_{21} & w_{22} & 0\\ 0 & 0 & w_{33}^{-1}\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ . & . & 1\\ . & . & .\end{pmatrix}$

Normal form of C	Stabilizer W	Normal forms of A
$\begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}$ $\lambda \neq 0$	$\begin{pmatrix} w_1 & 0 & 0 \\ w_2 & w_1 & 0 \\ 0 & 0 & w_1^{-2} \end{pmatrix}$	$\begin{pmatrix} . & . & 1 \\ . & . & 0 \\ . & . & . \end{pmatrix}$, $\begin{pmatrix} 0 & . & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} w_1 & 0 & 0 \\ w_2 & w_1 & w_3 \\ w_4 & 0 & w_1^{-2} \end{pmatrix}$	$\begin{pmatrix} 0 & . & 0 \\ . & . & 1 \\ . & 0 & . \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ . & 0 & 0 \\ . & . & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ w_2 & 1 & 0 \\ w_3 & w_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & . \\ . & . & . \\ . & . & . \end{pmatrix}$, $\begin{pmatrix} 0 & . & 0 \\ . & . & . \\ . & 0 & . \end{pmatrix}$

Table 2.

On normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations

P. Sebestyen, On normal forms of irreducible $\mathfrak{sl}_n$-valued zero curvature representations *Reports on Mathematical Physics*, **62** (2008), 57–68.

In this work we consider ZCR's for which the Jordan normal form J of the corresponding characteristic element C consists of a single cell. Then J and an arbitrary matrix W of the corresponding stabilizer subgroup H_J are

$$J = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ w_1 & 1 & \dots & 0 & 0 \\ w_2 & w_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{n-1} & w_{n-2} & \dots & w_1 & 1 \end{pmatrix}.$$

Here w_j are parameters of $(n-1)$-dimensional subgroup H_J. Denoting by w^i_j elements of the matrix W we have $w_k = w^l_{l-k}$ for all l.

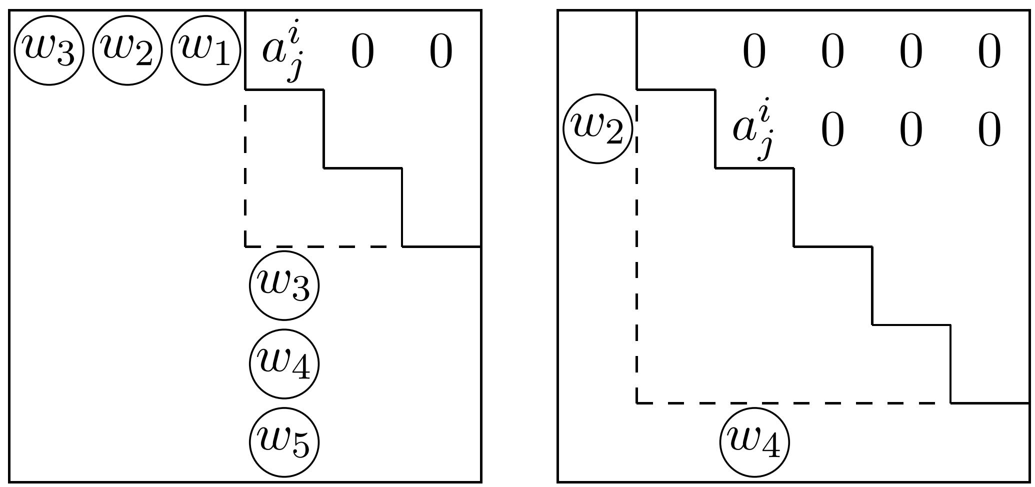
We denote

$$\begin{aligned} \widehat{A} := A^W &= D_x W \cdot W^{-1} + W \cdot A \cdot W^{-1} \\ \widehat{a}^i_j &= D_x w^i_s \cdot \bar{w}^s_j + w^i_s \cdot a^s_r \cdot \bar{w}^r_j, \end{aligned}$$

where $\widehat{a}^i_j$ (resp. w^i_j) are elements of $\widehat{A}$ (resp. W) and $\bar{w}^i_j$ (resp. $\bar{w}_j$) are elements of W^{-1} (resp. parameters of W^{-1} in H_J).

Lemma 1 *Let $a^i_j \neq 0, i < j$ be an element of A such that all elements $a^k_l, 1 \leq k \leq i, n \geq l \geq j$ except a^i_j are zero. Then $\widehat{a}^i_{j-t} = a^i_j w_t + f^i_{j-t}$ for all $t = i, \ldots, j-1$ and $\widehat{a}^{i+t}_j = a^i_j w_t + g^{i+t}_j$ for all $t = n+1-j, \ldots, n-i$. The expressions f^i_{j-t}, g^{i+t}_j do not depend on w_s, $s \geq t$.*

The next picture shows two possible situations in Lemma 1.



$n = 6$ $n = 6$

w_t denotes a linear polynomial in w_t, independent of w_s, $s > t$.

We use again the automorphism of $\mathfrak{sl}_n$:

$$A \mapsto -P \cdot A^\top \cdot P^{-1} =: A^*,$$

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix},$$

which can be described by the rule: $a_k^l \mapsto -a_r^s, r = (n+1) - l, s = (n+1) - k$. In this case the Jordan normal form of the characteristic element of the ZCR (A^*, B^*) is again J.

Definition 3 We say that the ZCR (A, B) is equivalent with a ZCR (C, D) and write $(A, B) \sim (C, D)$, if (A, B) is gauge equivalent with the ZCR (C, D) or with the ZCR (C^*, D^*).

Proposition 2 *The relation $\sim$ is reflexive, symmetric and transitive.*

Construction 1 Let $((i_1, j_1), \ldots, (i_m, j_m))$ be an m-tuple of couples of natural numbers , $1 \le m \le \lfloor n/2 \rfloor$, satisfying the following inequalities:

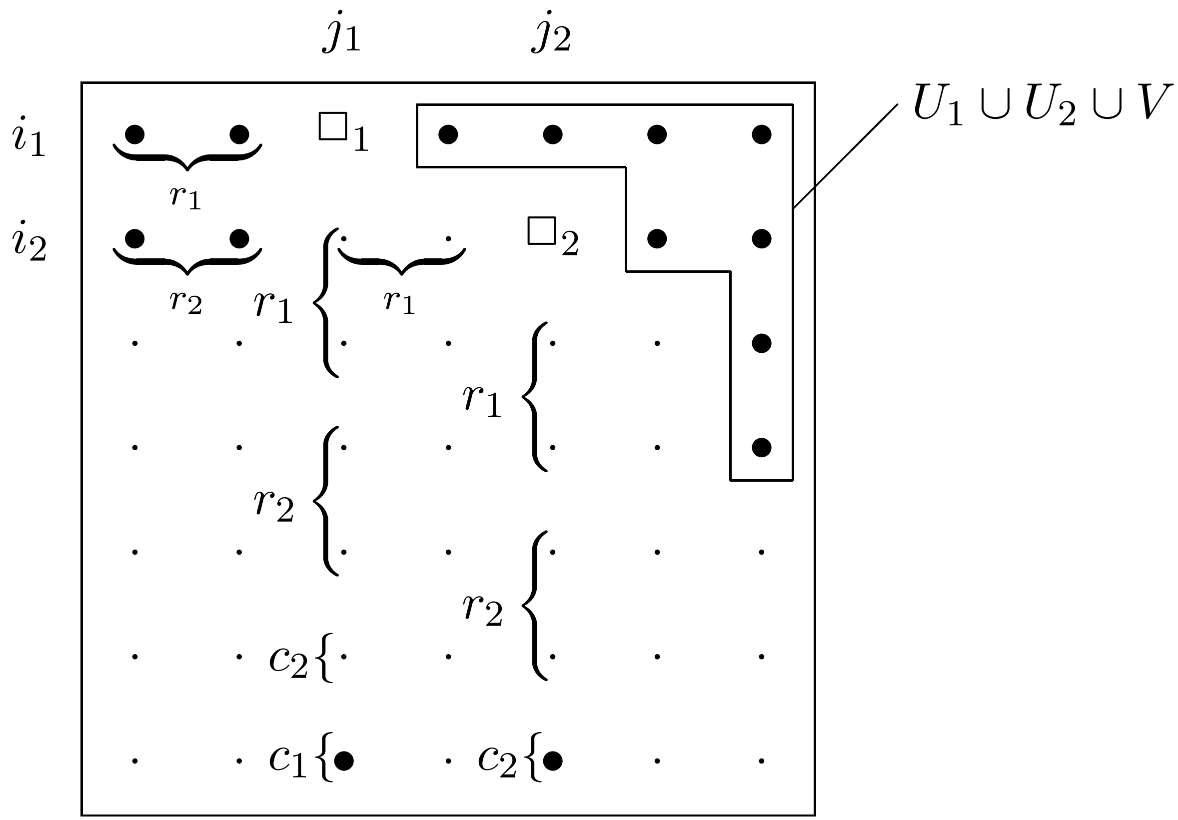
$$1 = i_1 < i_2 < \cdots < i_m \le \lfloor n/2 \rfloor,$$
$$1 < j_1 < j_2 < \cdots < j_m \le n,$$
$$i_\alpha < j_\alpha \text{ for } \alpha = 1, \ldots, m,$$
$$j_m > \lfloor n/2 \rfloor,$$
$$j_m \le n - 1 \text{ and } j_\alpha \le \lfloor n/2 \rfloor \text{ for } \alpha = 1, \ldots, m-1 \text{ and } m > 1.$$

We construct the type $N^{i_1,\ldots,i_m}_{j_1,\ldots,j_m}$ as the union of the sets:

$$\begin{aligned}
U_\alpha &:= \{(k,l) \mid k \le i_\alpha, l \ge j_\alpha\} \setminus \{(i_\alpha, j_\alpha)\}, \\
V &:= \{(k,n) \mid k < n + 1 - i_1\}, \\
P_\alpha &:= \{(i_\alpha, q) \mid q = 1, \ldots, j_\alpha - j_{\alpha-1}\}, \\
Q_\alpha &:= \{(p, j_\alpha) \mid p = n + 1 - (\widehat{i}_{\alpha+1} - i_\alpha), \ldots, n\},
\end{aligned}$$

where $\alpha = 1, \ldots, m$, $j_0 := 1$, $i_{m+1} := n + 1 - j_m$ and $\widehat{i}_\alpha := min(i_\alpha, n + 1 - j_m)$. If $\widehat{i}_{\alpha+1} - i_\alpha < 1$ then $Q_\alpha = \emptyset$.

The next picture shows one of the possible cases in $\mathfrak{sl}_7$:



Here $\bullet \in N^{i_1,i_2}_{j_1,j_2}$, $\square_\alpha = (i_\alpha, j_\alpha)$, further $r_\alpha = \#P_\alpha$ and $c_\alpha = \#Q_\alpha$.

Definition 4 A *type* is a subset N of the set $\{(i,j) \mid i = 1,\dots,n, j = 1,\dots,n\}$. A matrix A is said to be of type N if for every couple $(i,j) \in N$ we have $a^i_j = 0$.

Definition 5 We say, that a ZCR (A,B) with single cell of Jordan normal form of its characteristic element is in a *normal form*, if the matrix A is of one of the types $N^{i_1,\dots,i_m}_{j_1,\dots,j_m}$ from Construction 1. For shortness, we denote the normal form by the same symbol $N^{i_1,\dots,i_m}_{j_1,\dots,j_m}$.

Theorem 1 *Let (A,B) be an irreducible $\mathfrak{sl}_n$-valued ZCR with Jordan normal form of the corresponding characteristic element equal to J. Then A either belongs to one of the subalgebras*

$$L_k = \{(a^i_j) \in \mathfrak{sl}_n \mid a^i_j = 0 \text{ for all } i = 1,\dots,k, j = k+1,\dots,n\}, \tag{4}$$

where $k = 1,\dots,n-1$, or is equivalent to a matrix in the normal form.

Proof of Theorem 1

We introduce a procedure of computation of the gauge matrix W wich sends the matrix A to the corresponding normal form by gauge transformation. The procedure is a sequence of three simple algorithms.

In Algorithm 1 we choose between A and A^* to lower the number of normal forms.

In Algorithm 2 we establish the m-tuple from Construction 1.

Algorithm 3 computes the parameters of the gauge matrix W and returns the normal form $N^{i_1,\ldots,i_m}_{j_1,\ldots,j_m} := A^W$.

Algorithm 1 Choose between A and A^*

Input: $A \in \mathfrak{sl}_n$. *Output*: A.

1: Find the highest column index k in the first row of A such that $a^1_k \neq 0$; if all elements in the first row are zero then put $k = 0$
2: Find the highest column index l in the first row of A^* such that ${a^*}^1_l \neq 0$; if all elements in the first row are zero then put $l = 0$
3: **if** $k < l$ **then**
4: $\quad A := A^*$
5: **end if**
6: **return** A

Algorithm 2 Find the m-tuple $((i_1, j_1), \ldots, (i_m, j_m))$

Input: $A \in \mathfrak{sl}_n$. *Output*: m-tuple $((i_1, j_1), \ldots, (i_m, j_m))$.

$m := 0$, $i_0 := 0$ and $j_0 := 1$
repeat
    $r := i_m$
    repeat
        $r := r + 1$
        Find the highest column index k in the r-th row of A such that $a^r_k \neq 0$; if all elements in the r-th row are zero then put $k = 0$
    until $k > j_m$ or $r > \lfloor n/2 \rfloor$
    $m := m + 1$
    $(i_m, j_m) := (r, k)$
    if $i_m \geq j_m$ or $i_m > j_{m-1}$ **then**
        STOP: A in subalgebra L_{i_m} or $L_{i_m - 1}$
    end if
until $j_m > \lfloor n/2 \rfloor$
return $((i_1, j_1), \ldots, (i_m, j_m))$

Algorithm 3 Compute the gauge matrix W and the normal form $N^{i_1,\ldots,i_m}_{j_1,\ldots,j_m}$

Input: m-tuple $((i_1, j_1), \ldots, (i_m, j_m))$. *Output*: $N^{i_1,\ldots,i_m}_{j_1,\ldots,j_m}$.

$j_0 := 1$, $i_{m+1} := n + 1 - j_m$
for $\alpha = 1, \ldots, m$ **do**
  for $t = j_{\alpha-1}, \ldots, j_\alpha - 1$ **do**
    solve $\widehat{a}^{i_\alpha}_{j_\alpha - t} = 0$ for w_t and insert w_t back into W
  end for
end for
for $\alpha = m, \ldots, 1$ **do**
  if $i_\alpha \geq n + 1 - j_m$ **then**
    $i_\alpha := n + 1 - j_m$
  else
    for $t = n + 1 - i_{\alpha+1}, \ldots, n - i_\alpha$ **do**
      solve $\widehat{a}^{i_\alpha + t}_{j_\alpha} = 0$ for w_t and insert w_t back into W
    end for
  end if
end for
$N^{i_1,\ldots,i_m}_{j_1,\ldots,j_m} := A^W$
return $N^{i_1,\ldots,i_m}_{j_1,\ldots,j_m}$

In [5] we showed that Algorithm 2 always ends. We also proved the correctness of Algorithm 3.

As a conclusion we can say that our algorithms find for every $\mathfrak{sl}_n$ matrix A the corresponding normal form, or stop when A belongs to some subalgebra L_i.

Remark 1 The normal forms constructed in this paper are not the only possible. Lemma 1 allows us to construct different normal forms. But every normal form of the other type will be equivalent (in sense of Definition 3) with some normal form of our type.

Remark 2 Normal forms of a ZCR with the characteristic element possessing a single Jordan cell provide a base for the classification of all ZCRs. This, however, requires further research.

Examples

Example 1 The set of normal forms for $\mathfrak{sl}_5$:

$$N_5^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & a_5^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, N_4^1 = \begin{pmatrix} 0 & 0 & 0 & a_4^1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot \end{pmatrix}, N_3^1 = \begin{pmatrix} 0 & 0 & a_3^1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \end{pmatrix}$$

$$N_{2,4}^{1,2} = \begin{pmatrix} 0 & a_2^1 & 0 & 0 & 0 \\ 0 & 0 & \cdot & a_4^2 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot \end{pmatrix}, \quad N_{2,3}^{1,2} = \begin{pmatrix} 0 & a_2^1 & 0 & 0 & 0 \\ 0 & \cdot & a_3^2 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & \cdot & \cdot \end{pmatrix}.$$

Example 2 The Kupershmidt equation [1]:

$$u_t = u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x \,.$$

The matrix A of the corresponding ZCR has the normal form N_2^1, namely,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -u & 0 & 1 \\ m & -u & 0 \end{pmatrix}, \quad A^W = \begin{pmatrix} 0 & 1 & 0 \\ -2u & 0 & 1 \\ u_x + m & 0 & 0 \end{pmatrix}.$$

Normal forms of irreducible $\mathfrak{sl}_4$-valued zero curvature representations

Unpublished results

Jordan normal form of C	Stabilizer W
$J_{11} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$ $\lambda_1 \neq \lambda_2 \neq \lambda_3,\ \lambda_4 = -\lambda_1 - \lambda_2 - \lambda_3$	$W_{11} = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ 0 & w_{22} & 0 & 0 \\ 0 & 0 & w_{33} & 0 \\ 0 & 0 & 0 & w_{44} \end{pmatrix}$ $w_{44} = 1/w_{11}w_{22}w_{33}$
$J_{12} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$ $\lambda_1 \neq \lambda_2, \quad \lambda_3 = -2\lambda_1 - \lambda_2$	$W_{12} = \begin{pmatrix} w_{11} & w_{12} & 0 & 0 \\ w_{21} & w_{22} & 0 & 0 \\ 0 & 0 & w_{33} & 0 \\ 0 & 0 & 0 & w_{44} \end{pmatrix}$ $w_{44} = 1/(w_{11}w_{22} - w_{12}w_{21})w_{33}$
$J_{13} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -3\lambda \end{pmatrix}$ $\lambda \neq 0$	$W_{13} = \begin{pmatrix} w_{11} & w_{12} & w_{13} & 0 \\ w_{21} & w_{22} & w_{23} & 0 \\ w_{31} & w_{32} & w_{33} & 0 \\ 0 & 0 & 0 & w_{44} \end{pmatrix}$ $w_{44} = 1/det(W_{3\times 3})$

$J_{21} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$ $\lambda_3 = -2\lambda_1 - \lambda_2$	$W_{21} = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ w_{21} & w_{11} & 0 & 0 \\ 0 & 0 & w_{33} & 0 \\ 0 & 0 & 0 & w_{44} \end{pmatrix}$ $w_{44} = 1/w_{11}^2 w_{33}$
$J_{22} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -3\lambda \end{pmatrix}$	$W_{22} = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ w_{21} & w_{11} & w_{23} & 0 \\ w_{31} & 0 & w_{33} & 0 \\ 0 & 0 & 0 & w_{44} \end{pmatrix}$ $w_{44} = 1/w_{11}^2 w_{33}$
$J_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$W_{23} = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ w_{21} & w_{11} & w_{23} & w_{24} \\ w_{31} & 0 & w_{33} & w_{34} \\ w_{41} & 0 & w_{43} & w_{44} \end{pmatrix}$
$J_{31} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 \\ 0 & 0 & 1 & -\lambda_1 \end{pmatrix}$	$W_{31} = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ w_{21} & w_{11} & 0 & 0 \\ 0 & 0 & 1/w_{11} & 0 \\ 0 & 0 & w_{43} & 1/w_{11} \end{pmatrix}$
$J_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$W_{32} = \begin{pmatrix} w_{11} & 0 & w_{13} & 0 \\ w_{21} & w_{11} & w_{23} & w_{13} \\ w_{31} & 0 & w_{33} & 0 \\ w_{41} & w_{31} & w_{43} & w_{33} \end{pmatrix}$

$J_{41} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 \\ 0 & 1 & \lambda_1 & 0 \\ 0 & 0 & 0 & -3\lambda_1 \end{pmatrix}$	$W_{41} = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ w_{21} & w_{11} & 0 & 0 \\ w_{31} & w_{21} & w_{11} & 0 \\ 0 & 0 & 0 & 1/w_{11}^3 \end{pmatrix}$
$J_{42} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$W_{42} = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ w_{21} & w_{11} & 0 & 0 \\ w_{31} & w_{21} & w_{11} & w_{34} \\ w_{41} & 0 & 0 & 1/w_{11}^3 \end{pmatrix}$
$J_{51} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$W_{51} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ w_{21} & 1 & 0 & 0 \\ w_{31} & w_{21} & 1 & 0 \\ w_{41} & w_{31} & w_{21} & 1 \end{pmatrix}$

Table 3.

C	Normal form of A
J_{11}	$\begin{pmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ . & . & . & . \end{pmatrix}$ $\quad$ $\begin{pmatrix} . & 0 & 0 & 1 \\ 0 & . & 0 & 1 \\ 0 & 0 & . & 1 \\ . & . & . & . \end{pmatrix}$
J_{12}	$\begin{pmatrix} 0 & 1 & 0 & . \\ . & . & 1 & . \\ . & . & . & 1 \\ . & . & . & . \end{pmatrix}$ $\quad$ $\begin{pmatrix} . & 0 & . & 0 \\ 0 & . & . & . \\ 0 & 1 & . & . \\ 1 & 1 & . & . \end{pmatrix}$
J_{13}	?
J_{21}	$\begin{pmatrix} . & . & 1 & . \\ . & . & 0 & . \\ . & . & . & . \\ . & 1 & . & . \end{pmatrix}$ $\quad$ $\begin{pmatrix} . & . & 0 & . \\ . & . & . & . \\ 1 & 0 & . & . \\ 0 & 1 & . & . \end{pmatrix}$ $\quad$ $\begin{pmatrix} 0 & . & 0 & 0 \\ . & . & 1 & . \\ . & 0 & . & . \\ 1 & 0 & . & . \end{pmatrix}$ $\quad$ $\begin{pmatrix} 0 & . & 0 & 0 \\ . & . & . & 0 \\ 1 & 0 & . & . \\ 0 & 0 & 1 & . \end{pmatrix}$
J_{22}	?
J_{23}	?

C	Normal form of A
J_{31}	$\begin{pmatrix} . & . & 0 & 1 \\ . & . & . & 0 \\ . & . & . & . \\ . & . & . & . \end{pmatrix}$ $\begin{pmatrix} . & . & 1 & 0 \\ . & . & 0 & . \\ . & 0 & 0 & . \\ . & . & . & . \end{pmatrix}$
J_{32}	?
J_{41}	$\begin{pmatrix} . & . & . & 1 \\ . & . & . & 0 \\ . & . & . & 0 \\ . & . & . & . \end{pmatrix}$ $\begin{pmatrix} . & 0 & . & 0 \\ . & . & . & 1 \\ . & . & 0 & . \\ . & . & 0 & . \end{pmatrix}$ $\begin{pmatrix} . & . & 0 & 0 \\ . & . & . & 1 \\ . & 0 & . & 0 \\ . & . & 0 & . \end{pmatrix}$ $\begin{pmatrix} 0 & . & 0 & 0 \\ 0 & . & . & 0 \\ . & . & . & . \\ 1 & 0 & 0 & . \end{pmatrix}$
J_{42}	?
J_{51}	$\begin{pmatrix} 0 & 0 & 0 & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & . & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & 0 & . \end{pmatrix}$ $\begin{pmatrix} 0 & . & 0 & 0 \\ 0 & . & . & 0 \\ . & . & . & . \\ . & 0 & . & . \end{pmatrix}$

Table 4.

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