THE 1^{st} SUMMER SCHOOL ON GEOMETRY OF DIFFERENTIAL EQUATIONS SILESIAN UNIVERSITY IN OPAVA SEPTEMBER 17–21, 2012 LECTURE NOTES OF THE ADVANCED COURSE: RECURSION OPERATORS

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Abstract. These *Lecture Notes* contain the basic material on geometric approach to recursion operators for symmetries of differential equations, including an introductory part on the geometry of jet spaces and infinite prolongations of partial differential equations, [1]. A link to the Jets software can be found at [2].

Contents

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1. Introduction: finite jets, equations, classical symmetries

A differential equation is a functional relation imposed on unknown functions, their arguments and derivatives up to a certain order. So, a naïve way of a geometrization of the equation, say $F(x, u, u', u'') = 0$, $u = u(x)$, will be to consider the hypersurface

$$\mathscr{E}_F = \{F(x, u_0, u_1, u_2) = 0\} \subset \mathbb{R}^4$$

in $\mathbb{R}^4$ with the coordinates x, u_0, u_1, u_2 corresponding to independent and dependent variables and derivatives of the latter. But how to reflect "physical meaning" of these coordinates? The answer is given by the construction of *jet spaces*.

1.1. **Jets on $\mathbb{R}^n$.** Consider the spaces $\mathbb{R}^n$ with the coordinates $x^1, \dots, x^n$ (independent variables) and $\mathbb{R}^m$ with the coordinates $u^1, \dots, u^m$. Unknown functions will be understood as smooth[1] sections of the trivial bundle $\pi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Denote by $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ the space of such functions and let $[f]^0 \subset \mathbb{R}^n \times \mathbb{R}^m$ denote the graph of $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$.

For any $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$, $x \in \mathbb{R}^n$, and $k \geq 0$, define the class

$$[f]^k_x = \{\, g \in C^\infty(\mathbb{R}^n, \mathbb{R}^m) \mid [g]^0 \text{ is tangent to } [f]^0 \text{ at } (x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^m \text{ with order } k \,\}$$

and define $J^k_x(\mathbb{R}^n, \mathbb{R}^m)$ as the set consisting of all classes $[f]^k_x$ for a given $x \in \mathbb{R}^n$. For any two functions f, g and a number $\alpha \in \mathbb{R}$ set

$$[f]^k_x + [g]^k_x = [f + g]^k_x, \qquad \alpha[f]^k_x = [\alpha f]^k_x.$$

Exercise 1. Prove that the above operations are well defined.

Thus $J^k_x(\mathbb{R}^n, \mathbb{R}^m)$ is a vector space.

Exercise 2. Prove that

$$\dim J^k_x(\mathbb{R}^n, \mathbb{R}^m) = m\binom{n+k}{k}.$$

Define the *jet space* $J^k(\mathbb{R}^n, \mathbb{R}^m)$ of order k by

$$J^k(\mathbb{R}^n, \mathbb{R}^m) = \bigcup_{x \in \mathbb{R}^n} J^k_x(\mathbb{R}^n, \mathbb{R}^m).$$

For any $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$, define its *$k$-jet*

$$j_k(f)\colon \mathbb{R}^n \to J^k(\mathbb{R}^n, \mathbb{R}^m)$$

by $j_k(f)(x) = [f]^k_x$ and endow $J^k(\mathbb{R}^n, \mathbb{R}^m)$ with the minimal topology under which all the maps $j_k(f)$ are continuous.

For any multi-index σ of length $l \leq k$, define the functions $u^j_\sigma\colon J^k(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$, $j = 1, \dots, m$, as follows. Let $\theta = [f]^k_x \in J^k(\mathbb{R}^n, \mathbb{R}^m)$. Then

$$u^j_\sigma(\theta) = \frac{\partial^{|\sigma|} f^j}{\partial x^\sigma},$$

where $f = (f^1, \dots, f^m)$.

[1]The word 'smooth' means of the class C^∞" everywhere below.

Exercise 3. Prove that all the u^j_σ are well defined and continuous, while the correspondence

$$\theta = [f]^k_x \mapsto (x^1, \dots, x^n, \dots, u^j_\sigma, \dots),$$

where $(x^1, \dots, x^n)$, are the coordinates of x, is an homeomorphism between $J^k(\mathbb{R}^n, \mathbb{R}^m)$ and the space $\mathbb{R}^n \times \mathbb{R}^N$, where N is the dimension of $J^k_x(\mathbb{R}^n, \mathbb{R}^m)$ (see Exercise 2).

Thus we can consider the jet spaces as smooth manifolds (diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^N$). In particular, $J^0(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^N$.

In what follows, we shall write J^k instead of $J^k(\mathbb{R}^n, \mathbb{R}^m)$, when the context is clear.

Exercise 4. Prove that the maps $j_k(f)$ are smooth.

Define the maps $\pi_k \colon J^k \to \mathbb{R}^n$ and $\pi_{k,l} \colon J^k \to J^l$, $k \geq l$, by

$$\pi_k([f]^k_x) = x \quad \text{and} \quad \pi_{k,l}([f]^k_x) = [f]^l_x$$

respectively.

Exercise 5. Prove that these maps are smooth surjections.

1.2. **The Cartan distribution.** Denote by $[f]^k \subset J^k$ the graph of $j_k(f)$. Take a point $\theta \in J^k$ and define $\mathscr{C}_\theta = \mathscr{C}^k_\theta \subset T_\theta J^k$ as the hull of all $T_\theta [f]^k$, where $[f]^k$ passes through θ. Thus, we obtain the correspondence

$$J^k \ni \theta \mapsto \mathscr{C}^k_\theta \subset T_\theta J^k. \tag{1}$$

This is a particular case of a general construction. Let M be a smooth manifold. A *distribution* $\mathscr{D}$ on M is a vector subbundle of the tangent bundle $TM \to M$. A distribution may be understood as a smooth correspondence $M \ni x \mapsto \mathscr{D}_x \subset T_x M$, where $\dim \mathscr{D}_x = \operatorname{rank} \mathscr{D}$ does not depend on x.

Exercise 6. Prove that $\mathscr{C}^k \colon \theta \mapsto \mathscr{C}^k_\theta$ is a distribution on $J^k(\mathbb{R}^n, \mathbb{R}^m)$ and

$$\operatorname{rank} \mathscr{C}^k = n + m \binom{n+k-1}{k-1}.$$

This is the *Cartan distribution.*

Essentially, there are two ways to describe a distribution locally. The first one is to indicate differential 1-forms that vanish on $\mathscr{D}$. The space of such forms is denoted by $\Lambda(\mathscr{D})$.

Exercise 7. Show that $\Lambda(\mathscr{C}^k)$ is generated by the forms

$$\omega^j_\sigma = du^j_\sigma - \sum_{i=1}^n u_{\sigma i}\, dx^i, \qquad |\sigma| < k, \quad j = 1, \dots, m. \tag{2}$$

They are called *Cartan* (or *contact*) *forms* on J^k.

Another, dual, way is to indicate those vector fields that lie in $\mathscr{D}$. We shall not distinguish between notation for the space of such vector fields and for $\mathscr{D}$ itself.

Exercise 8. Show that the Cartan distribution $\mathscr{C}^k$ spans the fields

$$D^{(k)}_{x^i} = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^i_\sigma}, \tag{3}$$

where summation is taken over all $j = 1, \dots, m$ and σ such that $|\sigma| < k$, and

$$V^j_\sigma = \frac{\partial}{\partial u^j_\sigma}, \qquad |\sigma| = k, \quad j = 1, \dots, m \tag{4}$$

A submanifold $N \subset M$ is an *integral manifold* of $\mathscr{D}$ if

$$T_\theta N \subset \mathscr{D}_\theta$$

for any $\theta \in N$.

Exercise 9. Show that N is an integral manifold if and only if $\omega|_N = 0$ for any $\omega \in \Lambda(\mathscr{D})$.

Exercise 10. Prove that a section $s\colon \mathbb{R}^n \to J^k(\mathbb{R}^n, \mathbb{R}^m)$ of the bundle π_k is of the form $j_k(f)$ if and only if $s^*(\omega^j_\sigma) = 0$ for all the Cartan forms ω^j_σ.

A distribution $\mathscr{D}$ is *integrable* if for any point $\theta \in M$ there exists a unique local integral manifold N_θ of dimension $\dim N_\theta = \operatorname{rank} \mathscr{D}$ that passes through this point.

Frobenius Theorem. *A distribution $\mathscr{D}$ on M is integrable if and only if one of the two equivalent conditions holds*:

(1) *$[X, Y] \in \mathscr{D}$ whenever X and $Y \in \mathscr{D}$,*
(2) *$d\omega(X, Y) = 0$ for all $\omega \in \Lambda(\mathscr{D})$ and X, $Y \in \mathscr{D}$.*

Exercise 11. Prove equivalence of (1) and (2).

If $X_1, \dots, X_l$, $l = \operatorname{rank} \mathscr{D}$, is a (local) basis in $\mathscr{D}$ then Condition (1) reads

$$[X_i, X_j] = \sum_{\alpha=1}^{l} a^\alpha_{ij} X_\alpha, \qquad 1 \le i < j \le l,$$

where $a_{ij} \in C^\infty(M)$. If $\omega_1, \dots, \omega_r$, $r = \dim M - \operatorname{rank} \mathscr{D}$, form a basis of $\Lambda(\mathscr{D})$ then Condition (2) is equivalent to

$$d\omega_i = \sum_{\alpha=1}^{r} \rho_{i\alpha} \wedge \omega_\alpha, \qquad i = 1, \dots, r,$$

where $\rho_{i\alpha}$ are 1-forms on M.

Exercise 12. Show that $\mathscr{C}^k$ is not integrable.

1.3. **Lie transformations.** The Cartan distribution is the main geometric structure on J^k; its symmetries, i.e., diffeomorphisms $G\colon J^k \to J^j$ such that

$$G_*(\mathscr{C}^k_\theta) = \mathscr{C}^k_{G(\theta)}, \qquad \theta \in J^k,$$

are called *Lie transformations*. A complete description of Lie transformations is given by

Lie-Bäcklund Theorem. *For any Lie transformation $G\colon J^k \to J^k$ there exists a unique Lie transformation $G^{(1)}\colon J^{k+1} \to J^{k+1}$ (its lift) such that the diagram*

$$\begin{array}{ccc} J^{k+1} & \xrightarrow{G^{(1)}} & J^{k+1} \\ \downarrow & & \downarrow \\ J^k & \xrightarrow{G} & J^k \end{array}$$

is commutative.

Any Lie transformation G is of the form:

- *For $m > 1$: $G_0^{(k)}$, where $G_0\colon J^0 \to J^0$ is an arbitrary diffeomorphism.*
- *For $m = 1$: $G_1^{(k-1)}$, where $G_1\colon J^1 \to J^1$ is an arbitrary Lie transformation of J^1,*

where, by induction, $G^{(s)} = (G^{(s-1)})^{(1)}$.

Lie transformations of J^1 are called *contact transformations*.

If ω^j_σ are the Cartan forms on J^k then a diffeomorphism G is a Lie transformation if and only if

$$G^*(\omega^j_\sigma) = \sum_{\alpha\tau} \lambda^{j\tau}_{\sigma\alpha} \omega^\alpha_\tau,$$

where $\lambda^{j\tau}_{\sigma\alpha} \in C^\infty(J^k)$.

Exercise 13. Let $\tilde{x}^i = f^i(x, u)$, $\tilde{u}^j = g(x, u)$ be a diffeomorphism of $J^0(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m$. Describe its lift to J^1 in coordinates.

Let X be a vector field on J^k. It is called a *Lie field* if the corresponding one-parameter group consists of Lie transformations. For Lie fields, an infinitesimal analog of the Lie-Bäcklund Theorem is valid. A field X is a Lie field if and only if

$$X(\omega^j_\sigma) = \sum_{\alpha\tau} \mu^{j\tau}_{\sigma\alpha}\omega^\alpha_\tau,$$

where $\mu^{j\tau}_{\sigma\alpha} \in C^\infty(J^k)$.

Exercise 14. Let $X = \sum_i a_i\partial/\partial x^i + \sum_i b_j\partial/\partial u^j$ be a vector field on J^0. Describe in coordinates its lift to J^1.

In the case $m = 1$, Lie fields on J^1 are called *contact fields*,

Exercise 15. Show that any contact field X is of the form

$$X_f = -\sum_{i=1}^n \frac{\partial f}{\partial u_i}\frac{\partial}{\partial x^i} + \left(f - \sum_{i=1}^n u_i\frac{\partial f}{\partial u_i}\right)\frac{\partial}{\partial u} + \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i} + u_i\frac{\partial f}{\partial u}\right)\frac{\partial}{\partial u_i}, \tag{5}$$

where f is an arbitrary smooth function on J^1 called the *generating function* (or *characteristic*) of X.

Obviously, Lie fields form a Lie algebra. In particular, given two contact fields X_f and X_g by Equation (5), one has

$$[X_f, X_g] = X_{\{f,g\}} \tag{6}$$

for some function $\{f, g\}$ which is called the *Jacobi bracket* of f and g.

Exercise 16. Compute the Jacobi bracket in coordinates.

1.4. **Equations and classical symmetries.** Since coordinates in J^k are "partial derivatives", a differential equation is natural to be understood as a submanifold $\mathscr{E} \subset J^k$.

Example 1. Let

$$F\left(x^1, x^2, u, \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2}\right) = 0$$

be a scalar equation in two independent variables. Then the manifold

$$\mathscr{E}_F = \{F(x^1, x^2, u, u_1, u_2) = 0\} \subset J^1(\mathbb{R}^2, \mathbb{R}^1)$$

is the corresponding geometric image of this equation.

Let $f\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)$ be a smooth function. Then f is a solution of an equation $F = 0$ of order k if and only if $[f]^k \subset \mathscr{E} = \mathscr{E}_F$. We shall assume that the system of function $F = (F^1, \dots, F^r)$ is *generic* in the following sense: for any $H \in C^\infty(J^k)$ the condition $H|_{\mathscr{E}_F} = 0$ implies $H = \sum_j \lambda_j F^j$.

Define the *Cartan distribution* on $\mathscr{E} \subset J^k$ by

$$\mathscr{C}_\theta(\mathscr{E}) = T_\theta\mathscr{E} \cap \mathscr{C}^k_\theta.$$

Then solutions can be understood as n-dimensional integral manifolds of $\mathscr{C}(\mathscr{E})$ that project surjectively to $\mathbb{R}^n$ by π_k.

Let $\mathscr{E} \subset J^k$ be an equation. A Liie transformation $G\colon J^k \to J^k$ is a (*classical*) *symmetry* of $\mathscr{E}$ if $G(\mathscr{E}) = \mathscr{E}$. When $\mathscr{E} = \mathscr{E}_F$, $F = (F^1, \dots, F^r)$, then G is a symmetry if and only if $G^*(F^j)\big|_{\mathscr{E}} = 0$ for all j, or

$$G^*(F^j) = \sum_\alpha \lambda^j_\alpha F^\alpha.$$

A Lie field X is an *infinitesimal symmetry* if is tangent to $\mathscr{E}$, which amounts to $X(F^j)\big|_{\mathscr{E}} = 0$, or equivalently

$$X(F^j) = \sum_\alpha \mu^j_\alpha F^j.$$

There is another, intrinsic, way to define symmetries: a diffeomorphism $G\colon \mathscr{E} \to \mathscr{E}$ is a symmetry if it preserves the restricted Cartan distribution $\mathscr{C}_{\mathscr{E}}$. Similarly, a vector field X on $\mathscr{E}$ is an infinitesimal symmetry if $[X, \mathscr{C}_{\mathscr{E}}] \subset \mathscr{C}_{\mathscr{E}}$. Of course, any symmetry in the first sense defines a one in the second, but not vice versa.

Exercise 17. Give examples of equations whose intrinsic symmetries are not induced by extrinsic ones.

It can be shown, see [1], that when the equation at hand is not "too overdetermined" (i.e., when $\operatorname{codim}\mathscr{E}$ is not "too big") intrinsic symmetries coincide with extrinsic ones, except for the cases $n = 1$ or $k = 1$.

1.5. **Globalization.** Until now, we considered jets of vector-valued smooth functions on $\mathbb{R}^n$ and differential equations imposed on such functions. Nevertheless, there exist examples of a more general, global nature.

Example 2. Let M be a smooth manifold. Then $d\omega = 0$, where $\omega \in \Lambda^1(M)$ is a 1-form, is a first order equation imposed on sections of the cotangent bundle.

Exercise 18. Find other "natural" examples of differential equations defined globally.

To pass from the above described local viewpoint to the global one, the following changes are needed:

Local version	**Global version**
The space $\mathbb{R}^n$	A smooth manifold M
The trivial bundle $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$	A vector bundle $\pi\colon E \to M$
$\mathbb{R}^m$-valued functions on $\mathbb{R}^n$	(Local) sections of π

Then the manifolds $J^k(\pi)$ are defined, and all other definitions and results remain valid. The space of sections will be denoted by $\Gamma(\pi)$ below.

Exercise 19. Give rigorous definitions and reformulate all the above results for the global case.

2. Infinite jets and higher symmetries

Equations exist together with their differential and algebraic consequences. The corresponding geometric setting is given by infinite jets and infinitely prolonged equations.

2.1. **Infinite jets.** Consider a vector bundle[2] $\pi\colon E \to M$, $\dim M = n$, $\dim E = n + m$. Then one has the following infinite sequence of smooth surjections

$$M \xleftarrow{\pi} E = J^0(\pi) \longleftarrow \dots \longleftarrow J^k(\pi) \xleftarrow{\pi_{k+1,k}} J^{k+1}(\pi) \longleftarrow \dots$$

Its inverse limit $J^\infty(\pi)$ is the space of *infinite jets* associated with π. The above sequence defines the embeddings (algebra monomorphisms) of the function algebras

$$C^\infty(M) \subset C^\infty(J^0) \subset \dots \subset C^\infty(J^k) \subset C^\infty(J^{k+1}) \subset \dots,$$

and we define $C^\infty(J^\infty) = \cup_k C^\infty(J^k)$. Thus, a smooth function on J^∞ is a smooth function of $x^1, \dots, x^n$ and arbitrary but *finite* number of u^j_σ. A point $\theta \in J^\infty$ may be understood as the class of tangency $[s]^\infty_x$ or as the Taylor series of s at $x \in M$. Evident surjections

$$\pi_\infty\colon J^\infty(\pi) \to M \text{ and } \pi_{\infty,k}\colon J^\infty(\pi) \to J^k(\pi)$$

are defined.

Let X be a vector field on M. Define the vector field $\mathscr{C}_X$ on $J^\infty(\pi)$ by

$$j^*_\infty(s)\big(\mathscr{C}_X(f)\big) = X(j^*_\infty(s)(f)), \qquad f \in C^\infty(J^\infty), \quad s \in \Gamma(\pi).$$

[2]Those ones who prefer the local picture can think of π as of the trivial bundle $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

Locally, for $X = a_1\partial/\partial x^1 + \dots a_n\partial/\partial x^n$,

$$\mathscr{C}_X = a_1 D_{x^1} + \dots a_n D_{x^n},$$

where

$$D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u^j_{\sigma i}\frac{\partial}{\partial u^j_\sigma}$$

are the *total derivatives*.

Exercise 20. Prove that

$$[\mathscr{C}_X, \mathscr{C}_Y] = \mathscr{C}_{[X,Y]} \tag{7}$$

for any two vector fields X and Y on M.

The *Cartan distribution* $\mathscr{C}$ on J^∞ is defined as the span of the fields $\mathscr{C}_X$. Due to (7), one has

$$[\mathscr{C}, \mathscr{C}] \subset \mathscr{C}.$$

Thus, keeping in mind the Frobenius Theorem, the Cartan distribution on J^∞ may be considered as "formally integrable".

Exercise 21. Show that $\operatorname{rank}\mathscr{C} = \dim M$ and every Cartan plane $\mathscr{C}_\theta$ is horizontal with respect to $\pi_\infty\colon J^\infty(\pi) \to M$.

Exercise 22. Show that integral manifolds of $\mathscr{C}$ of maximal dimension are graphs of infinite jets $[s]^\infty \subset J^\infty(\pi)$, $s \in \Gamma(\pi)$.

Thus, the Cartan distribution on J^∞ is *not* integrable in the classical sense, since there exists infinitely many maximal integral manifolds passing through a given point $\theta \in J^\infty$.

2.2. **Prolongations.** Let an equation $\mathscr{E} \subset J^k(\pi)$ be locally given by the relations $F^1 = 0, \dots, F^r = 0$, $F^j \in C^\infty(J^\infty)$. Its *lth prolongation* $\mathscr{E}^l \subset J^{k+l}$ is given by

$$D_\sigma(F^j) = 0, \qquad |\sigma| \le l, \quad j = 1, \dots, r,$$

where $D_\sigma = D_{x^{\sigma_1}} \circ \dots \circ D_{x^{\sigma_p}}$, $p = |\sigma|$, is the corresponding composition of total derivatives. One has the sequence

$$\mathscr{E} \longleftarrow \mathscr{E}^1 \longleftarrow \dots \longleftarrow \mathscr{E}^l \longleftarrow \mathscr{E}^{l+1} \longleftarrow \dots$$

Its inverse limit is denoted by $\mathscr{E}^\infty \subset J^\infty$ and called the *infinite prolongation* of $\mathscr{E}$. Points of the infinite prolongation are *formal solutions* of the equation. Not all the maps above may be surjective.

Exercise 23. Invent examples of equations for which $\mathscr{E}^{l+1} \to \mathscr{E}^l$ are not surjections.

In the case when all these maps are smooth surjections, the equation is called *formally integrable*. From now on, we deal with infinite prolongations of formally integrable equations only and shall denote them by $\mathscr{E}$, omitting the "infinity" superscript. We assume that the map $\mathscr{E} \to M$ is surjective also.

By the definition of the infinite prolongation, all total derivatives are tangent to $\mathscr{E}$; thus, the Cartan distribution is tangent to $\mathscr{E}$ and induces the distribution $\mathscr{C}(\mathscr{E})$ which is also (formally) integrable.

2.3. **Higher symmetries.** Theory of higher symmetries (symmetries of the Cartan distribution) on J^∞ is essentially infinitesimal due to infinite dimension of the manifold.

A *Lie field* on $J^\infty(\pi)$ is a vector field such that $[X, \mathscr{C}] \subset \mathscr{C}$. Since $\mathscr{C}$ is a formally integrable distribution, any $X \in \mathscr{C}$ is a Lie field and such fields are tangent to any graph $[s]^\infty \subset J^\infty$.

Exercise 24. Prove that $\mathscr{C}$ is an ideal in the Lie algebra of all Lie fields on J^∞.

Elements of the quotient algebra are called *symmetries* of $\mathscr{C}$. Let

$$X = \sum_i a_i \frac{\partial}{\partial x^i} + \sum_{j\sigma} b^j_\sigma \frac{\partial}{\partial u^j_\sigma}$$

be a Lie field. Then

$$X^v = X - \sum_i a_i D_{x^i}$$

is a Lie field as well which is vertical with respect to the projection π_∞. Thus, the algebra of symmetries can be identified with the algebra of π_∞-vertical Lie fields. These fields admit an explicit description.

Theorem. *There is a one-to-one correspondence between symmetries of the Cartan distribution on $J^\infty(\pi)$ and sections of the pull-back $\pi^*_\infty(\pi)$ locally given by*

$$\varphi \in \varkappa(\pi) = \Gamma(\pi^*_\infty(\pi)) \mapsto \mathbf{E}_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma}. \tag{8}$$

Fields of the form (8) are called *evolutionary derivations.*

Exercise 25. Let $\pi'\colon E' \to M$ be another vector bundle and $\mathbf{E}_\varphi$, $\varphi \in \varkappa(\pi)$, be an evolutionary derivation. Consider $s \in \Gamma(\pi^*_\infty(\pi'))$ locally presented in the form $s = (s^1, \dots, s^{m'})$, $m' = \dim \pi'$. Prove that the component-wise action

$$\mathbf{E}^{\pi'}_\varphi(s) = (\mathbf{E}_\varphi(s^1), \dots, \mathbf{E}_\varphi(s^{m'}))$$

is well defined globally.

For φ_1, $\varphi_2 \in \varkappa$ their (*higher*) *Jacobi bracket* is defined by

$$\{\varphi_1, \varphi_2\} = \mathbf{E}^\pi_{\varphi_1}(\varphi_2) - \mathbf{E}^\pi_{\varphi_2}(\varphi_1).$$

Exercise 26. Prove that

$$[\mathbf{E}_{\varphi_1}, \mathbf{E}_{\varphi_2}] = \mathbf{E}_{\{\varphi_1,\varphi_2\}}$$

for any φ_1, $\varphi_2 \in \varkappa$.

Thus, $\varkappa(\pi)$ is a Lie algebra with respect to the Jacobi bracket.

Consider the situation of Exercise 25. A section $s \in \Gamma(\pi^*_\infty(\pi'))$ is identified with a nonlinear differential operator acting from $\Gamma(\pi)$ to $\Gamma(\xi)$ by

$$f \in \Gamma(\pi) \mapsto \Delta_s(f) = j^*_\infty(s) \in \Gamma(\pi').$$

Its *linearization* $\ell_s\colon \varkappa(\pi) \to \Gamma(\pi^*_\infty(\pi'))$ is defined by

$$\ell_s(\varphi) = \mathbf{E}^{\pi'}_\varphi(s).$$

In coordinates, if $s = (s^1, \dots, s^m)$ then

$$\ell_s = \left\| \sum_\sigma \frac{\partial s^\alpha}{\partial u^\beta_\sigma} D_\sigma \right\|.$$

A Lie field is a (*higher infinitesimal*) *symmetry* of $\mathscr{E}$ if it is tangent to $\mathscr{E}$. They form a Lie algebra $\operatorname{sym}\mathscr{E}$ with respect to the Jacobi bracket.

Exercise 27. Let $\Delta\colon \Gamma(\pi^*_\infty(\pi')) \to \Gamma(\pi^*_\infty(\pi''))$ be a differential operator in total derivatives [3] and $\mathscr{E} \subset J^\infty(\pi)$ be an infinite prolongation. Show that any such a Δ can be restricted to $\mathscr{E}$, i.e., there exists a unique operator $\bar\Delta$ such that the diagram

$$\begin{array}{ccc} \Gamma(\pi^*_\infty(\pi')) & \xrightarrow{\Delta} & \Gamma(\pi^*_\infty(\pi'')) \\ \Big\downarrow{\scriptstyle \mathrm{r}} & & \Big\downarrow{\scriptstyle \mathrm{r}} \\ \Gamma(\pi^*_\infty(\pi')|_{\mathscr{E}}) & \xrightarrow{\bar\Delta} & \Gamma(\pi^*_\infty(\pi'')|_{\mathscr{E}}) \end{array}$$

[3]Such operators are called *total*, or *$\mathscr{C}$-differential* operators.

is commutative, where $\mathrm{r}(s) = s|_{\mathscr{E}}$.

Theorem. *Let $\mathscr{E}$ be given by $\{F = 0\}$ for $F \in \Gamma(\pi_\infty^*(\xi))$ and assume that $\mathscr{E}$ projects to $J^0(\pi)$ surjectively*[4]. *Then*

$$\operatorname{sym} \mathscr{E} = \ker \ell_{\mathscr{E}},$$

where $\ell_{\mathscr{E}} = \ell_F|_{\mathscr{E}}$.

Thus, to find symmetries of $\mathscr{E}$, one needs to solve the following system of linear equations in total derivatives

$$\sum \frac{\partial F^1}{\partial u_s^1} D_\sigma(\varphi^1) + \dots + \sum \frac{\partial F^1}{\partial u_s^m} D_\sigma(\varphi^m) = 0,$$
$$\dots$$
$$\sum \frac{\partial F^r}{\partial u_s^1} D_\sigma(\varphi^1) + \dots + \sum \frac{\partial F^r}{\partial u_s^m} D_\sigma(\varphi^m) = 0,$$

where the sums are taken over all the so-called *internal coordinates* (see examples below) on $\mathscr{E}$ and $\varphi = (\varphi^1, \dots, \varphi^m)$ is the unknown symmetry.

3. Examples: the Burgers and Korteweg-de Vries (KdV) equations

Consider two illustrative examples.

3.1. **Burgers equation.** The Burgers equation

$$u_t + uu_x = \nu u_{xx}$$

is popular to model processes of gas dynamics and fluid mechanics (ν being viscosity). When $\nu \neq 0$, it can be transformed to the form

$$u_t = uu_x + u_{xx}.$$

Internal coordinates on $\mathscr{E}$ may be chosen as

$$x,\ t,\ u_0 = u,\ u_1 = u_x,\ u_2 = u_{xx}, \dots, u_k, \dots$$

Then symmetries are defined by

$$D_t(\varphi) = u_1\varphi + uD_x(\varphi) + D_x^2(\varphi), \tag{9}$$

where

$$D_x = \frac{\partial}{\partial x} + \sum_k u_{k+1}\frac{\partial}{\partial u_k}, \quad D_t = \frac{\partial}{\partial t} + \sum_k D_x^k(uu_1 + u_2)\frac{\partial}{\partial u_k}.$$

Of course, this choice of internal coordinates is no unique. For example, an alternative choice is

$$x,\ t,\ u_{00} = u,\ u_{10} = u_t, \dots, u_{0k},\ u_{1k}, \dots,$$

where

$$u_{0k} = u_{\underbrace{t\dots t}_{k \text{ times}}}, \qquad u_{1k} = u_{x\underbrace{t\dots t}_{k \text{ times}}}.$$

Exercise 28. Write down total derivatives on $\mathscr{E}$ for this choice of internal coordinates.

Any solution $\varphi(x, t, u, \dots, u_k)$ of Equation (9) for $k > 1$ is of the form

$$\varphi_k[a] = au_k + \frac{1}{2}(\dot{a}x + kua + \bar{a})u_{k-1} + O(k-2),$$

a, $\bar{a}$ being functions of t, and

$$\{\varphi_k[a], \varphi_l[b]\} = \varphi_{k+l-2}[c], \qquad c = \frac{1}{2}(l\dot{a}b - k\dot{b}a).$$

[4] This only means that there is no functional relation in our equation.

Here and below $O(l)$ denotes a function on $\mathscr{E}$ that does not depend on $u_{l+1},\dots,u_k$ and "dot" is the t-derivative.

For small k we have explicit solutions

$$\varphi_1^0 = u_1, \quad \varphi_1^1 = tu_1 + 1,$$

$$\varphi_2^0 = u_2 + uu_1, \quad \varphi_2^1 = tu_2 + \Bigl(tu + \frac{1}{2}x\Bigr)u_1 + \frac{1}{2}u, \quad \varphi_2^2 = t^2u_2 + (t^2u + tx)u_1 + tu + x,$$

(these are classical symmetries) and also

$$\varphi_3^1 = tp_3 + \frac{1}{2}(x+3tu)u_2 + \frac{3}{2}tu_1^2 + \Bigl(\frac{1}{2}x + \frac{3}{4}tu\Bigr)uu_1 + \frac{1}{4}u^2.$$

From the identity

$$\{\varphi_k[a], \varphi_1^0\} = \frac{1}{2}\varphi_{k-1}[\dot a]$$

it follows that the coefficient a in $\varphi_k[a]$ is a polynomial in t of order $\le k$.

One has

$$\{\varphi_k[a], \varphi_3^1\} = \frac{1}{2}(2\dot a t - ka)u_{k+1} + O(k);$$

applying this formula to $\varphi_1^0 = u_1$ we obtain existence of symmetries

$$\varphi_k^0 = u_k + O(k-1).$$

On the other hand,

$$\{\varphi_k[a], \varphi_2^2\} = t(t\dot a - ka)u_k + O(k-1).$$

Applying this to φ_k^0 one obtains existence of symmetries

$$\varphi_k^i = t^iu_k + O(k-1), \qquad i = 0,\dots,k.$$

The computations above lead to the following result:

Proposition. *The algebra* $\operatorname{sym}\mathscr{E}$ *for the Burgers equation is infinite-dimensional and consists of the functions* $\varphi_k^i = t^iu_k + \dots$ *for all* $k \ge 1$ *and* $t = 0,1,\dots,k$*. As a Lie algebra it is generated by* φ_1^0*,* φ_2^2*, and* φ_3^1*. One also has*

$$\{\varphi_k^i, \varphi_l^i\} = \frac{1}{2}(li - kj)\varphi_{k+l-2}^{i+j-1} + \varphi_{k+l-3},$$

where φ_{k+l-3} *is a symmetry of order* $< k+l-2$*.*

Exercise 29. Prove that all symmetries of the Burgers equation are polynomial in all variables.

Let us assign the following weights

$$|x| = 1, \quad |t| = 2, \quad |u_k| = k-1$$

to the internal coordinate functions and for a monomial define its weight to be equal the sum of its factors weights. Due to Exercise 29, all the symmetries φ_k^i can be regarded as homogeneous with respect to these weights.

Exercise 30. Prove that the homogeneous symmetries φ_k^0 pair-wise commute.

3.2. **The Korteweg-de Vries (KdV) equation.** This famous equation describes waves on shallow water surfaces and is of the form[5]

$$u_t = uu_x + u_{xxx} \tag{10}$$

and the linearization is

$$D_t(\varphi) = u_1\varphi + uD_x(\varphi) + D_x^3(\varphi),$$

where

$$D_x = \frac{\partial}{\partial x} + \sum_k u_{k+1}\frac{\partial}{\partial u_k}, \quad D_t = \frac{\partial}{\partial t} + \sum_k D_x^k(uu_1 + u_3)\frac{\partial}{\partial u_k}.$$

[5] The equation is often written in the equivalent form $u_t + 6uu_x + u_{xxx} = 0$.

Exercise 31. Analyze other possible choices of internal coordinates and write down the corresponding formulas for the total derivatives.

Similar to Subsection 3.1, it can be shown that any symmetry is to be of the form

$$\varphi_{2k-1}[a] = au_{2k-1} + O(2k-2), \qquad k > 0,$$

$a = a(t)$ being a polynomial in t. Classical symmetries of KdV are

$$\begin{aligned}
&\varphi_1^0 = u_1, && x\text{-translation},\\
&\varphi_1^1 = tu_1 + 1, && \text{Galilean boost},\\
&\varphi_3^0 = u_3 + uu_1, && t\text{-translation},\\
&\varphi_3^1 = tu_3 + \Big(tu + \frac{1}{3}x\Big)u_1 + \frac{2}{3}u, && \text{scaling}.
\end{aligned}$$

Exercise 32. Prove that there exists no symmetry of order > 3 with $a \neq \mathrm{const}$.

But symmetries of the form $u_{2k-1} + \dots$ do exist for all $k \geq 1$ and this will be established in Section 5.

4. An application: 1-solitons as invariant solutions of the KdV equation

A solution s of an equation $\mathscr{E}$ is *invariant* with respect to a symmetry $\varphi \in \operatorname{sym}\mathscr{E}$ if the field $\mathbf{E}_\varphi$ is tangent to $[s]^\infty \subset \mathscr{E}$.

Exercise 33. Prove that to find φ-invariant solutions one is to solve the equation $\varphi = 0$ together with the initial one.

Both the Burgers and KdV equations possess x- and t-translation symmetries (u_x and u_t, respectively.). Solutions invariant with respect to the symmetry $cu_x + u_t$ are called *traveling waves*. They are of the form $u = u(x - ct)$, where $c = \mathrm{const}$ is the velocity. Let $x - ct = \tau$.

4.1. **Traveling waves of the Burgers equation.** The defining equation for traveling wave solutions is

$$-cu' = uu' + u'',$$

where "prime" denotes the τ-derivative. Hence,

$$\frac{2\,du}{(u-c)^2 \pm \alpha^2} = -\,d\tau,$$

where $\alpha \geq 0$ is the integration constant. There are three families of solutions:

$$\begin{aligned}
u^0 &= \frac{2}{x - ct + A} + c,\\
u^+ &= \alpha \cot \frac{\alpha(x - ct + A)}{2} + c,\\
u^- &= \alpha \frac{1 + \beta \exp(-\alpha(x - ct))}{1 + \beta \exp(-\alpha(x - ct))} + c,
\end{aligned}$$

where A, β are constants, $\beta \geq 0$.

4.2. 1**-soliton solutions of the KdV equation.** To comply with the standard notation (see Footnote 5), rescale the KdV equation by $x \mapsto -x$, $u \mapsto 6u$. Then the defining equation for traveling waves is

$$-cu' + 6uu' + u''' = 0.$$

Hence,

$$-cu + 3u^2 + u'' = A, \qquad A = \mathrm{const};$$

multiplying by u' and integrating, one obtains

$$-\frac{c}{2}u^2 + u^3 + \frac{1}{2}(u')^2 = Au + B, \qquad B = \mathrm{const}.$$

By physical reasons, u, u', $u'' \to 0$ as soon as $\tau \to \pm\infty$; consequently, $A = B = 0$ and

$$\int \frac{du}{u\sqrt{c-2u}} = \tau + a. \tag{11}$$

Using the substitution

$$u = \frac{c}{2}\operatorname{sech}^2 w,$$

we integrate (11) and get

$$u = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct + a)\right). \tag{12}$$

This is the (right moving) *soliton solution* of the KdV equation. It was observed and reported by John Scott Russel, see, e.g., [3] in 1845.

Exercise 34. Find other traveling wave solutions of the KdV equation.

5. Back to Section 3: Lenard's recursion operator

We shall now present a method to establish existence of symmetries $u_{2k-1} + \dots$ for the KdV equation, see Section 3.2. Namely, assume that there exists an operator R such that

$$\ell_{\mathscr{E}} \circ R = \bar{R} \circ \ell_{\mathscr{E}}, \tag{13}$$

where $\bar{R}$ is another operator. Then, obviously, for any $\varphi \in \operatorname{sym}\mathscr{E}$ one has $R(\varphi) \in \operatorname{sym}\mathscr{E}$. E.g., for the heat equation $u_t = u_{xx}$ any operator $R = \sum a_i D_x^i$, $a_i = \text{const}$, possesses this property. But linear equations with constant coefficients provide probably the only example with such a simple solution (see below).

The operator $\ell_{\mathscr{E}}$ is expressed in total derivatives and so both R and $\bar{R}$ are reasonable to be expressed in the same form. A $\mathscr{C}$-differential operator satisfying (13) is called a *recursion operator* for symmetries of $\mathscr{E}$.

Note that if $\mathscr{E}$ is an evolution equation then D_t may be excluded from the expression for R.

Exercise 35. Prove that $\bar{R} = R$ for evolution equations, provided R being a $\mathscr{C}$-differential operator in D_x.

5.1. Burgers equation.

Consider the solutions to (13) of the form $\sum_{k\ge 0} a_k D_x^k$ for the Burgers equation.

Exercise 36. Prove that the only solutions of this form are $R = \text{const}$.

Certainly, these solutions are not interesting. Nevertheless, in a wider setting, nontrivial solutions do exist.

Exercise 37. Prove that the operator

$$R = D_x + \frac{1}{2}u + \frac{1}{2}u_x D_x^{-1}$$

satisfies Equation (13).

In this particular case, it can be shown by elementary means (though it is not simple) that D_x^{-1} acts on symmetries in a well-defined way, but generally more complicated techniques are needed. We expose the latter briefly in Subsection 5.2.

5.2. Digression: conservation laws and cosymmetries.

We consider scalar evolution equations $\mathscr{E} = \{u_t - F(u) = 0\}$ for simplicity, where $F(u)$ is a smooth function of x, t, u, $u_x, \dots$

Let $\Delta = \sum_{i\ge 0} a_i D_x^i$. The operator

$$\Delta^* = \sum (-D_x)^i \circ a_i$$

is called the *formally adjoint* to Δ. The *Green formula* reads

$$(\Delta^*\psi)\cdot\varphi - \psi\cdot(\Delta\varphi) \in \operatorname{im} D_x. \tag{14}$$

Exercise 38. Prove formula (14).

The operator

$$\delta(a) = \ell_a^*(1) = \frac{\partial a}{\partial u} - D_x\left(\frac{\partial a}{\partial u_x}\right) + D_x^2\left(\frac{\partial a}{\partial u_{xx}}\right) - \dots$$

is called the *Euler operator*.

A 1-form $\omega = X\,dx + T\,dt$ is a *conservation law* of the equation $\mathscr{E}$ if

$$D_x(T) = D_t(X).$$

Indeed, assume that $\lim_{x\to\pm\infty} T = 0$. Then

$$\frac{d}{dt}\int_{-\infty}^{+\infty} X\,dx = \int_{-\infty}^{+\infty} D_t X\,dx = \int_{-\infty}^{+\infty} D_x T\,dx = \lim_{x\to+\infty} T - \lim_{x\to-\infty} T = 0$$

on solutions of $\mathscr{E}$.

A conservation law is *trivial* if $X = D_x(p)$, $T = D_t(p)$ for some function p.

Exercise 39. Prove that a conservation laws is trivial if and only if $\delta(X) = 0$.

A function ψ is called a *cosymmetry* of $\mathscr{E}$ if

$$\ell_{\mathscr{E}}^*(\psi) = 0.$$

Exercise 40. Prove that for any conservation law, the function $\delta(X)$ is a cosymmetry.

Let φ and ψ be symmetry and cosymmetry of $\mathscr{E}$, respectively, i.e.,

$$D_t(\varphi) = \ell_F(\varphi), \qquad D_t(\psi) = -\ell_F^*(\psi).$$

Then

$$D_t(\psi\varphi) = D_t(\psi)\varphi + \psi D_t(\varphi) = -\varphi\ell_F^*(\psi) + \psi\ell_F(\varphi) = D_x(T_{\varphi,\psi})$$

for some $T_{\varphi,\psi}$ by (14), i.e.,

$$\omega_{\varphi,\psi} = \psi\varphi\,dx + T_{\varphi,\psi}\,dt$$

is a conservation law. Hence, $D_x^{-1}\circ\psi$ acts on φ in a well-defined way if and only if this conservation law is trivial. Using this fact, consider two examples.

5.3. **KdV.** Consider the KdV equation in the form (10).

Exercise 41. Check that that the operator

$$R = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1} \tag{15}$$

satisfies Equation (13).

Let us prove that it generates an infinite hierarchy of symmetries for the KdV equation. What we actually need to show is that the action of D_x^{-1} on symmetries is well defined.

To this end, note that

$$\varphi_3^1 = tu_3 + \left(tu + \frac{1}{3}x\right)u_1 + \frac{2}{3}u$$

is a scaling symmetry; due to this fact the equation becomes homogeneous with respect to the weights

$$|x| = 1, \quad |t| = 3, \quad |u_k| = -(k+2);$$

consequently, polynomial symmetries and cosymmetries can be considered to be homogeneous as well.

The weight of the symmetry $\varphi_{2k-1}^0 = u_{2k-1} + \dots$ (if such a symmetry exists) is $-(2k+1)$.

Exercise 42. Show that all (x,t)-independent solutions of the equation

$$D_t(\psi) = uD_x(\psi) + D_x^3(\psi),$$

i.e., cosymmetries are of the form

$$\psi_0 = 1, \quad \psi_k = u_{2k-2} + \dots, \qquad k = 1, 2, \dots;$$

Hence, $|\psi_k| = -2k$. But

$$\left|\delta(\psi_0 \cdot \varphi^0_{2k-1})\right| = \left|\varphi^0_{2k-1}\right| - |u| = 1 - 2k.$$

Consequently, $\delta(\psi_0 \cdot \varphi^0_{2k-1}) = 0$ and the conservation law $\omega_{1,\varphi^0_{2k-1}}$ is trivial. Thus, the result follows by induction.

Exercise 43. Prove that the action of the operator from Exercise 37 on symmetries of the Burgers equation is also well defined.

6. What happens when applying Lenard's operator to (x,t)-dependent symmetries?

Let us now apply the operator (15) to the symmetry φ^1_1 of the KdV equation. One has

$$R(\varphi^1_1) = \left(D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}\right)(tu_1 + 1) = tu_3 + \frac{2}{3}u(tu_1 + 1) + \frac{1}{3}u_1(tu + x) = \varphi^1_3.$$

But the next step leads to

$$\begin{aligned} R(\varphi^1_3) &= \left(D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}\right)\left(tu_3 + \left(tu + \frac{1}{3}x\right)u_1 + \frac{2}{3}u\right) \\ &= t(\underbrace{u_5 + uu_3 + u_1u_2 + u^2u_1}_{\varphi^0_5}) + x(\underbrace{u_3 + uu_1}_{\varphi^0_3}) + \frac{4}{3}u_2 + \frac{4}{9}u^2 + \frac{1}{9}u_1 D_x^{-1}(u). \end{aligned}$$

Of course, the most interesting part of this expression is the last summand: objects of such a type do not belong to the geometrical picture we had until now. Formally, one can introduce a new variable u_{-1} such that

$$D_x(u_{-1}) = u,$$

but at the moment there is no place for such a nonlocal quantity in our current world. Let us extend the latter.

7. Geometrization: coverings

Geometric setting for nonlocalities is provided by the notion of a covering.

7.1. **Definitions.** Let $\mathscr{E}$ be an infinitely prolonged differential equation with the Cartan distribution $\mathscr{C} = \mathscr{C}(\mathscr{E})$, $\dim \mathscr{C} = n$. Consider another manifold $\tilde{\mathscr{E}}$ endowed with a formally integrable[6] distribution $\tilde{\mathscr{C}}$ of the same dimension. We say that $\tilde{\mathscr{E}}$ *covers* $\mathscr{E}$ if a vector bundle $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ is given such that

$$d\tau(\tilde{\mathscr{C}}_{\tilde{\theta}}) = \mathscr{C}_{\tau(\tilde{\theta})}$$

for any $\tilde{\theta} \in \tilde{\mathscr{C}}$. The map τ is called a *covering* and the dimension of fibers is called the *dimension* of this covering.

Let in local coordinates $D_{x^1}, \dots, D_{x^n}$ be the total derivatives in $\mathscr{E}$ and $w^1, w^2, \dots, w^l, \dots$ be coordinates in the fiber of τ (*nonlocal variables*). Due to the definition, D_{x^i} can be uniquely lifted to the fields

$$\tilde{D}_{x^i} = D_{x^i} + X_i, \qquad X_i = \sum X^\alpha_i \frac{\partial}{\partial w^\alpha}, \tag{16}$$

lying in $\tilde{\mathscr{C}}$ and such that

$$[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = D_{x^i}(X_j) - D_{x^j}(X_i) + [X_i, X_j] \tag{17}$$

for all $1 \le i < j \le n$.

Exercise 44. Prove that locally, $\tilde{\mathscr{E}}$ is always an infinitely prolonged equation determined by the relations

$$\frac{\partial w^\alpha}{\partial x^i} = X^\alpha_i. \tag{18}$$

[6] I.e., such that $[\tilde{\mathscr{C}}, \tilde{\mathscr{C}}] \subset \tilde{\mathscr{C}}$.

System (18) is overdetermined and compatible modulo the equation $\mathscr{E}$.

A *morphism* of two coverings is a map $g\colon \tilde{\mathscr{E}} \to \tilde{\mathscr{E}}'$ such that the diagram

$$\begin{tikzcd} \tilde{\mathscr{E}} \arrow[rr, "g"] \arrow[dr, "\tau"'] & & \tilde{\mathscr{E}}' \arrow[dl, "\tau'"] \\ & \mathscr{E} & \end{tikzcd}$$

is commutative and $dg(\tilde{\mathscr{C}}_{\tilde{\theta}}) \subset \tilde{\mathscr{C}}'_{g(\tilde{\theta})}$. Two coverings are equivalent if g is a diffeomorphism.

A covering is trivial if for any $\theta \in \tilde{\mathscr{E}}$ there exist coordinates in which the fields X_i in (16) vanish. This means that the covering is locally equivalent to

$$\tau\colon \mathscr{E} \times \mathbb{R}^{\dim \tau} \to \mathscr{E}, \qquad \tilde{D}_{x^i} = D_{x^i}, \qquad i = \dots, n.$$

7.2. **Examples.** Consider several examples.

Example 3. Geometric treating of the situation considered in the end of Section 6 is as follows.

Consider the KdV equation denoted by $\mathscr{E}$. Let $\tilde{\mathscr{E}} = \mathscr{E} \times \mathbb{R}$ with the variable w being a coordinate in $\mathbb{R}$. Put

$$\tilde{D}_x = D_x + u\frac{\partial}{\partial w}, \qquad \tilde{D}_t = D_t + \left(\frac{1}{2}u^2 + u_2\right)\frac{\partial}{\partial w}.$$

Exercise 45. Check that $[\tilde{D}_x, \tilde{D}_t] = 0$ and w enjoys the equation

$$w_t = \frac{1}{2}w_x^2 + w_{xxx}$$

(this the so-called the *potential Korteweg-de Vries*, or pKdV, *equation*). Obviously, w is a formalization of the quantity u_{-1} above.

Exercise 46. Prove that a similar construction,

$$\tilde{D}_x = D_x + u\frac{\partial}{\partial w}, \qquad \tilde{D}_t = D_t + \left(\frac{1}{2}u^2 + u_1\right)\frac{\partial}{\partial w},$$

applied to the Burgers equation leads to the equation

$$w_t = \frac{1}{2}w_x^2 + w_{xx},$$

which is linearizable.

Example 4. Consider the sine-Gordon equation

$$u_{xy} = \sin u.$$

Exercise 47. Prove that

$$\tilde{D}_x = D_x + \left(2\lambda \sin\frac{w+u}{2} + u_x\right)\frac{\partial}{\partial w}, \quad \tilde{D}_y = D_y + \left(\frac{2}{\lambda}\sin\frac{w-u}{2} - u_y\right)\frac{\partial}{\partial w}, \qquad \lambda \neq 0,$$

is a covering and w satisfies the same equation for any nonvanishing λ.

Exercise 48. Prove that for these coverings are pair-wise inequivalent for different values of λ.

Example 5. Consider the pKdV equation in the form

$$u_t + 3u_x^2 + u_{xxx} = 0. \tag{19}$$

Exercise 49. Prove that the fields

$$\begin{aligned}\tilde{D}_x =&D_x + \left(\lambda - u_x - \frac{1}{2}(u-w)^2\right)\frac{\partial}{\partial w}, \\ \tilde{D}_t =&D_t + \big((u-w)(u_{xx}-w_{xx}) - 2(u_x^2 + u_x w_x + w_x^2)\big)\frac{\partial}{\partial w}\end{aligned} \tag{20}$$

determine a covering structure in $\mathscr{E}\times\mathbb{R}$ for any λ and the new w satisfies the pKdV equation.

Exercise 50. Prove that for these coverings are pair-wise inequivalent for different values of λ.

Example 6. Let $\mathscr{E}$ be an equation in two independent variables x and y and ω be a conservation law, i.e., a form $X\,dx + Y\,dy$ such that $D_x(Y) = D_y(X)$. Then by

$$\tilde{D}_x = D_x + X, \qquad \tilde{D}_y = D_y + Y$$

we introduce a covering structure τ_ω in $\mathscr{E}\times\mathbb{R}$. E.g., the coverings from Example 3 and Exercise 46 are of this kind.

Exercise 51. Prove that two coverings τ_ω and $\tau_{\omega'}$ are equivalent if and only if ω and ω' differ by a trivial conservation law.

Example 7. Let $X \in \mathscr{C}(\mathscr{E})$ and $\tilde{X} \in \tilde{\mathscr{C}}$ be its (unique) lift to $\tilde{\mathscr{E}}$. Denote by $L \subset C^\infty(\tilde{\mathscr{E}})$ the space of fiber-wise (with respect to τ) linear functions. The covering is *linear* if $\tilde{X}(L) \subset L$ for any $X \in \mathscr{C}$. This means that the fields X_i in (16) are of the form

$$X_i = \sum_{\alpha\beta} a_i^{\alpha\beta} w^\alpha \frac{\partial}{\partial w^\beta}.$$

If the matrices $A_i = (a_i^{\alpha\beta})$ belong to a Lie algebra $\mathfrak{g}$ we say that τ is a $\mathfrak{g}$-*valued zero-curvature representation* and write $\tilde{D}_{x^i} = D_{x^i} + A_i$. In this case, equation (18) reads

$$D_{x^i}(A_j) - D_{x^j}(A_i) + [A_i, A_j] = 0,$$

where $[A_i, A_j] = A_i \circ A_j - A_j \circ A_i$.

Exercise 52. Prove that if we take the KdV equation in the form $u_t + uu_x + u_{xxx} = 0$ then the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -u+\lambda & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -u_1 & 2u+4\lambda \\ -u_2 - 2u^2 - 2\lambda u + 4\lambda^2 & u_1 \end{pmatrix}$$

provide a λ-parametric family of $\mathfrak{sl}_2$-valued ZCR.

8. The Wahlquist-Estabrook algebra

Until now, we had no regular way to describe coverings. In a particular case of two-dimensional evolution equations $u_t = f(u, u_1, \dots, u_k)$, where f does not depend on x and t explicitly, there is a method that allows to describe a class of coverings. It goes as follows.

Assume that the τ-vertical fields X and T do not depend of x and t either and, in addition, they are independent of all u_i for $i \geq k$. Then such coverings (WE-coverings) are in one-to-one correspondence with representations of a certain Lie algebra (the *Wahlquist-Estabrook algebr*a) in the Lie algebra of vector fields on the fiber of τ; we denote this algebra by $\mathfrak{we}(\mathscr{E}) = \mathfrak{we}$.

Consider examples.

8.1. **The Burgers equation and the Cole-Hopf transformation.** For the Burgers equation $u_t = uu_x + u_{xx}$ the ansatz reads

$$X = X(w, u, u_1), \qquad T = T(w, u, u_1),$$

where w is the collection of all nonlocal variables. Then

$$u_1 \frac{\partial T}{\partial u} + u_2 \frac{\partial T}{\partial u_1} - (uu_1 + u_2)\frac{\partial X}{\partial u} - (u_1^2 + uu_2 + u_3)\frac{\partial X}{\partial u_1} + [X, T] = 0,$$

from where it follows that $X = X(w, u)$ and

$$\frac{\partial T}{\partial u_1} = \frac{\partial X}{\partial u}, \quad u_1 \frac{\partial T}{\partial u} - uu_1 \frac{\partial X}{\partial u} + [X, T] = 0. \tag{21}$$

Hence,

$$T = u_1 \frac{\partial X}{\partial u} + A(w, u).$$

Substituting this expression to the second equation in (21), one obtains

$$u_1 \left(u_1 \frac{\partial^2 X}{\partial u^2} + \frac{\partial A}{\partial u} \right) - uu_1 \frac{\partial X}{\partial u} + \left[X, u_1 \frac{\partial X}{\partial u} + A \right] = 0,$$

from where it follows that

$$\frac{\partial^2 X}{\partial u^2} = 0, \quad \frac{A}{\partial u} - u \frac{\partial X}{\partial u} + \left[X, \frac{\partial X}{\partial U} \right], \quad [X, A] = 0.$$

By similar computations, we finally obtain that any WE-covering is of the form

$$\tilde{D}_x = D_x + ua + b, \qquad \tilde{D}_t = D_t + \left(\frac{1}{2} u^2 + u_1 \right) a + u[a, b] + c,$$

where a, b, c are vector fields that depend on w only.

Exercise 53. Prove that these fields satisfy the relations

$$[a, [a, b]] = \frac{1}{2}[a, b], \quad [a, c] = [b, [b, a]], \quad [b, c] = 0. \tag{22}$$

They define the algebra $\mathfrak{we}$ for the Burgers equation.

Using these relations it becomes possible to describe all one-dimensional coverings up to an equivalence. Two cases are to be considered: (i) $a = 0$ and (ii) $a \neq 0$.

Exercise 54. Show that all the coverings corresponding to the first case are trivial.

In Case (ii), in a neighborhood of a generic point there is an equivalence transformation under which the field a transforms to $\partial/\partial w$ and (22) becomes

$$\beta'' = \frac{1}{2}\beta', \quad (\beta')^2 - \beta\beta'' = \gamma', \quad \beta'\gamma = \beta\gamma',$$

where "prime" denotes the w-derivative and $b = \beta\partial/\partial w$, $c = \gamma\partial/\partial w$.

Exercise 55. Solving this system of ODEs, show that all WE-coverings are equivalent to the following ones:

$$\begin{aligned}
\tau^0 : \tilde{D}_x &= D_x + u\frac{\partial}{\partial w}, \\
\tilde{D}_t &= D_t + \left(u_1 + \frac{1}{2}u^2 \right), \\
\tau_\lambda^+ : \tilde{D}_x &= D_x + (u + \mathrm{e}^{\frac{w}{2}} + \lambda)\frac{\partial}{\partial w}, \\
\tilde{D}_t &= D_t + \left(u_1 + \frac{1}{2}u^2 + \frac{1}{2}(u - \lambda)\,\mathrm{e}^{\frac{1}{2}w} - \frac{\lambda^2}{2} \right)\frac{\partial}{\partial w}, \\
\tau_\lambda^- : \tilde{D}_x &= D_x + (u - \mathrm{e}^{\frac{w}{2}} + \lambda)\frac{\partial}{\partial w},
\end{aligned}$$

$$\tilde{D}_t = D_t + \left(u_1 + \frac{1}{2}u^2 - \frac{1}{2}(u-\lambda)\,\mathrm{e}^{\frac{1}{2}w} - \frac{\lambda^2}{2}\right)\frac{\partial}{\partial w},$$

which are pair-wise inequivalent, $\lambda \in \mathbb{R}$.

The covering τ^0 was discussed in Exercise 46 and is equivalent to the one with

$$w_x = \frac{1}{2}wu.$$

This is the so-called *Cole-Hopf transformation.*

Exercise 56. Show the covering equation in this case is $w_t = w_{xx}$, i.e., the heat equation.

8.2. **KdV and the Miura transformation.** Consider the KdV equation $u_t = uu_x + u_{xxx}$.

Exercise 57. Using computations similar to those from Section 8.1 show that any WE-covering over the KdV equation is of the form

$$\tilde{D}_x = D_x + u^2 a + ub + c,$$

$$\tilde{D}_x = D_x + \left(2uu_2 - u_1^2 + \frac{2}{3}u^3\right) + \left(u_2 + \frac{1}{2}u^2\right)b + +u_1[b,c] - \frac{1}{2}u^2[b,[b,c]] + u[c,[c,b]] + d.$$

The algebra $\mathfrak{we}$ for the KdV equation is generated by elements a, b, c, and d with the relations

$$[a,b] = [a,c] = [c,d] = 0,$$
$$[b,[b,[b,c]]] = 0, \quad [b,d] + [c,[c,[c,b]]] = 0,$$
$$[a,d] + \frac{1}{2}[c,b] + \frac{3}{2}[b,[c,[c,b]]] = 0.$$

Up to an equivalence, pair-wise nonequivalent WE-coverings have the following description:

$$\tau^0 : \tilde{D}_x = D_x + u\frac{\partial}{\partial w},$$
$$\tilde{D}_t = D_t + \left(u_2 + \frac{1}{2}u^2\right)\frac{\partial}{\partial w},$$
$$\tau^1_\lambda : \tilde{D}_x = D_x + \left(u + \frac{1}{6}w^2 + \lambda\right)\frac{\partial}{\partial w},$$
$$\tilde{D}_t = D_t + \left(u_2 + \frac{1}{3}wu_1 + \frac{1}{3}u^2 + \frac{1}{3}\left(\frac{1}{6}w^2 - \lambda\right) - \frac{2}{3}\left(\frac{1}{6}w^2 + \lambda\right)\right)\frac{\partial}{\partial w},$$
$$\tau^2_\lambda : \tilde{D}_x = D_x + \left(u^2 + \lambda u\right)\frac{\partial}{\partial w},$$
$$\tilde{D}_t = D_t + \left(2uu_2 - u_1^2 + \frac{2}{3}u^3 + \lambda\left(u_2 + \frac{1}{2}u^2\right)\right)\frac{\partial}{\partial w},$$

$\lambda \in \mathbb{R}$.

The covering equation for τ^1_λ is the *modified KdV equation* (mKdV)

$$w_t = w^2 w_x + w_{xxx},$$

while the corresponding substitution

$$u = w_x - \frac{1}{6}w^2 - \lambda \tag{23}$$

is the *Miura transformation.*

Exercise 58. Describe the algebra $\mathfrak{we}$ for the mKdV equation and one-dimensional coverings over it.

Exercise 59. Describe the algebra $\mathfrak{we}$ for the pKdV equation (19) and one-dimensional coverings over it.

Relation between the KdV and mKdV equations by the Miura transformation allows one to establish existence of infinite number of conservation laws for KdV. Namely, let us change the nonlocal variable in such a way that (23) becomes

$$u = w - \varepsilon w_x - \frac{1}{6}\varepsilon^2 w^2. \tag{24}$$

Then

$$\begin{aligned} u_t - uu_x - u_{xxx} &= \left(1 - \varepsilon D_x - \frac{1}{3}\varepsilon^2 w\right)\left(w_t + \left(\frac{1}{3}\varepsilon^2 w^2 - w\right) w_x - w_{xxx}\right) \\ &= \left(1 - \varepsilon D_x - \frac{1}{3}\varepsilon^2 w\right)\left(D_t(w) - D_x\left(w_{xx} + \frac{1}{2}w^2 - \frac{1}{9}\varepsilon^2 w^3\right)\right) \end{aligned}$$

and the form

$$\omega = w\,dx + \left(w_{xx} + \frac{1}{2}w^2 - \frac{1}{9}\varepsilon^2 w^3\right) dt$$

is a conservation law of the covering equation with the density w.

Let us expand w in formal series

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots$$

Then the quantities w_i must also be densities of conservation laws. Inserting this expansion to (24) one obtains

$$u = \sum_i \varepsilon^i w_i - \varepsilon\left(\sum_i \varepsilon^i w_i\right)_x - \frac{1}{6}\varepsilon^2\left(\sum_i \varepsilon^i w_i\right)^2,$$

from where the recurrent relations for w_i follow:

$$\begin{aligned} &u = w_0, \\ &0 = w_1 - (w_0)_x, \\ &0 = w_2 - (w_1)_x - \frac{1}{6}w_0^2, \\ &0 = w_3 - (w_2)_x - \frac{1}{3}w_0 w_1, \\ &0 = w_4 - (w_3)_x - \frac{1}{6}(w_1^2 + 2w_0 w_2), \\ &\quad \dots, \end{aligned}$$

or

$$\begin{aligned} &w_0 = u, \\ &w_1 = u_x, \\ &w_2 = u_{xx} + \frac{1}{6}u^2, \\ &w_3 = u_{xxx} + \frac{2}{3}uu_x, \\ &w_4 = u_{xxxx} + \frac{5}{6}u_x^2 + uu_{xx} + \frac{1}{18}u^3, \\ &\quad \dots \end{aligned}$$

This provides the infinite hierarchy of conservation laws for the KdV equation.

Exercise 60. Prove that the odd conservation laws obtained in this way are trivial while the even ones are not.

9. An application: Bäcklund transformations

Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two (infinitely prolonged) equations. A *Bäcklund transformation* (BT) between $\mathscr{E}_1$ and $\mathscr{E}_2$ is the diagram

$$\begin{tikzcd} & \tilde{\mathscr{E}} \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ \mathscr{E}_1 & & \mathscr{E}_2, \end{tikzcd} \tag{25}$$

where τ_1 and τ_2 are coverings.

Assume that these coverings are finite-dimensional and f is a solution of $\mathscr{E}_1$. Then $\tilde{\mathscr{C}}$ induces on the $(n \cdot \dim \tau_1)$-dimensional manifold $\tau_1^{-1}([f]^\infty) \subset \tilde{\mathscr{E}}$ an integrable n-dimensional distribution. Consequently $\tau_1^{-1}([f]^\infty)$ is fibered by its integral manifolds. In a neighborhood of a generic point they are projected by τ_2 to solutions of $\mathscr{E}_2$. And vice versa.

Let τ_λ, $\lambda \in \mathbb{R}$, be a family of pair-wise inequivalent coverings such that the covering equations $\tilde{\mathscr{E}}_\lambda$ are pair-wise isomorphic. We shall be interested in the case when in (25) $\mathscr{E}_1 = \mathscr{E}_2$, $\tau_1 = \tau_\lambda$, $\tau_2 = \tau_\mu$, i.e.,

$$\begin{tikzcd} & \tilde{\mathscr{E}} \arrow[dl, "\tau_\lambda"'] \arrow[dr, "\tau_\mu"] & \\ \mathscr{E} & & \mathscr{E}. \end{tikzcd}$$

Denote such a BT by $\mathscr{B}_{\lambda,\mu}$.

The first BT of this type was constructed for the sine-Gordon equation (see Example 4), and in 1892 Luigi Bianchi discovered the following fact: let u be a solution of the sine-Gordon equation. Consider the following families of solutions:

u_λ	obtained from	u	by	$\mathscr{B}_{\lambda,\lambda}$
u_μ	obtained from	u	by	$\mathscr{B}_{\mu,\mu}$
$u_{\lambda,\mu}$	obtained from	u_λ	by	$\mathscr{B}_{\mu,\mu}$
$u_{\mu,\lambda}$	obtained from	u_μ	by	$\mathscr{B}_{\lambda,\lambda}$.

Then there exists a solution $u_{\lambda,\mu} = u_{\mu,\lambda}$,

$$\begin{tikzcd} & \boxed{u_\lambda} \arrow[dr, "\mu"] & \\ \boxed{u} \arrow[ur, "\lambda"] \arrow[dr, "\mu"'] & & \boxed{u_{\lambda,\mu} = u_{\mu,\lambda}} \\ & \boxed{u_\mu} \arrow[ur, "\lambda"'] & \end{tikzcd} \tag{26}$$

and this solution can be explicitly written in terms of u; it can be also understood as *nonlinear superposition* of solutions u_λ and u_μ. This is the *Bianchi permutability theorem*, and later it was established that this remarkable fact holds for a number of interesting equations of geometry and mathematical physics.

Consider examples.

9.1. **Sine-Gordon.** Let $\mathscr{E}$ be the sine-Gordon equation and τ_λ be the covering from Example 4, i.e.,

$$w_x = 2\lambda \sin\frac{w+u}{2} + u_x, \quad w_y = \frac{2}{\lambda}\sin\frac{w-u}{2} - u_y, \qquad \lambda \neq 0.$$

Exercise 61. Show that taking in (26) $u = 0$ one obtain the solutions

$$u_\lambda = 4\arctan\left(\lambda x + \frac{y}{\lambda} + \alpha\right), \qquad \alpha = \text{const}.$$

They are called 1-*kinks*.

Exercise 62. Continue the Bianchi procedure with the Bäcklund parameter μ and obtain the 2-*kink solution*

$$u_{\lambda,\mu} = 4\arctan\left(\frac{\lambda+\mu}{\lambda-\mu}\tan\frac{u_\lambda - v_\mu}{4}\right), \qquad \lambda \neq \mu.$$

See [4] for the animated presentation of kinks.

9.2. **pKdV and KdV.** Consider the covering (20):

$$w_x = \lambda - u_x - \frac{1}{2}(u-w)^2, \qquad w_t = (u-w)(u_{xx} - w_{xx}) - 2(u_x^2 + u_x w_x + w_x^2)$$

over the pKdV equation $u_t + 3uu_x + u_{xxx}$.

Exercise 63. Show that starting with the zero solution, i.e.,

$$u_x = \lambda - \frac{1}{2}u^2, \qquad u_t = uu_x - 2u_x^2.$$

one obtains

$$u_\lambda = \sqrt{2\lambda}\tanh\left(\sqrt{\frac{\lambda}{2}}(x - 2\lambda t)\right).$$

This is the *one-soliton solution* of the pKdV equation.

Exercise 64. Show that the superposition of two solutions of this type by the Bäcklund transformation is

$$u_{\lambda,\mu} = \frac{2(\lambda-\mu)}{u_\lambda - u_\mu}, \qquad \lambda \neq \nu$$

(the two-soliton solution).

Exercise 65. Using the covering from Example 3 obtain the corresponding solutions of the KdV equation.

10. Back to Section 6: nonlocal symmetries and shadows

Let us return back to Section 6 and recall that by applying the recursion operator to the scaling symmetry we obtained the expression

$$\varphi_5^1 = R(\varphi_3^1) = t(\underbrace{u_5 + uu_3 + u_1u_2 + u^2u_1}_{\varphi_5^0}) + x(\underbrace{u_3 + uu_1}_{\varphi_3^0}) + \frac{4}{3}u_2 + \frac{4}{9}u^2 + \frac{1}{9}u_1u_{-1},$$

where u_{-1} is a new variable that satisfies $(u_{-1})_x = u$. What kind of a quantity is this object?

On one hand, it lives on the space $\tilde{\mathscr{E}}$, being and element of $\varkappa$ (see Section 2.3) lifted to the covering equation. On the other hand, it satisfies the equation

$$\tilde{\ell}_{\mathscr{E}}(\varphi) = 0, \tag{27}$$

where $\tilde{\ell}_{\mathscr{E}}$ is obtained from $\ell_{\mathscr{E}}$ with changing all total derivatives by their lifts[7] to $\tilde{\mathscr{E}}$. But it is not a symmetry: to this end, we shall have to restore the coefficient at $\partial/\partial u_{-1}$ and this may give rise to new nonlocal objects.

Analyzing this example, let us give the following definition: take an equation $\mathscr{E}$ and its covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$.

- A symmetry of the covering equation is called a τ-*nonlocal symmetry* of $\mathscr{E}$.
- A derivation $S\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$ is a *nonlocal τ-shadow* if for any $X \in \mathscr{C}(\mathscr{E})$ one has

  $$\tilde{X} \circ S - S \circ X \in \mathscr{C}(\tilde{\mathscr{E}}),$$

  where $\tilde{X}$ is the natural lift of X to $\tilde{\mathscr{E}}$.

[7]Obviously, any $\mathscr{C}$-differential operator can be lifted to the covering equation in such a way.

To compute nonlocal symmetries and shadows, the following result is used:

Theorem. *Nonlocal symmetries are in one-to-one correspondence with solutions of the equation*

$$\ell_{\tilde{\mathscr{E}}}(\varphi) = 0,$$

where $\ell_{\tilde{\mathscr{E}}}$ is the linearization of the covering equation. Nonlocal shadows are the solutions of

$$\tilde{\ell}_{\mathscr{E}}(\varphi) = 0,$$

where $\tilde{\ell}_{\mathscr{E}}$ is the natural lift of $\ell_{\mathscr{E}}$ to $\tilde{\mathscr{E}}$.

Exercise 66. Prove the theorem above.

Thus, φ_5^1 is a shadow.

Obviously, any nonlocal symmetry defines a shadow, but is the converse true?

11. The reconstruction theorem

Consider an equation $\mathscr{E}$ and a covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ given by

$$\tilde{D}_{x^i} = D_{x^i} + \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}.$$

Let S be a τ-shadow and we want to construct a nonlocal τ-symmetry $\tilde{S}$ such that $\tilde{S}\Big|_{C^\infty(\mathscr{E})} = S$, or, in local coordinates,

$$\tilde{S} = S + \sum_\alpha \varphi^\alpha \frac{\partial}{\partial w^\alpha}.$$

Since $\tilde{S}$ is to be a symmetry, one has $[\tilde{D}_{x^i}, \tilde{S}] = 0$ for all i, or

$$\tilde{D}_{x^i}(\varphi^\alpha) = \tilde{S}(X_i^\alpha) = S(X_i^\alpha) + \sum_\beta \varphi^\beta \frac{\partial X_i^\alpha}{\partial w^\beta}. \tag{28}$$

Exercise 67. Prove that Equations (28) are compatible over $\mathscr{E}$.

Consequently, these are defining equations for a new covering $\tau_1\colon \tilde{\mathscr{E}}_1 \to \tilde{\mathscr{E}}$.

Two options are possible:

- The covering τ_1 is trivial. Then (28) can be solved and the symmetry $\tilde{S}$ does exist.
- If τ_1 is nontrivial then $\tilde{S}$ is a τ_1-shadow such that $\tilde{S}\Big|_{C^\infty(\tilde{\mathscr{E}})} = S$.

Thus, the following result is valid:

Theorem. *For any covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ and a τ-shadow S there exists a covering $\tau_1\colon \tilde{\mathscr{E}}_1 \to \tilde{\mathscr{E}}$ and a τ_1-shadow $\tilde{S}$ such that $\tilde{S}\Big|_{C^\infty(\tilde{\mathscr{E}})} = S$.*

Continuing this procedure *ad infinitum* we obtain a covering $\tau_*\colon \tilde{\tilde{\mathscr{E}}} \to \tilde{\mathscr{E}}$, where S reconstructs to a nonlocal τ_*-symmetry.

Let $n = 2$, $\dim\tau = 1$, and X_i^α do not depend on w (i.e., τ is associated with a conservation law[8] on $\mathscr{E}$). Then Equations (28) define a conservation law on $\tilde{\mathscr{E}}$. If this conservation law is trivial then the shadow is reconstructed, otherwise it determines a new shadow. So, obstructions to reconstruction are classes of conservation laws modulo trivial ones.

Consider an equation $\mathscr{E}$ and let $\Omega = \{\omega_1, \omega_2, \dots\}$ be a basis in the space of nontrivial conservation laws of $\mathscr{E}$. Associate to Ω the covering $\mathfrak{a}_1\colon \mathscr{A}_1 \to \mathscr{E}$ obtained by adding the nonlocal variables coorresponding to all $\omega \in \Omega$. Let $\mathscr{A}_2 = \mathscr{A}_1(\mathscr{A}_1)$, etc. The inverse limit of

$$\mathscr{E} \xleftarrow{\mathfrak{a}_1} \mathscr{A}_1 \xleftarrow{\mathfrak{a}_2} \mathscr{A}_2 \xleftarrow{\mathfrak{a}_3} \dots$$

is denoted by $\mathfrak{a}_*\colon \mathscr{A}_* \to \mathscr{E}$ called the *universal Abelian covering.*

[8]Such coverings are called *Abelian.*

Exercise 68. Prove the following result:

Theorem. *For any Abelian covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ there exists a covering $\tau'\colon \mathscr{A}_* \to \tilde{\mathscr{E}}$ such that the digram*

$$\begin{tikzcd} \mathscr{A}_* \arrow[rr, "\tau'"] \arrow[dr, "\mathfrak{a}"'] & & \tilde{\mathscr{E}} \arrow[dl, "\tau"] \\ & \mathscr{E}. & \end{tikzcd}$$

is commutative. Any τ-shadow can be reconstructed to a τ'-nonlocal symmetry.

The possibility to reconstruct shadows is practically important because there is no well-defined construction for the commutator of shadows while their reconstructions are "good" vector fields.

Example 8. Consider the KdV equation and the shadow φ_5^1.

Exercise 69. Prove that, when reconstructed in an appropriate covering, it acts recursively on φ_1^0 generating the hierarchy φ_k^0.

Symmetries with this property are called *master-symmetries.*

The only (but important!) problem that arises in examples of such a type is that the reconstruction procedure, in general, is not unique.

12. Back to recursion operators: the tangent covering υ

Consider an equation $\mathscr{E} \subset J^\infty(E) \to M$ and its *vertical tangent bundle* $\upsilon\colon \mathscr{V}(\mathscr{E}) \to \mathscr{E}$, i.e., the subbundle of the tangent bundle consisting of vectors that project to zero on M. For any function $f \in C^\infty(\mathscr{E})$ define its *vertical differential* ω_f by

$$i_X(\omega_f) = X^v(f) \tag{29}$$

where X^v is the projection of X to the fiber parallel to the Cartan distribution.

Exercise 70. Prove that this construction is well defined.

In coordinates, vertical vectors are

$$v = \sum_{j\gamma} a_\sigma^j \frac{\partial}{\partial u_\sigma^j},$$

where a_σ^j are smooth functions on $\mathscr{E}$, while

$$\omega_f = \sum_{j\sigma} \frac{\partial f}{\partial u_\sigma^j} \omega_\sigma^j,$$

where ω_σ^j are the Cartan forms.

Any field $X \in \mathscr{C}(\mathscr{E})$ can be canonically lifted to $\mathscr{V}(\mathscr{E})$: the forms (29) may be understood as fiber-wise linear functions on $\mathscr{V}(\mathscr{E})$ and we set

$$\tilde{X}(\omega_f) = L_X(\omega_f).$$

Exercise 71. Prove that $\widetilde{gX} = g\tilde{X}$ for any $X \in \mathscr{C}$ and $g \in C^\infty(\mathscr{E})$.

Thus, υ is a covering (the so-called *tangent covering* of $\mathscr{E}$).

Exercise 72. Prove that any symmetry of $\mathscr{E}$ is reconstructed to a symmetry of $\mathscr{V}(\mathscr{E})$. Is this true for υ-shadows?

Exercise 73. Prove that analytically the tangent covering is given by the equations

$$\ell_{\mathscr{E}}(p) = 0$$

added to $\mathscr{E}$, where $p = (p^1, \dots, p^m)$ is a new dependent variable.

Exercise 74. Prove that there exists a one-to-one correspondence between *holonomic sections*[9] of υ and symmetries of $\mathscr{E}$.

Example 9. If $\mathscr{E}$ is an evolution equation

$$u_t = f(x, t, u, u_1, \dots, u_k)$$

then we must consider additionally the equation

$$p_t = \frac{\partial f}{\partial u} p + \frac{\partial f}{\partial u_1} p_1 + \dots + \frac{\partial f}{\partial u_k} p_k.$$

In particular,

$$p_t = u_x p + u p_x + p_{xx}$$

for the Burgers equation and

$$p_t = u_x p + u p_x + p_{xxx}$$

for KdV.

Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be equations. Consider the diagram of coverings

$$\begin{tikzcd} & \mathscr{V}(\mathscr{E}_1) \arrow[r, "\upsilon_1"] & \mathscr{E}_1 \\ \mathscr{R} \arrow[ur, "\tau_1"] \arrow[dr, "\tau_2"'] & & \\ & \mathscr{V}(\mathscr{E}_2) \arrow[r, "\upsilon_2"'] & \mathscr{E}_2. \end{tikzcd} \tag{30}$$

Due to the properties of tangent coverings (see Exercise 74), the BT $\mathscr{R}$ must relate symmetries of the equations at hand with each other. In particular, for $\mathscr{E}_1 = \mathscr{E}_2$ it plays the role of a recursion operator.

Example 10 (M. Marvan)**.** The Lenard recursion operator for the KdV equation is presented in these terms as follows:

$$\begin{tikzcd} & \boxed{\begin{array}{c} w_x = p, \\ w_t = p_{xx} + up, \\ p_t = p_{xxx} + up_x + u_x p, \\ u_t = uu_x + u_{xxx} \end{array}} \arrow[dl, "p=w_x"'] \arrow[dr, "q=p_{xx}+\frac{2}{3}up+\frac{1}{3}u_x w"] & \\ \boxed{\begin{array}{c} p_t = p_{xxx} + up_x + u_x p \\ u_t = uu_x + u_{xxx} \end{array}} \arrow[dr, "\upsilon"'] & & \boxed{\begin{array}{c} q_t = q_{xxx} + uq_x + u_x q \\ u_t = uu_x + u_{xxx} \end{array}} \\ & \boxed{u_t = uu_x + u_{xxx}} \arrow[ur, "\upsilon"'] & \end{tikzcd}$$

Note that the geometric definition of recursion operators, i.e., Diagram (30), does not indicate the *direction* of action: one may start with the equation $\mathscr{E}_1$ and obtain symmetries of $\mathscr{E}_2$. Denoting thus obtained operator by R_{12} and starting with $\mathscr{E}_2$, we shall obtain the recursion operator R_{21} acting in the opposite direction which is natural to be understood as the *inverse* to R_{12}.

Exercise 75. Find the operator inverse to the Lenard recursion operator (Example 10).

[9] I.e, sections $\varphi\colon \mathscr{E} \to \mathscr{V}(\mathscr{E})$ such that $\varphi_*(\mathscr{C}(\mathscr{E})) \subset \mathscr{C}(\mathscr{V})$.

13. An alternative viewpoint: shadows of symmetries in υ

The tangent covering possesses yet another property that helps to construct recursion operators. We formulate it for the case of scalar evolution equations, though it is of a general nature.

Theorem. *Let $\mathscr{E}$ be a scalar evolution equation and $\upsilon\colon \mathscr{V} \to \mathscr{E}$ be its tangent covering. Consider a $\mathscr{C}$-differential operator $A = \sum a_{ij} D_x^i D_t^j$ and its lift $\tilde{A}$ to $\mathscr{V}$. Then solutions to the equation*

$$\tilde{A}(\varphi) = 0, \qquad \varphi = \sum_i b_i p_i,$$

are in one-to-one correspondence with the classes of operators $B = \sum_i b_i D_x^i$ such that

$$\square \circ \ell_{\mathscr{E}} = A \circ B \text{ for some } \square,$$

modulo operators of the form $B = \square' \circ \ell_{\mathscr{E}}$.

Exercise 76. Prove the theorem.

Thus, the above constructed operators B possess the following property:

$$B\colon \ker \ell_{\mathscr{E}} \to \ker A.$$

In particular, when one sets $A = \ell_{\mathscr{E}'}$ then B takes symmetries of the equation $\mathscr{E}$ to those of $\mathscr{E}'$.

Example 11. Consider the heat equation $u_t = u_{xx}$. Then the tangent covering is given by the diagram

$$\boxed{\begin{matrix} p_t = p_{xx} \\ u_t = u_{xx} \end{matrix}} \xrightarrow{\quad \upsilon \quad} \boxed{u_t = u_{xx}}$$

Solving the equation

$$\tilde{D}_t(b) = u_1 b + u\tilde{D}_x(b) + \tilde{D}_x^2(b)$$

for $b = b_0 p + b_1 p_1$, we obtain a solution

$$b_0 = -\frac{2}{u^2}, \qquad b_1 = \frac{2u}{u^2},$$

to which the operator

$$B = \frac{2}{u^2}(uD_x - u_x)$$

corresponds. It takes symmetries of the heat equation to those of the Burgers equation and is the linearization of the Cole-Hopf transformation.

If $A = \ell_{\mathscr{E}}$ we obtain recursion operators for symmetries of $\mathscr{E}$. In other words, recursion operators may be understood as shadows in the tangent covering.

Example 12. Consider the previous example and take $A = D_t - D_x^2$. Solving

$$\tilde{D}_t(b) = \tilde{D}_x^2(b)$$

for $b = b_0 p + b_1 p_1$, we obtain solutions

$$b = p, \quad b = p_1 \quad b = \frac{x}{2} + tp_1,$$

to which the recursion operators

$$\mathrm{id}, \quad R_0 = D_x, \quad R_1 = tD_x + \frac{x}{2}$$

correspond.

Exercise 77. Prove that these operators generate the entire associative algebra of recursion operators for the heat equation with

$$[R_0, R_1] = \frac{1}{2}\,\mathrm{id}$$

being the only relation.

If we do similar computations for the KdV equation the result will be trivial: the only solution is $b = \alpha p$, i.e., $R = \alpha\,\mathrm{id}$, $\alpha \in \mathbb{R}$. This is not surprising, because the Lenard recursion operator is nonlocal. Let us add nonlocalities to the picture.

Example 13. Consider the tangent covering of the KdV equation and the covering

$$q_x = p, \qquad q_t = up + p_{xx} \tag{31}$$

over $\mathscr{V}$. Then a nontrivial solution in this covering arises,

$$b = p_2 + \frac{2}{3}up + \frac{1}{3}u_1 q,$$

to which the Lenard operator corresponds.

The way of introducing nonlocalities used in the previous example is a part of a general scheme. Namely, consider a cosymmetry ψ (in the previous example, $\psi = 1$).

Exercise 78. Prove that the quantity $\psi \cdot p$ is the density of a conservation law on $\mathscr{V}(\mathscr{E})$ (compare with computations in Section 5.2).

Thus, to any cosymmetry of $\mathscr{E}$ there corresponds a "canonical" nonlocal variable in $\mathscr{V}$. When adding such variables, the recursion operator (if any) will arise in the form

$$R = \text{Local part } + \varphi_1 D_x^{-1} \circ \psi_1 + \dots \varphi_l D_x^{-1} \circ \psi_l,$$

where $\varphi_1, \dots, \varphi_l$ are symmetries and $\psi_1, \dots, \psi_l$ are cosymmetries. Such operators are called *weakly nonlocal.*

14. Commutativity of hierarchies

Our last topic concerns with the question: when do hierarchies of symmetries generated by a recursion operator commute?

Consider an operator R. Its *Nijenhuis torsion* is defined by

$$N_R(X, Y) = [R(X), R(Y)] - R(R(X) \circ Y - R(Y) \circ X).$$

An operator is called *hereditary* if $N_R = 0$. For a symmetry X, define the Lie derivative of R with respect to X by

$$L_X(R) = [R, \mathrm{ad}_X],$$

i.e.,

$$(L_X R)(Y) = R[Y, X] - [R(Y), X].$$

An operator is X-*invariant* if $L_X(R) = 0$. For a symmetry X and a recursion operator R, denote $R^n(X) = X_n$.

Theorem. *Let R be a hereditary operator and X and Y be symmetries. Then*

$$[X_m, X_n] = [X, Y]_{m+n} + \sum_{i=0}^{n-1} \big((L_X R)(Y_i)\big)_{m+n-i-1} - \sum_{j=0}^{m-1} \big((L_Y R)(Y_j)\big)_{m+n-j-1}.$$

In particular, if R is both X- and Y-invariant one has

$$[X_m, X_n] = [X, Y]_{m+n}.$$

Taking $X = Y$ we see that a hereditary operator generate commutative hierarchies when it is invariant with respect to the seed symmetry $X = X_0$.

Exercise 79. Prove the theorem.

Exercise 80. Prove that the Lenard operator for the KdV equation is hereditary.

15. Perspectives.

The next course (2013) will be concerned with the theory of Hamiltonian and symplectic operators for nonlinear PDEs.

References

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