THE 1^{st} SUMMER SCHOOL ON GEOMETRY OF DIFFERENTIAL EQUATIONS SILESIAN UNIVERSITY IN OPAVA SEPTEMBER 17–21, 2012 AN EXERCISE FROM THE BASIC COURSE: SYMMETRIES AND CONSERVATION LAWS

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Abstract. This is a step–by–step computation of the 5D Lie algebra of classical symmetries of the Burgers equation, and of its infinite family of higher symmetries.

Contents

Introduction

In the first part of this exercise we compute step–by–step the 5D Lie algebra of classical symmetries of the Burgers equation. The method is conventional, and consists in repeatedly using the identity principle for polynomials. We stress how the structure of jet spaces, which allows to assign to a smooth function both an *order* (i.e., the maximum derivative order it depends on) and, in the cases of our interest, also a *degree* (w.r.t. the polynomial dependency on the fiber coordinates), is essential to make the method effective.

In the second part, the structure of function algebras is examined more carefully from a theoretical viewpoint, and some results are presented, needed to estimate the powers of total derivatives (here "estimating" does not mean to control some norm, but rather the polynomial dependency) which appear

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in the defining equations for higher symmetries. Then we compute the commutator $[\partial_\beta, \overline{\ell}_F]\big|_{\mathcal{F}_k(\mathcal{E})}$, which in turn allows to split the linearized equation into a system of easier equations; this is done for any equation of the form $u_t = \Phi(u, u_x, u_{xx})$, and then specialized to the case $\Phi = u_{xx} + f(u, u_x)$. In this last setting, we introduce the estimates $\phi_k[a]$, and compute their Jacobi brackets, stressing that these are not yet solutions of the defining equations, but rather their approximations modulo some lower–order jet function algebra. Finally, commuting the classical symmetries from the first part with the $\phi_k[a]$'s, we obtain further restrictions on the dependency of $\phi_k[a]$ on jet variables, which eventually leads to a true (i.e., not approximated) expression for infinitely many higher symmetries.

The computations deliberately follows the scheme used by the computer algebra program Jets [2], in order to better understand (and appreciate) the behavior of the software itself.

The material collected here is but an extended exposition of the analogous exercises which can be found in [1].

1. Computing classical symmetries of Burgers equation

On the 8D jet space

$$J^2(\mathbb{R}^3, 2) = \{(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})\}$$

the Lie field with generating function

$$W \in C^\infty(J^1(\mathbb{R}^3, 2))$$

is given by

$$
\begin{aligned}
X_W = &- W_{u_x}\partial_x - W_{u_t}\partial_t && (1.1)\\
&+ (W - u_x W_{u_x} - u_t W_{u_t})\partial_u && (1.2)\\
&+ D_x^{[1]}(W)\partial_{u_x} && (1.3)\\
&+ D_t^{[1]}(W)\partial_{u_t} && (1.4)\\
&+ \Big(D_x^{[2]}(D_x^{[1]}(W)) + u_{xx}D_x^{[2]}(W_{u_x}) + u_{xt}D_x^{[2]}(W_{u_t})\Big)\,\partial_{u_{xx}} && (1.5)\\
&+ \Big(D_x^{[2]}(D_t^{[1]}(W)) + u_{tx}D_x^{[2]}(W_{u_x}) + u_{tt}D_t^{[2]}(W_{u_t})\Big)\,\partial_{u_{xt}} && (1.6)\\
&+ \Big(D_t^{[2]}(D_t^{[1]}(W)) + u_{tx}D_t^{[2]}(W_{u_x}) + u_{tt}D_t^{[2]}(W_{u_t})\Big)\,\partial_{u_{tt}} && (1.7)
\end{aligned}
$$

Observe that (1.1) is the "shadow" of X_W on the base manifold $X = \mathbb{R}^2$, and (1.1)–(1.2) is the "shadow" of X_W on $J^0(\mathbb{R}^3, 2) \equiv \mathbb{R}^3$. This means that we obtain *relative* vector fields. On the contrary, (1.1)–(1.4) is a true vector field on $J^1(\mathbb{R}^3, 2) \equiv \mathbb{R}^5$.

Compute

$$
\begin{aligned}
D_x^{[2]}(D_x^{[1]}(W)) &= D_x^{[2]}(W_x + u_x W_u) \\
&= D_x^{[2]}(W_x) + u_{xx} W_u + u_x D_x^{[2]}(W_u) \\
&= W_{xx} + u_x W_{xu} + u_{xx} W_{xu_x} + u_{tx} W_{xu_t} \\
&+ u_{xx} W_u \\
&+ u_x \left(W_{ux} + u_x W_{uu} + u_{xx} W_{uu_x} + u_{tx} W_{uu_t}\right)
\end{aligned}
$$

and add to it

$$
u_{xx} D_x^{[2]}(W_{u_x}) = u_{xx} \left(W_{u_x x} + u_x W_{u_x u} + u_{xx} W_{u_x u_x} + u_{tx} W_{u_x u_t}\right)
$$

and

$$
u_{xt} D_x^{[2]}(W_{u_t}) = u_{xt} \left(W_{u_t x} + u_x W_{u_t u} + u_{xx} W_{u_t u_x} + u_{tx} W_{u_t u_t}\right)
$$

The result consists of 17 terms

$$
\begin{aligned}
& W_{xx} + u_x W_{xu} + u_{xx} W_{xu_x} + u_{tx} W_{xu_t} \\
& + u_{xx} W_u \\
& + u_x \left(W_{ux} + u_x W_{uu} + u_{xx} W_{uu_x} + u_{tx} W_{uu_t}\right) \\
& + u_{xx} \left(W_{u_x x} + u_x W_{u_x u} + u_{xx} W_{u_x u_x} + u_{tx} W_{u_x u_t}\right) \\
& + u_{xt} \left(W_{u_t x} + u_x W_{u_t u} + u_{xx} W_{u_t u_x} + u_{tx} W_{u_t u_t}\right)
\end{aligned}
$$

which can be collected as

$$
\begin{aligned}
u_{xx} W_u + W_{xx} + 2u_x W_{xu} + 2u_{xx} W_{xu_x} + 2u_{xt} W_{xu_t} \\
+u_x^2 W_{uu} + 2u_x u_{xx} W_{uu_x} + 2u_x u_{tx} W_{uu_t} + u_{xx}^2 W_{u_x u_x} \\
+2u_{xx} u_{tx} W_{u_x u_t} + u_{xt}^2 W_{u_t u_t}
\end{aligned}
$$

as in [**?**] (example 4.1, pag 94).
Let now $F := uu_x + u_{xx} - u_t$. Then

$$
\begin{cases}
F_x & = 0 \\
F_t & = 0 \\
F_u & = u_x \\
F_{u_x} & = u \\
F_{u_t} & = -1 \\
F_{u_{xx}} & = 1 \\
F_{u_{xt}} & = 0 \\
0 & = 0
\end{cases}
$$

hence

$$\begin{aligned}
X_W(F) = &(W - u_x W_{u_x} - u_t W_{u_t}) \cdot u_x \\
&+ D_x^{[1]}(W) \cdot u \\
&+ D_t^{[1]}(W) \cdot (-1) \\
&+ \Big(D_x^{[2]}(D_x^{[1]}(W)) + u_{xx} D_x^{[2]}(W_{u_x}) + u_{xt} D_x^{[2]}(W_{u_t}) \Big) \cdot 1
\end{aligned}$$

so that the equation $X_W(F) - \lambda F = 0$ reads

$$\begin{aligned}
&u_x W - u_x^2 W_{u_x} - u_x u_t W_{u_t} \\
&+ u W_x + u u_x W_u \\
&- W_t - u_t W_u \\
&+ u_{xx} W_u + W_{xx} + 2u_x W_{xu} + 2u_{xx} W_{xu_x} + 2u_{xt} W_{xu_t} \\
&+ u_x^2 W_{uu} + 2u_x u_{xx} W_{uu_x} + 2u_x u_{tx} W_{uu_t} + u_{xx}^2 W_{u_x u_x} \\
&+ 2u_{xx} u_{tx} W_{u_x u_t} + u_{xt}^2 W_{u_t u_t} \\
&- \lambda \left(u u_x + u_{xx} - u_t \right) = 0
\end{aligned}$$

Organize the LHS of last equation as a 2nd order polynomial on J^2 with coefficients in J^1:

$$(u_x W - u_x^2 W_{u_x} - u_x u_t W_{u_t} + u W_x + u u_x W_u - W_t - u_t W_u$$

$$+ W_{xx} + 2u_x W_{xu} + u_x^2 W_{uu} - \lambda u u_x + \lambda u_t) \tag{1.8}$$

$$+ \left(W_u + 2W_{xu_x} + 2u_x W_{uu_x} - \lambda \right) u_{xx} \tag{1.9}$$

$$+ \left(2W_{xu_t} + 2u_x W_{uu_t} \right) u_{xt} \tag{1.10}$$

$$+ \left(W_{u_x u_x} \right) u_{xx}^2 \tag{1.11}$$

$$+ \left(W_{u_t u_t} \right) u_{xt}^2 \tag{1.12}$$

$$+ \left(2W_{u_x u_t} \right) u_{xx} u_{xt} \tag{1.13}$$

Since this polynomial has to be identically zero, we obtain

$$W_{u_x u_x} = 0, \quad W_{u_t u_t} = 0, \quad W_{u_x u_t} = 0$$

i.e., W is a 1st order polynomial on J^1 with coefficients in J^0:

$$W = A u_x + B u_t + C, \quad A, B, C \in J^0(\mathbb{R}^3, 2) \equiv \mathbb{R}^3 \tag{1.14}$$

Now, from (1.9) it follows

$$\begin{aligned}
\lambda &= W_u + 2W_{xu_x} + 2u_x W_{uu_x} \\
&= A_u u_x + B_u u_t + C_u + 2A_x + 2u_x A_u \\
&= C_u + 2A_x + 3A_u u_x + B_u u_t
\end{aligned}$$

Also, from (1.10), we get

$$\begin{aligned}
0 &= W_{xu_t} + u_x W_{uu_t} \\
&= B_x + u_x B_u
\end{aligned}$$

i.e. a polynomial on J^1, with coefficients in J^0, equated to zero. Hence,

$$B_x = 0, \quad B_u = 0,$$

i.e.,

$$B = B(t)$$

Consequently,

$$\lambda = C_u + 2A_x + 3A_u u_x$$

and (1.8) gives

$$\begin{aligned} 0 &= u_x W - u_x^2 W_{u_x} - u_x u_t W_{u_t} + u W_x \\ &+ u u_x W_u - W_t - u_t W_u + W_{xx} + 2u_x W_{xu} \\ &+ u_x^2 W_{uu} + (C_u + 2A_x + 3A_u u_x)(-u u_x + u_t) \end{aligned}$$

i.e.,

$$\begin{aligned} 0 &= u_x(Au_x + Bu_t + C) - u_x^2 A - u_x u_t B + u(Au_x + Bu_t + C)_x \\ &+ u u_x (Au_x + Bu_t + C)_u - (Au_x + Bu_t + C)_t - u_t(Au_x + Bu_t + C)_u \\ &+ (Au_x + Bu_t + C)_{xx} + 2u_x(Au_x + Bu_t + C)_{xu} \\ &+ u_x^2 (Au_x + Bu_t + C)_{uu} + (C_u + 2A_x + 3A_u u_x)(-u u_x + u_t) \end{aligned}$$

hence

$$\begin{aligned} 0 &= u_x(Au_x + Bu_t + C) - u_x^2 A - u_x u_t B + u(A_x u_x + C_x) \\ &+ u u_x (A_u u_x + C_u) - (A_t u_x + B' u_t + C_t) - u_t(A_u u_x + C_u) \\ &+ (A_{xx} u_x + C_{xx}) + 2u_x(A_{xu} u_x + C_{xu}) \\ &+ u_x^2 (A_{uu} u_x + C_{uu}) + (C_u + 2A_x + 3A_u u_x)(-u u_x + u_t) \end{aligned}$$

Collecting terms,

$$\begin{aligned} 0 &= uC_x - C_t + C_{xx} \\ &+ (A_{xx} + C - A_t + 2C_{xu} - uA_x)u_x \\ &+ (-B' + 2A_x)u_t \\ &+ (2A_{xu} + C_{uu} - 2uA_u)u_x^2 \\ &+ 2A_u u_x u_t \\ &+ A_{uu} u_x^3 \end{aligned}$$

So,

$$A_u = 0 \tag{1.15}$$

and

$$0 = uC_x - C_t + C_{xx} \tag{1.16}$$
$$+ (A_{xx} + C - A_t + 2C_{xu} - uA_x)u_x \tag{1.17}$$
$$+ (-B' + 2A_x)u_t \tag{1.18}$$
$$+ (C_{uu})u_x^2 \tag{1.19}$$

whence $C_{uu} = 0$, i.e.,

$$C = r(x,t)u + s(x,t)$$

From (1.17) follows

$$0 = A_{xx} + ru + s - A_t + 2(ru + s)_{xu} - uA_x \tag{1.20}$$
$$= A_{xx} + s - A_t + 2r_x + (r - A_x)u \tag{1.21}$$

Since A, r, s do not depend on u, the last expression is a 1st order polynomial in u, and hence

$$A_x = r \tag{1.22}$$

On the other hand, by (1.18),

$$2r = B', \tag{1.23}$$

which is a function of t only, so that

$$r_x = 0$$

and

$$r = r(t).$$

Now, from (1.16) we obtain

$$\begin{aligned} 0 &= uC_x - C_t + C_{xx} \\ &= u(ru + s)_x - (ru + s)_t + (ru + s)_{xx} \\ &= us_x - r'u - s_t + s_{xx} \\ &= -s_t + s_{xx} + (s_x - r')u \end{aligned}$$

whence

$$s_t = s_{xx} \tag{1.24}$$
$$s_x = r' \tag{1.25}$$

Differentiating (1.25) w.r.t. x we get

$$s_{xx} = 0$$

and hence, by (1.24),

$$s_t = 0 \tag{1.26}$$

It follows that $s = s(x)$ and $s'' = 0$, i.e.,

$$s(x) = wx + v, \quad w, v \in \mathbb{R} \tag{1.27}$$

Similarly, by differentiating (1.25) w.r.t. t we get $r'' = s_{xt}$, which is zero by (1.26), hence

$$r(t) = wt + n, \quad n \in \mathbb{R} \tag{1.28}$$

since $r' = s_x = w$.

Together, (1.27) and (1.28) furnish

$$C = C(x, t, u) = (wt + n)u + wx + v \tag{1.29}$$

Then, because of (1.22), we obtain $A = rx + k$, where $k = k(t)$, in view of (1.15).

We didn't yet make use of the u–constant part of (1.21), which gives

$$\begin{aligned} 0 &= s - A_t + 2r_x \\ &= wx + v - (rx + k)_t \\ &= wx + v - wx + k' \end{aligned}$$

i.e., $k' = v$, and so

$$A = A(x, t) = (wt + n)x + vt + k_0, \quad k_0 \in \mathbb{R} \tag{1.30}$$

Finally, (1.23) implies $B' = 2wt + 2n$, i.e.,

$$B = B(t) = wt^2 + 2nt + l, \quad l \in \mathbb{R} \tag{1.31}$$

Plugging the results from (1.30), (1.31) and (1.29) into (1.14) we get

$$W = ((wt+n)x+vt+k_0)u_x+(wt^2+2nt+l)u_t+(wt+n)u+wx+v, \quad w, v, n, k_0, l \in \mathbb{R}$$

or, equivalently

$$\begin{aligned} W = w\cdot &\left(xtu_x + t^2u_t + tu + x\right) \\ +v\cdot &\left(xu_x + 2tu_t + u\right) \\ +n\cdot &\left(tu_x + 1\right) \\ +k_0\cdot &\left(u_x\right) \\ +l\cdot &\left(u_t\right) \end{aligned}$$

i.e., W in an element of the 5D linear subspace of $C^\infty(J^1(\mathbb{R}^3, 2))$

$$\mathrm{Sym}(\mathcal{E}) := \mathrm{Span}\left\{xtu_x + t^2u_t + tu + x,\ xu_x + 2tu_t + u,\ tu_x + 1,\ u_x,\ u_t\right\}$$

Plugging each generator of $\mathrm{Sym}(\mathcal{E})$ into (1.1)–(1.4), we obtain the 5 classical symmetries of the Burgers equation

$$\begin{array}{rl} (tu + x)\partial_u - tx\partial_x - t^2\partial_t & \\ u\partial_u - x\partial_x - 2t\partial_t & \text{scale} \\ \partial_u - t\partial_x & \text{Galilean} \\ \partial_x & x\text{–translation} \\ \partial_t & t\text{–translation} \end{array}$$

2. Computing higher symmetries of Burgers equation

2.1. **Preparatory results about evolutionary equations.**

2.1.1. *Coordinates.* Recall that

$$
\begin{aligned}
J^\infty := J^\infty(\mathbb{R}^3, 2) = \{ (x, t, u, & \qquad (2.1)\\
& u_x, u_t \qquad (2.2)\\
& u_{xx}, u_{xt}, u_{tt}, \qquad (2.3)\\
& u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt} \qquad (2.4)\\
& \vdots \quad)\} \qquad (2.5)
\end{aligned}
$$

where (2.1) are coordinates on $J^0(\mathbb{R}^3, 2) \equiv \mathbb{R}^3$, (2.2) are coordinates along the fibers of $\pi_{1,0}$, (2.3) are coordinates along the fibers of $\pi_{2,1}$, (2.4) are coordinates along the fibers of $\pi_{3,2}$, and so on so forth.

Let

$$\mathcal{E} := F = 0, \quad F = u_t - \Phi(u, u_x, u_{xx}) \tag{2.6}$$

be a 2nd order evolutionary PDE, not depending on x, u, and $\mathcal{E}^\infty \subseteq J^\infty$ its infinite prolongation. Observe that

$$
\begin{aligned}
J^\infty = \{ (x, t, u, & \qquad (2.7)\\
& u_x, u_{xx}, u_{xxx}, \ldots \qquad (2.8)\\
& u_t, u_{tx}, u_{txx}, u_{txxx}, \ldots \qquad (2.9)\\
& u_{tt}, u_{ttx}, u_{ttxx}, u_{ttxxx}, \ldots \qquad (2.10)\\
& \vdots \quad)\} \qquad (2.11)
\end{aligned}
$$

where, again, (2.7) are coordinates on $J^0(\mathbb{R}^3, 2) \equiv \mathbb{R}^3$, whereas (2.8) are the coordinates along the fibers of $\pi_{\infty,0}$ whose corresponding partial derivatives **does not contain** any derivative w.r.t. t, (2.9) are the coordinates along the fibers of $\pi_{\infty,0}$ whose corresponding partial derivatives **contains exactly one** derivative w.r.t. t, (2.10) are the coordinates along the fibers of $\pi_{\infty,0}$ whose corresponding partial derivatives **contains exactly two** derivatives w.r.t. t, etc.

However, on $\mathcal{E}^\infty$, we have

$$u_{tx} = D_x(u_t) = D_x(\Phi) = u_x\Phi_u + u_{xx}\Phi_{u_x} + u_{xxx}\Phi_{u_{xx}} \tag{2.12}$$

and, in general,

$$\underbrace{u_{txx\cdots x}}_{n \text{ times } \text{“}x\text{”}} = D_x^n(u_t) = D_x^n(\Phi) = \text{ function of } u, u_x, u_{xx}, \ldots, \underbrace{u_{xx\cdots x}}_{2+n \text{ times } \text{“}x\text{”}} \tag{2.13}$$

In other words, any coordinate from the set (2.9) can be expressed, on $\mathcal{E}^\infty$, as a function of coordinates picked from the set (2.8). Now, in view of (2.13),

$$u_{tt} = D_t(u_t) = D_t(\Phi) = u_t\Phi_u + u_{xt}\Phi_{u_x} + u_{xxt}\Phi_{u_{xx}} \tag{2.14}$$

is again a function of of coordinates picked from the set (2.8) and, in general,

$$\underbrace{u_{ttxx\cdots x}}_{n \text{ times } \text{“}x\text{”}} = D_x^n(u_{tt}) = \text{ function of } u, u_x, u_{xx}, \ldots, \underbrace{u_{xx\cdots x}}_{2+n \text{ times } \text{“}x\text{”}} \tag{2.15}$$

Hence, also the coordinates from the set (2.10) can be expressed, on $\mathcal{E}^\infty$, as functions of the coordinates from the set (2.8). Continuing in this fashion, one realizes that (2.8) (together, of course with (2.7)), form a **coordinate system** on $\mathcal{E}^\infty$.

2.1.2. *Smooth functions algebras.* Recall also that

$$J^\infty(\mathbb{R}^2, 1) = \{(x, p_0, p_1, p_2, \ldots)\} \tag{2.16}$$

and that

$$\mathcal{F}(\mathcal{E}) := C^\infty(\mathcal{E}^\infty) \supseteq \cdots \supseteq \mathcal{F}_k(\mathcal{E}) := C^\infty(\mathcal{E}^\infty) \cup C^\infty(J^k) \supseteq \cdots \tag{2.17}$$

is a filtered algebra, i.e.,

$$\mathcal{F}(\mathcal{E}) = \bigcup_{k\in\mathbb{N}} \mathcal{F}_k(\mathcal{E}) \tag{2.18}$$

On the other hand, setting

$$u_0 := u \tag{2.19}$$

$$u_n := \underbrace{u_{xx\cdots x}}_{n \text{ times "}x\text{"}} \tag{2.20}$$

we obtain a (noncanonical) isomorphism

$$\mathcal{E}^\infty \longrightarrow J^\infty(\mathbb{R}^2, 1) \times \mathbb{R} \tag{2.21}$$

$$(x, t, u, u_x, u_{xx}, \ldots) \longmapsto ((x, u_0, u_1, u_2, \ldots), t) \tag{2.22}$$

Consequently,

$$\mathcal{F}_k(\mathcal{E}) \supseteq \mathcal{F}_k^i(\mathcal{E}) := C^\infty(J^k(\mathbb{R}^2, 1)) \cdot t^i \tag{2.23}$$

i.e., we can speak of the **order** of a function $\phi \in \mathcal{F}(\mathcal{E})$, which is the smallest number k such that $\phi \in \mathcal{F}_k(\mathcal{E})$, and, if it is polynomial in t, also of its **degree** (in t). So, $\mathcal{F}_k^i(\mathcal{E})$ is the linear space of functions on $\mathcal{E}$ which have order k and degree i.

From now on, we set

$$\partial_n := \frac{\partial}{\partial u_n} \equiv \frac{\partial}{\underbrace{\partial_{u_{xx\cdots x}}}_{n \text{ times "}x\text{"}}} \tag{2.24}$$

in view of (2.20). Accordingly,

$$\phi_n := \partial_n \phi, \quad \forall \phi \in \mathcal{F}(\mathcal{E}^\infty) \tag{2.25}$$

2.1.3. *Total derivatives.* So,

$$\partial_x, \partial_t, \partial_0, \partial_1, \partial_2, \ldots\ldots \tag{2.26}$$

form a basis of coordinate vector field on $\mathcal{E}^\infty$.

Accordingly, the x–total derivative restricted to $\mathcal{E}^\infty$ reads

$$D_x|_{\mathcal{E}^\infty} = \partial_x + u_{n+1}\partial_n \tag{2.27}$$

On the other hand, in view of (2.13) and (2.20), we have

$$D_t(x) = 0 \tag{2.28}$$

$$D_t(t) = 1 \tag{2.29}$$

$$D_t(u_n) = \underbrace{u_{txx\cdots x}}_{n \text{ times "}x\text{"}} = D_x^n(\Phi) \tag{2.30}$$

so that by restricting the t–total derivative to $\mathcal{E}^\infty$ we obtain

$$D_t|_{\mathcal{E}^\infty} = \partial_t + D_x^n(\Phi)\partial_n \tag{2.31}$$

Since the restricted x–total derivative (2.27) corresponds to the total derivative of $J^\infty(\mathbb{R}^2,1)$ via isomorphism (2.21), we shall denote it simply by D. On the other hand, the restricted t–total derivative (2.31) will be denoted by $\overline{D}_t$.

Lemma 1. *If $\phi \in \mathcal{F}(\mathcal{E})$ is such that $D(\phi)=0$, then $\phi=\phi(t)$.*

Proof. Thanks to identification (2.21), it suffices to prove that a function $\phi \in C^\infty(J^\infty(\mathbb{R}^2,1))$ of order k, with $D(\phi=0)$ must be zero. To this end, compute

$$D(\phi)=\phi_x+u_1\phi_0+u_2\phi_1+\cdots+p_{k+1}\phi_k \tag{2.32}$$

We get a function on $J^{k+1}(\mathbb{R}^2,1)$ which is polynomial along the fibers of $\pi_{k+1,k}$. So, since it is zero, all its coefficients must vanish, in particular that of p_{k+1}, which is ϕ_k. But this contradicts the fact that ϕ has order k. □

The fact that total derivative gain a linear term on higher order jets is the key of all computation techniques of generating functions.

But what about composition of total derivatives? To this end, recall that

$$[\partial_n, D]=\partial_{n-1}, \quad \forall n>0 \tag{2.33}$$

Such a property admits a straightforward generalization.

Lemma 2. *On $\mathcal{F}_k(\mathcal{E})$ it holds*

$$\partial_\beta \circ D^\alpha=\sum_{i=0}^{k}\left(\begin{array}{c}\alpha\\ \alpha-\beta+i\end{array}\right)D^{\alpha-\beta+i}\circ\partial_i \tag{2.34}$$

where ***a negative coefficient in the binomial makes it zero****.*

Proof. See [**?**]. □

Corollary 3 (Estimate of powers of total derivatives)**.**

$$D^{2r}(\mathcal{F}_k(\mathcal{E})) \subseteq \mathcal{F}_{k+r-1}(\mathcal{E})\cdot\{p_{k+r}^2, p_{k+r+1},\ldots,p_{k+2r}\} \tag{2.35}$$

$$D^{2r+1}(\mathcal{F}_k(\mathcal{E})) \subseteq \mathcal{F}_{k+r-1}(\mathcal{E})\cdot\{p_{k+r}, p_{k+r+1},\ldots,p_{k+2r}\} \tag{2.36}$$

Paraphrasing: D^α raises the order of ϕ by α, but the dependency of the result in the last $\frac{\alpha}{2}$ orders is linear.

2.1.4. *Defining equation.* Now compute the universal linearization of (2.6):

$$\ell_F=D_t-\Phi_u-\Phi_{u_x}D_x-\Phi_{u_{xx}}D_x^2 \tag{2.37}$$

and restrict (2.37) to $\mathcal{E}^\infty$:

$$\overline{\ell}_F=\overline{D}_t-\Phi_0-\Phi_1 D-\Phi_2 D^2 \tag{2.38}$$

Then, a function $\phi\in\mathcal{F}(\mathcal{E})$ is a (higher) symmetry for $\mathcal{E}$ iff

$$\overline{D}_t(\phi)=\Phi_0\phi+\Phi_1 D(\phi)+\Phi_2 D^2(\phi) \tag{2.39}$$

Now compose both sides of (2.38) on the left with ∂_β, with $\beta>2$:

$$\partial_\beta\circ\overline{\ell}_F=\partial_\beta\circ\overline{D}_t-\partial_\beta\circ\Phi_0-\partial_\beta\circ\Phi_1\circ D-\partial_\beta\circ\Phi_2\circ D^2 \tag{2.40}$$

and observe that

$$[\partial_\beta, \Phi_i] = 0, \quad i = 0, 1, 2$$

hence (2.40) becomes

$$\partial_\beta \circ \bar{\ell}_F = \partial_\beta \circ \overline{D}_t - \Phi_0 \circ \partial_\beta - \Phi_1 \circ \partial_\beta \circ D - \Phi_2 \circ \partial_\beta \circ D^2 \tag{2.41}$$

and hence,

$$\partial_\beta \circ \bar{\ell}_F = \partial_\beta \circ \overline{D}_t - \Phi_0 \circ \partial_\beta - \Phi_1 \circ R^1_\beta - \Phi_2 \circ R^2_\beta \tag{2.42}$$

where

$$R^\alpha_\beta := \partial_\beta \circ D^\alpha \tag{2.43}$$

Now, recalling the definition (2.31) of $\overline{D}_t$, we obtain

$$\begin{aligned}
\partial_\beta \circ \overline{D}_t &= \partial_\beta \circ (\partial_t + D^n(\Phi)\partial_n) \\
&= \partial_t \circ \partial_\beta + D^n(\Phi)\partial_n \circ \partial_\beta + R^n_\beta(\Phi)\partial_n \\
&= \overline{D}_t \circ \partial_\beta + R^n_\beta(\Phi)\partial_n
\end{aligned}$$

So, (2.42) reads

$$\partial_\beta \circ \bar{\ell}_F = \overline{D}_t \circ \partial_\beta + R^n_\beta(\Phi)\partial_n - \Phi_0 \circ \partial_\beta - \Phi_1 \circ R^1_\beta - \Phi_2 \circ R^2_\beta \tag{2.44}$$

Let us focus on $R^n_\beta(\Phi)$. Recalling that $\Phi \in \mathcal{F}_2(\mathcal{E})$, and applying Lemma 2, we get

$$\begin{aligned}
R^n_\beta(\Phi) = &\underbrace{\binom{n}{n-\beta} D^{n-\beta}(\Phi_0)}_{i=0} + \underbrace{\binom{n}{n-\beta+1} D^{n-\beta+1}(\Phi_1)}_{i=1} \\
&+ \underbrace{\binom{n}{n-\beta+2} D^{n-\beta+2}(\Phi_2)}_{i=2}
\end{aligned} \tag{2.45}$$

Now, plug (2.45) into (2.44), and evaluate the result on $\phi \in \mathcal{F}_k(\mathcal{E})$. Recalling that $\phi_n = 0$ for $n > k$, we get

$$\begin{aligned}
\partial_\beta(\bar{\ell}_F(\phi)) = &\overline{D}_t(\phi_\beta) \\
&+ \sum_{n=\beta}^{k} \left[\binom{n}{n-\beta} D^{n-\beta}(\Phi_0) + \binom{n}{n-\beta+1} D^{n-\beta+1}(\Phi_1) \right. \\
&\left. + \binom{n}{n-\beta+2} D^{n-\beta+2}(\Phi_2) \right] \phi_n & (2.46) \\
&+ \Phi_1 \phi_{\beta-1} + (\beta-1)D(\Phi_2)\phi_{\beta-1} + \Phi_2\phi_{\beta-2} & (2.47) \\
&- \Phi_0\phi_\beta - \Phi_1 R^1_\beta(\phi) - \Phi_2 R^2_\beta(\phi) & (2.48)
\end{aligned}$$

Observe that only the "$i = 0$" term of the LHS of (2.45) is fully accounted by the the summation $n = \beta, \ldots, k$ on (2.46). Concerning the "$i = 1$" term of the LHS of (2.45), the $n = \beta - 1$ summand appears as the first term of

(2.47). Finally, the $n=\beta-2,\beta-1$ summands of the LHS of (2.45) appears as the last two terms of (2.47).

Again by Lemma 2, we get

$$R^1_\beta = D\circ\partial_\beta+\partial_{\beta-1} \tag{2.49}$$
$$R^2_\beta = D^2\circ\partial_\beta+2D\circ\partial_{\beta-1}+\partial_{\beta-2} \tag{2.50}$$

Then, last equation reads

$$\begin{aligned}
\partial_\beta(\overline{\ell}_F(\phi)) &= \overline{D}_t(\phi_\beta) && (2.51)\\
&+\sum_{n=\beta}^{k}\left[\binom{n}{n-\beta}D^{n-\beta}(\Phi_0)+\binom{n}{n-\beta+1}D^{n-\beta+1}(\Phi_1)\right.\\
&\left.+\binom{n}{n-\beta+2}D^{n-\beta+2}(\Phi_2)\right]\phi_n\\
&+\Phi_1\phi_{\beta-1}+(\beta-1)D(\Phi_2)\phi_{\beta-1}+\Phi_2\phi_{\beta-2} && (2.52)\\
&-\Phi_0\phi_\beta-\Phi_1(D(\phi_\beta)+\phi_{\beta-1})-\Phi_2(D^2(\phi_\beta)+2D(\phi_{\beta-1})+\phi_{\beta-2}) && (2.53)
\end{aligned}$$

Notice that the 1st term of (2.52) cancels with the 3rd term of (2.53), and similarly for the 3rd term of the first and the last one of the latter. On the other hand, the RHS of (2.51), combined with the 1st, the 2nd and the 4th term of (2.53) produces $\overline{\ell}_F(\phi_\beta)$. Summing up,

$$\begin{aligned}
\partial_\beta(\overline{\ell}_F(\phi)) &= \overline{\ell}_F(\phi_\beta)\\
&+\sum_{n=\beta}^{k}\left[\binom{n}{n-\beta}D^{n-\beta}(\Phi_0)+\binom{n}{n-\beta+1}D^{n-\beta+1}(\Phi_1)\right.\\
&\left.+\binom{n}{n-\beta+2}D^{n-\beta+2}(\Phi_2)\right]\phi_n\\
&+(\beta-1)D(\Phi_2)\phi_{\beta-1}\\
&-2\Phi_2D(\phi_{\beta-1}) && (2.54)
\end{aligned}$$

In other words, we've found the following expression for the commutator (recall $\beta>2$)

$$\begin{aligned}
[\partial_\beta,\overline{\ell}_F]\big|_{\mathcal{F}_k(\mathcal{E})} &= \sum_{n=\beta}^{k}\left[\binom{n}{n-\beta}D^{n-\beta}(\Phi_0)+\binom{n}{n-\beta+1}D^{n-\beta+1}(\Phi_1)\right.\\
&\left.+\binom{n}{n-\beta+2}D^{n-\beta+2}(\Phi_2)\right]\partial_n\\
&+(\beta-1)D(\Phi_2)\partial_{\beta-1}\\
&-2\Phi_2D\circ\partial_{\beta-1}
\end{aligned}$$

2.1.5. *The case of* $\Phi=u_{xx}+f(u,u_x)$. From now on,

$$\Phi=u_{xx}+f(u,u_x). \tag{2.55}$$

It follows

$$\begin{aligned}\Phi_0 &= f_0\\ \Phi_1 &= f_1\\ \Phi_2 &= 1.\end{aligned}$$

Accordingly, (2.54) reduces to

$$\begin{aligned}\partial_\beta(\overline{\ell}_F(\phi)) = \overline{\ell}_F(\phi_\beta)&\\ +\sum_{n=\beta}^{k}\left[\binom{n}{n-\beta}D^{n-\beta}(f_0)+\binom{n}{n-\beta+1}D^{n-\beta+1}(f_1)\right]\phi_n&\\ -2D(\phi_{\beta-1}).&\end{aligned} \tag{2.56}$$

Suppose that $\phi\in\mathcal{F}_k(\mathcal{E})$ is a symmetry. Then, setting $\beta=k+1$, (2.56) becomes

$$0=-2D(\phi_k). \tag{2.57}$$

With $\beta=k$,

$$0=\overline{\ell}_F(\phi_k)+(f_0+kD(f_1))\phi_k-2D(\phi_{k-1}). \tag{2.58}$$

Continuing by decreasing β, we stop with $\beta=3$,

$$0=\overline{\ell}_F(\phi_3)+\sum_{n=3}^{k}\left[\binom{n}{n-3}D^{n-3}(f_0)+\binom{n}{n-2}D^{n-2}(f_1)\right]\phi_n-2D(\phi_2). \tag{2.59}$$

Hence, the $k-2+1=k-1$ functions $\phi_2,\phi_3,\phi_4,\dots,\phi_k$ satisfies the system of $k+1-3+1=k-1$ equations (2.57), (2.58)–(2.59).

Observe that, in virtue of Lemma 1, equation (2.57) dictates strong restriction on ϕ_k, namely

$$\phi_k=a_k(t). \tag{2.60}$$

Before continuing, perform the change of variables

$$\psi_n:=2^{k-n}\phi_n. \tag{2.61}$$

Plugging (2.60) into (2.58) we obtain

$$D(\psi_{k-1})=\overline{\ell}_F(\psi_k)+(f_0+kD(f_1))\psi_k. \tag{2.62}$$

Since $\overline{\ell}_F(\psi_k)=a_k'-f_0\psi_k$, last equation reads

$$D(\psi_{k-1})=a_k'+kD(f_1)a_k. \tag{2.63}$$

Hence, again by Lemma 1,

$$\psi_{k-1}=a_k'x+kf_1a_k+a_{k-1} \tag{2.64}$$

where $a_{k-1}=a_{k-1}(t)$. Indeed, by applying D to (2.64), we get (2.63).

Together, equations (2.60) and (2.64) forces any symmetry ϕ of order k to be of the form

$$\phi \equiv a_k u_k + \frac{a_k' x + k f_1 a_k + a_{k-1}}{2} u_{k-1} \qquad \bmod \mathcal{F}_{k-2}(\mathcal{E}). \tag{2.65}$$

It is convenient to set

$$\phi_k[a] := a u_k + \frac{a_k' x + k f_1 a + \underline{a}}{2} u_{k-1} \tag{2.66}$$

where $a, \underline{a}$ are arbitrary functions of t. By using Corollary 3, one can compute the Jacobi bracket

$$\{\phi_k[a], \phi_l[b]\}_{\mathcal{E}} \equiv \phi_{k+l-2}[c] \qquad \bmod \mathcal{F}_{k+l-3}(\mathcal{E}) \tag{2.67}$$

where

$$c = \frac{l a' b - k b' a}{2} \tag{2.68}$$

2.2. **The Burgers equation.** Consider now the equation given by $f = u u_x$, and recall that the Lie algebra of its classical symmetries is

$$\mathrm{Sym}(\mathcal{E}) := \mathrm{Span}\left\{ x t u_x + t^2 u_t + t u + x,\ x u_x + 2 t u_t + u,\ t u_x + 1,\ u_x,\ u_t \right\} \tag{2.69}$$

Rewrite the five generators above in the coordinate of $\mathcal{E}$:

$$\phi_1^0 := u_1 \tag{2.70}$$
$$\phi_1^1 := t u_1 + 1 \tag{2.71}$$
$$\phi_2^0 := u_2 + u_0 u_1 \tag{2.72}$$
$$\phi_2^1 := t u_2 + (t u_0 + \frac{1}{2} x) u_1 + \frac{1}{2} u_0 \tag{2.73}$$
$$\phi_2^2 := t^2 u_2 + (t^2 u_0 + t x) u_1 + t u_0 + x \tag{2.74}$$

Notice that "ϕ_k^i" means that ϕ has order k and degree i. The symmetries listed above correspond to the 4th, the 3rd, the 5th, the 2nd and the 1st in (2.69), respectively.

Now the properties of the Jacobi bracket (2.67) and (2.67), combined with the knowledge of the classical symmetries, allow to clarify the structure of a higher symmetry $\phi_k[a]$. More precisely, being ϕ_1^0 of the form $\phi_1[1]$ (see (2.66)), formulae (2.67) and (2.67) give

$$\{\phi_k[a], \phi_1^0\}_{\mathcal{E}} \equiv \frac{a' u_{k-1}}{2} \qquad \bmod \mathcal{F}_{k-2}(\mathcal{E}) \tag{2.75}$$

So, one may commute $k-2$ times the symmetry $\phi_k[a]$ with ϕ_1^0,

$$\left(\{\cdot, \phi_1^0\}_{\mathcal{E}}\right)^{k-2} (\phi_k[a]) = \{\{\cdots\{\phi_k[a], \phi_1^0\}_{\mathcal{E}}, \cdots \phi_1^0\}_{\mathcal{E}}, \phi_1^0\}_{\mathcal{E}} \equiv \frac{a^{(k-2)} u_2}{2^{k-2}} \qquad \bmod \mathcal{F}_1(\mathcal{E}). \tag{2.76}$$

Hence, modulo $\mathcal{F}_1(\mathcal{E})$, the symmetry $\left(\{\cdot, \phi_1^0\}_{\mathcal{E}}\right)^{k-2} (\phi_k[a])$ must belong to the above list, so its degree in t cannot exceed 2. It follows that the degree of $\phi_k[a]$ in t cannot exceed k.

So far, we only know which degree should posses a symmetry, **provided it existed**. In order to show existence, one need a so–called **master symmetry**, i.e., a symmetry which generates all the remaining ones by means of Jacobi commutators. In our case, the master symmetry is

$$\phi_3^1 = tu_3 + \frac{x+3tu_0}{2}u_2 + \frac{3}{2}tu_1^2 + \left(\frac{x}{2} + \frac{3tu_0}{4}\right)u_0u_1 + \frac{u_0^2}{4} \tag{2.77}$$

Thanks to (2.68), one sees that

$$\{\phi_k[a], \phi_3^1\}_{\mathcal{E}} \equiv \frac{3a't - ka}{2}u_{k+1} \qquad \mathrm{mod}\ \mathcal{F}_k(\mathcal{E}) \tag{2.78}$$

Hence (recalling that ϕ_1^0 is of the form $\phi_1[1]$),

$$\left(\{\cdot, \phi_3^1\}_{\mathcal{E}}\right)^k (\phi_1^0) \equiv (-2)^k k! u_{k+1} \qquad \mathrm{mod}\ \mathcal{F}_k(\mathcal{E}) \tag{2.79}$$

is a symmetry of the form $\phi_{k+1}[1]$, i.e., with coefficient of u_{k+1} equal to 1, modulo $\mathcal{F}_k(\mathcal{E})$ (1st **existence result**). Finally, such a symmetry can be used to produce symmetries of the form $\phi_k[t^i]$, as follows

$$\left(\{\cdot, \phi_2^2\}_{\mathcal{E}}\right)^i (\phi_{k+1}[1]) \equiv \lambda t^i u_k \qquad \mathrm{mod}\ \mathcal{F}_{k-1}(\mathcal{E}), \lambda \in \mathbb{R} \tag{2.80}$$

and this is the 2nd **existence result**.

References

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